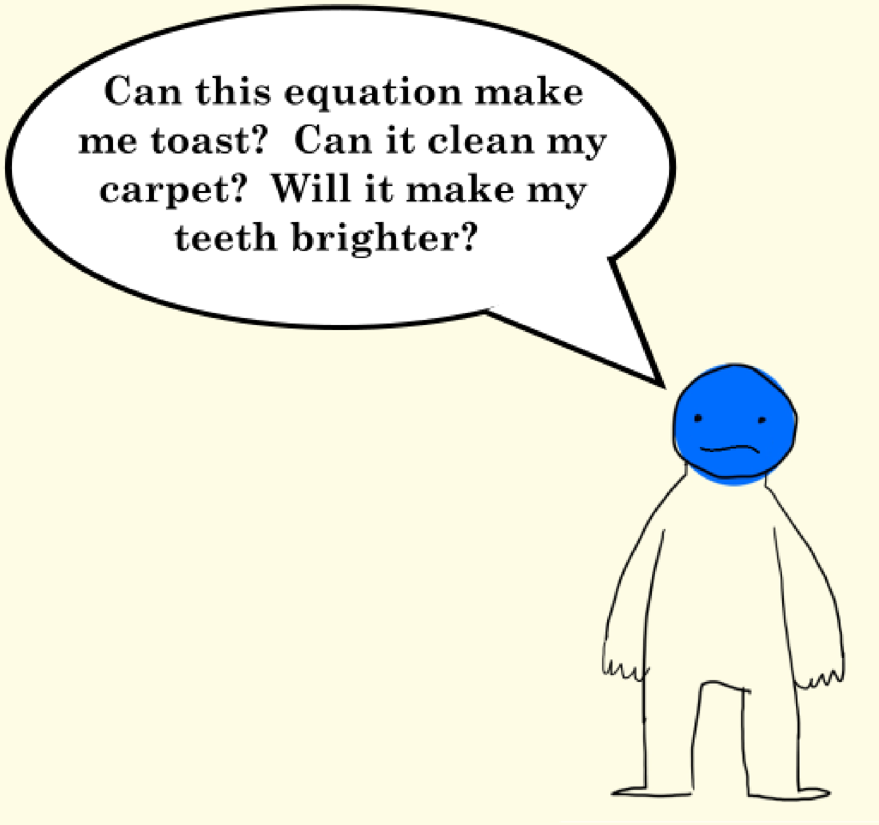
THE HEAT EQUATION

Solving the Heat Equation Analytically

Using the Heat Equation

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{1}{\rho C}\dot{q}$$



So we have the heat equation now. But how do we use it?

The equation describes factors (on the right hand side) that lead to a change in temperature (left hand side) at a single infinitesimal volume. These factors are the diffusion and the generation of thermal energy.

In a real world problem, this equation needs to solved at each point in the field. The overall solution will take the form of a time-dependent temperature field: $T(x, y, z, t)$.

We can find a solution analytically (i.e. with math!) or numerically (i.e. with a computer!).

