Compulsory 1

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(1) $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$

$\mu = E[X] = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \qquad \Sigma = Cov(X) = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

$Y = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} X$

a) By theorem from lectures, we have that since X is normal, any linear transformation $Y = Ax + b$ is also normal and distributed by

$$Y \sim \mathcal{N}(A\mu + b, A\Sigma A^T)$$

In our case $b = \vec{0}$, $A = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$

So we obtain

$$\nu = E[Y] = E[AX + b] = AE[X] = A\mu = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -\sqrt{2} \\ \sqrt{2} \end{pmatrix}$$

$$\Sigma_y = Cov(Y) = Cov(AX + b) = A\,Cov(X)A^T = A\Sigma A^T = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

So $Y \sim \mathcal{N}(\nu, \Sigma_Y) = \mathcal{N}\left(\begin{pmatrix} -\sqrt{2} \\ \sqrt{2} \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\right)$

From Σ_Y, we notice that $(\Sigma_Y)_{12} = Cov(Y_1, Y_2) = 0 = Cov(Y_2, Y_1) = (\Sigma_Y)_{21}$
so we conclude that Y_1 and Y_2 are independent random variables

b) As Σ is symmetric, with positive eigenvalues
$\Rightarrow \Sigma$ is symmetric positive definite and so is Σ^{-1}

$\Sigma^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$, we can find the eigenvalues of Σ^{-1} which gives $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{4}$

The Euclidean distance d between two points is defined as

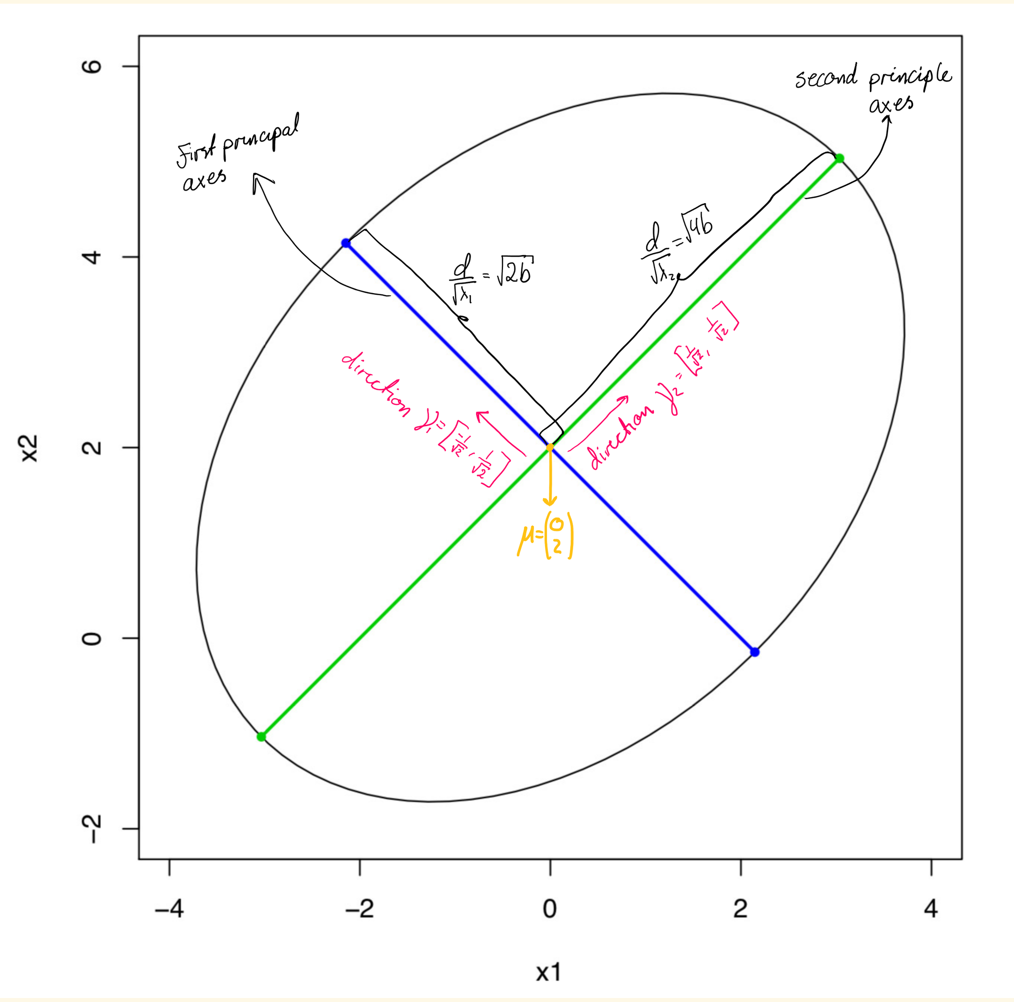
$$d^2(x,y) = (x-y)^T A (x-y)$$

Where A is a metric. So in our case, Σ^{-1} is the metric used to measure the distance between X and μ.