Boundary Conditions

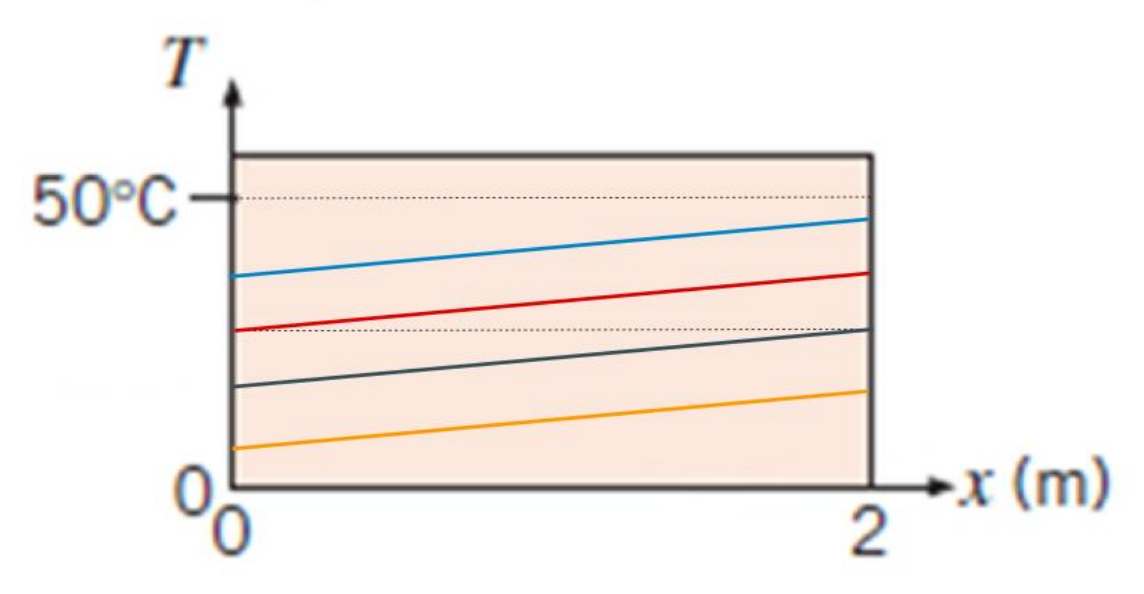
All of terms in the heat equation are *relative* terms: that is, they describe the change in temperature of a infinitesimal volume by comparing the volume to the volumes around it. This allows us to find the *shape* of the field: that is, the *general* solution.

To find a *particular* solution, we need *boundary conditions*: assumed values at the surface of the volume.

If we have a one-dimensional first-order differential equation, how many BCs do we need to know the temperature field?

$$\frac{dT}{dx} = 5\ \frac{°C}{m}$$

This equation is true for all of the lines shown in the image: but one BC ($T(0)$ *or* $T(2)$) would define a unique solution.



Boundary Conditions

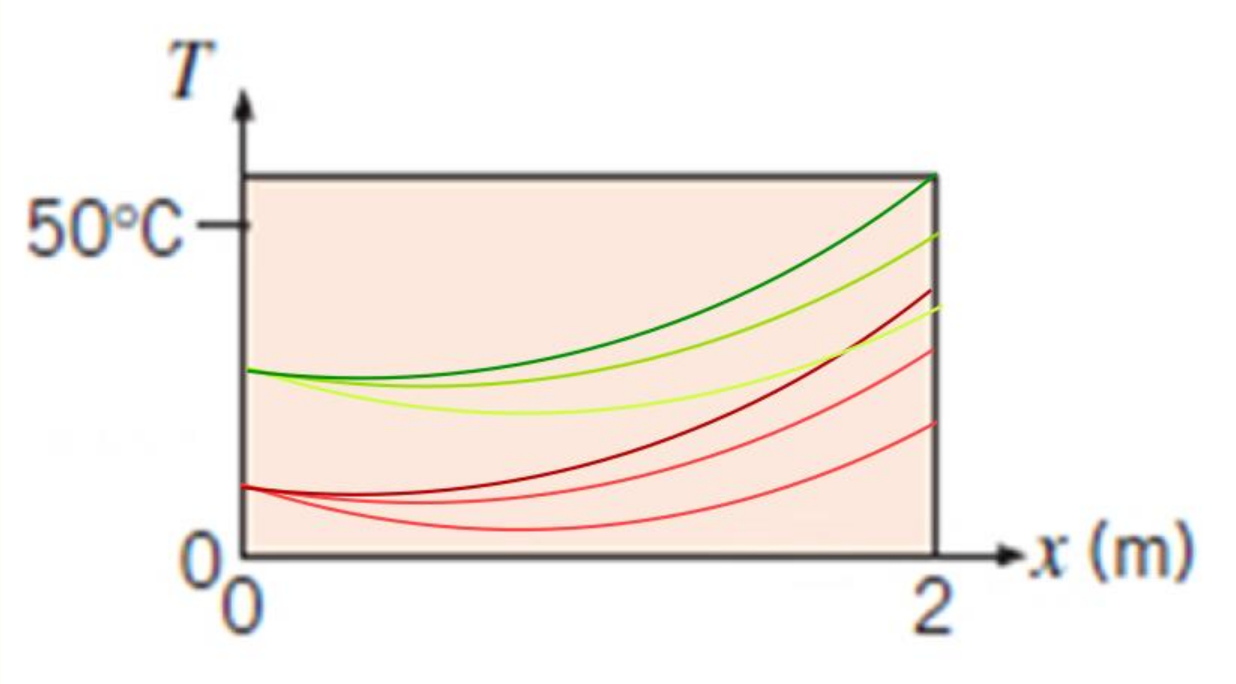
For a one-dimensional *second-order* differential equation, how many BCs do we need?

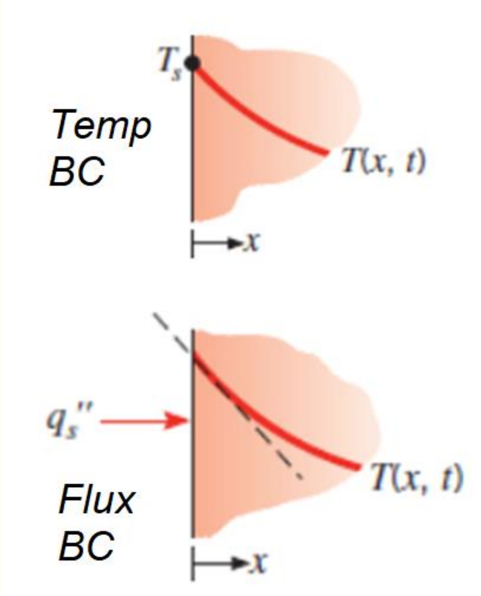
$$\frac{d^2T}{dx^2} = 15.0\ \frac{°C}{m^2}$$

The equation describes a curvature, but to place the curve, we need to know *two* BCs. Here are a couple possible combinations:

- $T(0)\ \ and\ \ T(2)$
- $\frac{dT}{dx}(0)\ \ and\ T(0)$
- $\frac{dT}{dx}(0)\ \ and\ \frac{dT}{dx}(2)$

These types of BCs are called *temperature BCs* $(T(0))$ and *flux BCs* $\left(\frac{dT}{dx}(0)\right)$. A flux BC that is set to 0 is an *insulated* or *adiabatic* BC.