The contour of f, that is

$$(x-\mu)^T \Sigma^{-1} (x-\mu) = b$$

is thus the equation for the ellipsoid spanned out by the half-lengths $\frac{d}{\sqrt{\lambda_1}}$, $\frac{d}{\sqrt{\lambda_2}}$, where $d = \sqrt{b}$



In the Figure, the blue line and the green line are the principal axes.
The principal axes lie in the direction of the eigenvector of Σ^{-1} corresponding to the eigenvalues.
i.e.

γ_1 corresponds to $\lambda_1 = \frac{1}{2}$

γ_2 —— " —— $\lambda_2 = \frac{1}{4}$

We see that

$$(x-\mu)^T \Sigma^{-1} (x-\mu) = (x-\mu)^T \Gamma \Lambda^{-1} \Gamma^T (x-\mu)$$
$$= (\Lambda^{-1/2}\Gamma^T(x-\mu))^T (\Lambda^{-1/2\,T}\Gamma^T(x-\mu)) = (\Lambda^{-1/2}\Gamma^T(x-\mu))^T(\Lambda^{-1/2}\Gamma^T(x-\mu))$$

by the Jordan decomposition.

We define $Z = \Lambda^{-1/2}\Gamma^T(x-\mu)$

This is a linear transformation of X, so Z is normal

Note that $E[Z] = E[\Lambda^{-1/2}\Gamma^T(x-\mu)] = \Lambda^{-1/2}\Gamma^T(E(x) - \mu) = 0$

and $Cov(Z) = \Lambda^{-1/2}\Gamma^T Cov(x)(\Lambda^{-1/2}\Gamma^T)^T - 0 = \Lambda^{-1/2}\Gamma^T \Sigma \Gamma \Lambda^{-1/2} = \Lambda^{-1/2}\Gamma^T\Gamma\Lambda\Gamma^T\Gamma\Lambda^{-1/2} = \Lambda^{-1/2}\Lambda\Lambda^{-1/2} = I$

So $Z \sim \mathcal{N}(0, I_2)$

We now have $Z^T Z = Z_1^2 + Z_2^2 \sim \chi_2^2$

The probability of X falling within the ellipse is given by

$P\{(X-\mu)\Sigma^{-1}(X-\mu) \le b\} = P\{Z^T Z \le b\} = F_{Z^TZ}(b) \approx 0.9$, as it is the the inverse of the quantile function which we have been given.

The probability that x falls within the ellipsoid is thus 0.9 as $b = 4.6$

(2) a) $\bar{X}$ = vector of means

Let $\vec{X} = (X_1, \dots, X_n)$

We want $\bar{X} = \frac{1}{n}1^T\vec{X}$

$$\frac{1}{n}1^T X = \frac{1}{n}\left((1,1,\dots,1)\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}\right) = \frac{1}{n}(X_1 + X_2 + \dots + X_n) = \frac{1}{n}\sum_{i=1}^n X_i \quad \text{as defined}$$

We now want

$$S^2 = \frac{1}{n-1}X^T C X = \frac{1}{n-1}X^T C C X = \frac{1}{n-1}X^T C^T C X$$

$$= \frac{1}{n-1}(X_1, \dots, X_n)\begin{pmatrix} 1-1/n & \cdots & -1/n \\ \vdots & \ddots & \vdots \\ -1/n & \cdots & 1-1/n \end{pmatrix}^T \begin{pmatrix} 1-1/n & \cdots & -1/n \\ \vdots & \ddots & \vdots \\ -1/n & \cdots & 1-1/n \end{pmatrix}\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$$

$$= \frac{1}{n-1}\begin{pmatrix} X_1(1-1/n) - X_2/n - \dots - X_n/n \\ \vdots \\ -X_1/n - X_2/n - \dots - X_n/n + X_n \end{pmatrix}^T \begin{pmatrix} (1-1/n)X_1 - X_2/n - \dots - X_n/n \\ \vdots \\ -X_1/n - X_2/n - \dots - X_n/n + X_n \end{pmatrix} = \frac{1}{n-1}\begin{pmatrix} X_1 - \sum_{i=1}^n X_i/n \\ \vdots \\ X_n - \sum_{i=1}^n X_i/n \end{pmatrix}^T \begin{pmatrix} X_1 - \sum_{i=1}^n X_i/n \\ \vdots \\ X_n - \sum_{i=1}^n X_i/n \end{pmatrix}$$

$$= \frac{1}{n-1}\begin{pmatrix} X_1 - \bar{X} \\ X_2 - \bar{X} \\ \vdots \\ X_n - \bar{X} \end{pmatrix}^T \begin{pmatrix} X_1 - \bar{X} \\ X_2 - \bar{X} \\ \vdots \\ X_n - \bar{X} \end{pmatrix} = \frac{1}{n-1}\left((X_1-\bar{x})(X_1-\bar{x}) + (X_2-\bar{x})(X_2-\bar{x}) + \dots + (X_n-\bar{x})(X_n-\bar{x})\right)$$