Boundary Conditions

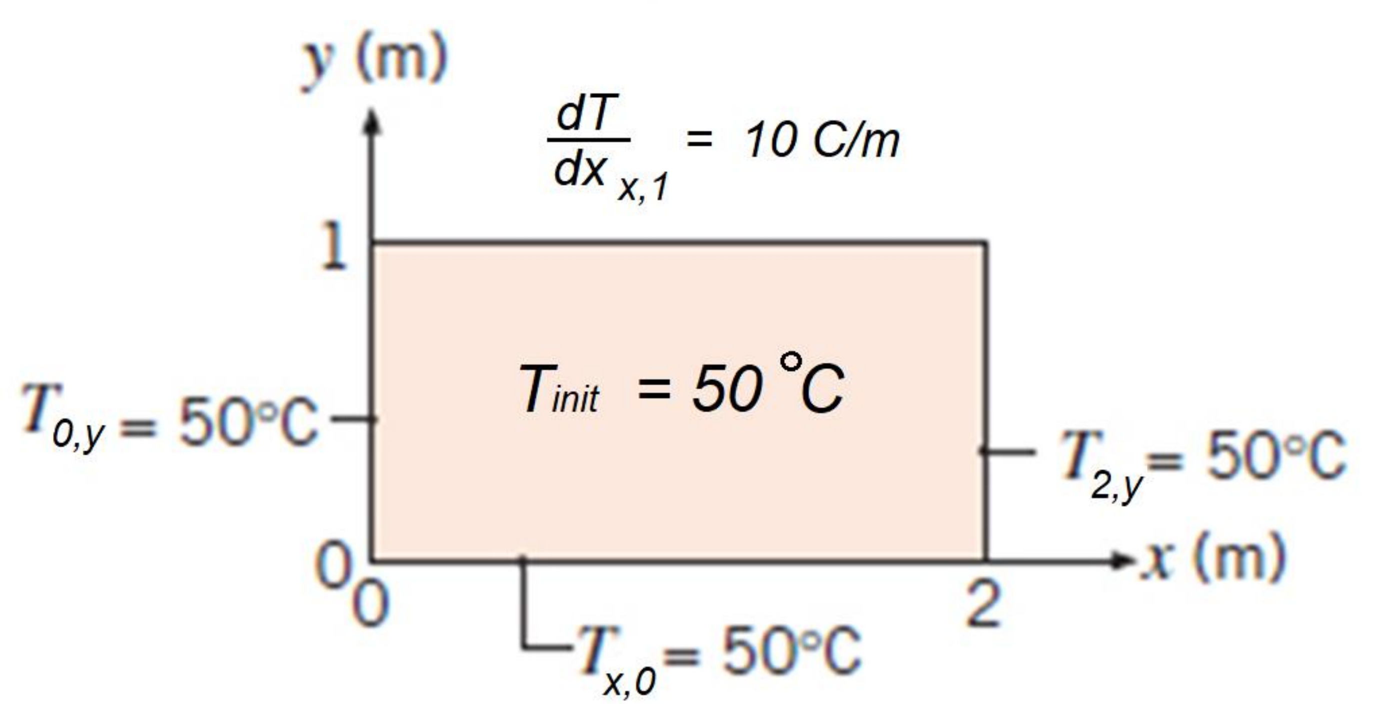
The heat equation is first-order in time, but second-order in space.

So to find a unique solution we need:

- 1 "boundary condition" for time (i.e. an initial condition that describes the initial temperature field)
- 2 BCs for *each* spatial dimension

In the image, you can see the 5 necessary BCs for a 2D problem.

With these and the heat equation, we can describe the temperature field T(x,y,z,t) at any given time!

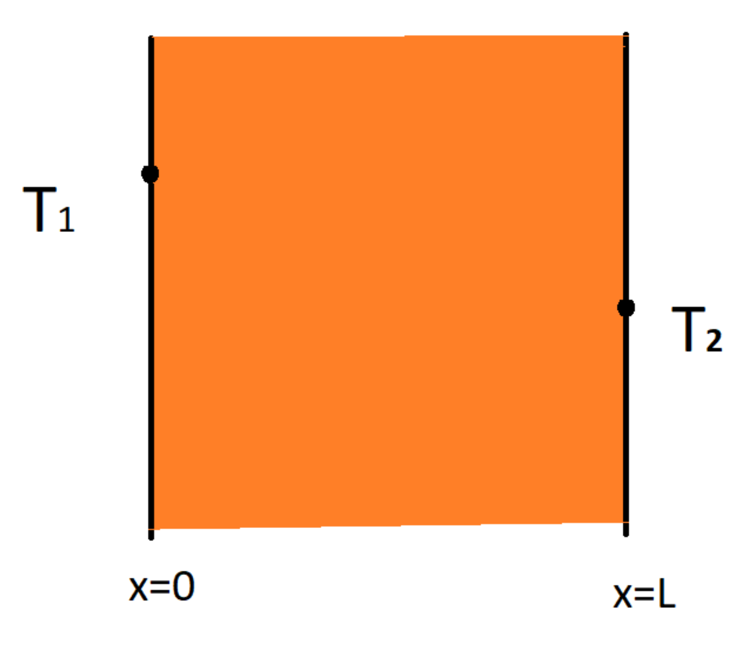


Analytical Solution

Analytical solutions give us exact solutions to a problem, but can only be used to solve simple problems. The simplest heat equation scenario involves these assumptions:

Assumptions:

- Steady-State (no need for an initial condition)
- 1D (No temperature flux in y and z directions)
- No heat generation
- Two temperature boundary conditions



The first step in solving a heat equation problem is simplifying the equation: which terms are equal to zero given our assumptions?