$$= \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{x})^2 \quad \text{which is the definition given}$$

b) $$\frac{1}{n}1^T C = \frac{1}{n}(1,1,\dots,1)\begin{pmatrix} 1-1/n & -1/n & \cdots & -1/n \\ \vdots & & & \\ -1/n & -1/n & \cdots & 1-1/n \end{pmatrix} = \frac{1}{n}\begin{pmatrix} \overbrace{1 - \frac{1}{n} - \frac{1}{n} - \dots - \frac{1}{n}}^{n\text{-times}} \\ \vdots \\ -1/n - 1/n - \dots - 1/n + 1 \end{pmatrix} = \frac{1}{n}\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \vec{0}$$

This would imply that if I can simplify $Cov(\bar{x}, S^2)$ to $\frac{1}{n}1^T C$, they are independent.

Independence of $\bar{x}$ and S^2:

We know that $\bar{x} \sim \mathcal{N}(\mu, \frac{1}{n}\Sigma)$ and $\bar{x} = \frac{1}{n}1^T X$

So both $\bar{X}$ and CX are linear transformations of X, hence $\bar{X}$ and CX are independent iff $Cov(\bar{x}, CX) = \frac{1}{n}1^T Cov(X) C^T = 0 \qquad (C^T = C)$

so

$$\frac{1}{n}1^T Cov(X)C^T = \frac{1}{n}1^T \cdot \sigma^2 I \cdot C = \frac{\sigma^2}{n} \cdot 1^T C = 0$$

$\Rightarrow \bar{x}$ and CX are independent.

We now how $\bar{x} = Ax$ and $S^2 = \frac{1}{n-1}x^T B x$, $A = \frac{1}{n}1^T$, $B = C$

which means $\bar{x}$ and S^2 are indep iff $AB = \frac{1}{n}1^T C = 0$ as shown above. Hence $\bar{X}$ and S^2 are independent

Alternatively, notice that:

$$S^2 = \frac{1}{n-1}X^T C X = \frac{1}{n-1}X^T C C X = \frac{1}{n-1}X^T C^T C X = \frac{1}{n-1}(Cx)^T Cx = \frac{1}{n-1}\|Cx\|^2$$

So as $\bar{x}$ and Cx are indep, we must have that $\bar{x}$ and S^2 are indep

c) We have: $X \sim \mathcal{N}(\mu 1, \sigma^2 I)$, $\Sigma = \sigma^2 I \Rightarrow \Sigma^{1/2} = \sigma I$ and $\Sigma^{-1/2} = \sigma I^{-1} = \sigma I$

Define $\vec{\mu} = (\mu, \dots, \mu)$ as $E[X]$

and then by Mahalanobis

$$Y = \Sigma^{-1/2}(x-\mu) \sim \mathcal{N}(0, I)$$

By the theorem given, we have

$$Y^T C Y \sim \chi^2$$

$$Y^T C Y = (\Sigma^{-1/2}(x-\mu))^T C (\Sigma^{-1/2}(x-\mu)) = \left(\frac{X-\mu}{\sigma}\right)^T C \left(\frac{X-\mu}{\sigma}\right) = \left(\frac{X-\mu}{\sigma}\right)^T C^T C \left(\frac{X-\mu}{\sigma}\right) = \left(\frac{CX - C\mu}{\sigma}\right)^T \left(\frac{CX - C\mu}{\sigma}\right)$$

where $$C\mu = \begin{pmatrix} \mu - \overbrace{\mu/n - \mu/n - \dots - \mu/n}^{n\text{-times}} \\ \vdots \\ -\mu/n - \mu/n - \dots - \mu/n + \mu \end{pmatrix} = \vec{0}$$

$$\Rightarrow Y^T C Y = \left(\frac{CX - C\mu}{\sigma}\right)^T\left(\frac{CX - C\mu}{\sigma}\right) = \left(\frac{CX}{\sigma}\right)^T\left(\frac{CX}{\sigma}\right) = \frac{1}{\sigma^2}X^T C^T C X = \frac{1}{\sigma^2}X^T C C X = \frac{1}{\sigma^2}X^T C X = \frac{(n-1)}{\sigma^2}S^2$$

So $Y = \frac{(n-1)}{\sigma^2}S^2$, and C is idempotent and symm, so

$\frac{(n-1)}{\sigma^2}S^2 \sim \chi_r^2$ where r is the rank of C.

we know that for idempotent matrices we have $rank(C) = tr(C)$

and $Tr(C) = \sum_{i=1}^n 1 - 1/n = n - 1$, and we obtain

$$Y = \frac{(n-1)}{\sigma^2}S^2 \sim \chi_{n-1}^2$$