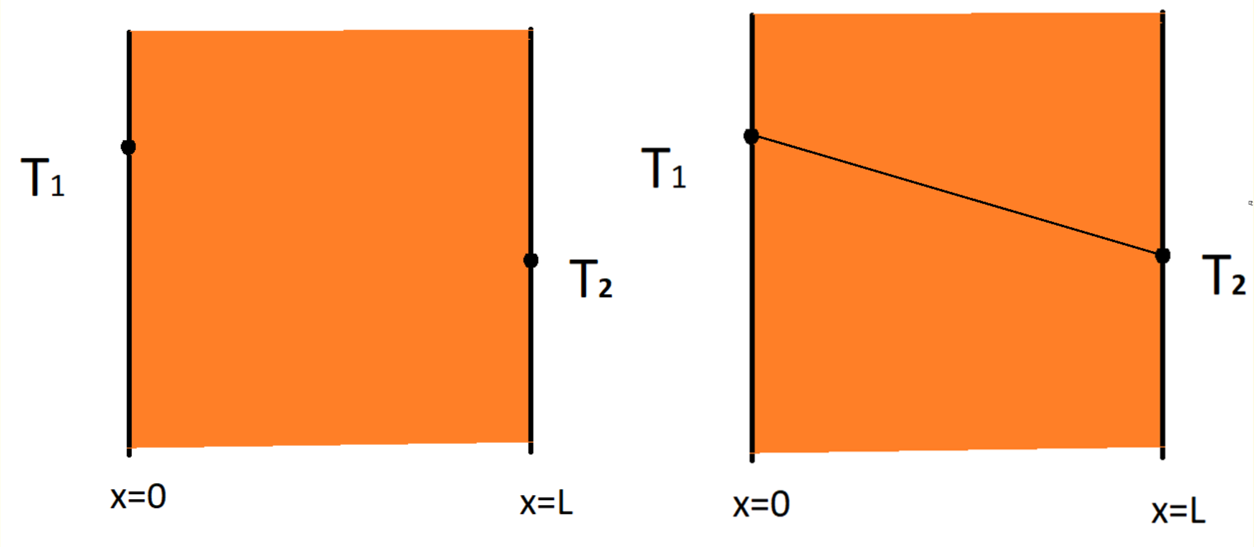
$$\rho C \frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \dot{q}$$

$$0 = k\frac{\partial^2 T}{\partial x^2}$$

Heat Equation Example

$$\cancel{\rho C \frac{\partial T}{\partial t}} = k\left(\frac{\partial^2 T}{\partial x^2} + \cancel{\frac{\partial^2 T}{\partial y^2}} + \cancel{\frac{\partial^2 T}{\partial z^2}}\right) + \cancel{\dot{q}}$$

$$0 = k\frac{\partial^2 T}{\partial x^2} \quad \rightarrow \quad \frac{\partial^2 T}{\partial x^2} = 0$$



This tells us that the field has no curvature (we could guess at the solution!). But to solve analytically, integrate both sides with respect to x:

$$\int \frac{\partial^2 T}{\partial x^2} dx = \int 0\, dx \quad \rightarrow \quad \frac{\partial T}{\partial x} = C_1$$

And integrate again:

$$\int \frac{\partial T}{\partial x} dx = \int C_1\, dx \quad \rightarrow \quad T = C_1 x + C_2$$

This is the general solution. Notice that the solution doesn't have anything to do with k!

Heat Equation Example

To apply the boundary conditions, we put the known condition into the general solution. We know at $x = 0$, $T = T_1$. Inserting this into our general solution yields:

$$T_1 = C_1(0) + C_2 \qquad \rightarrow \qquad C_2 = T_1$$

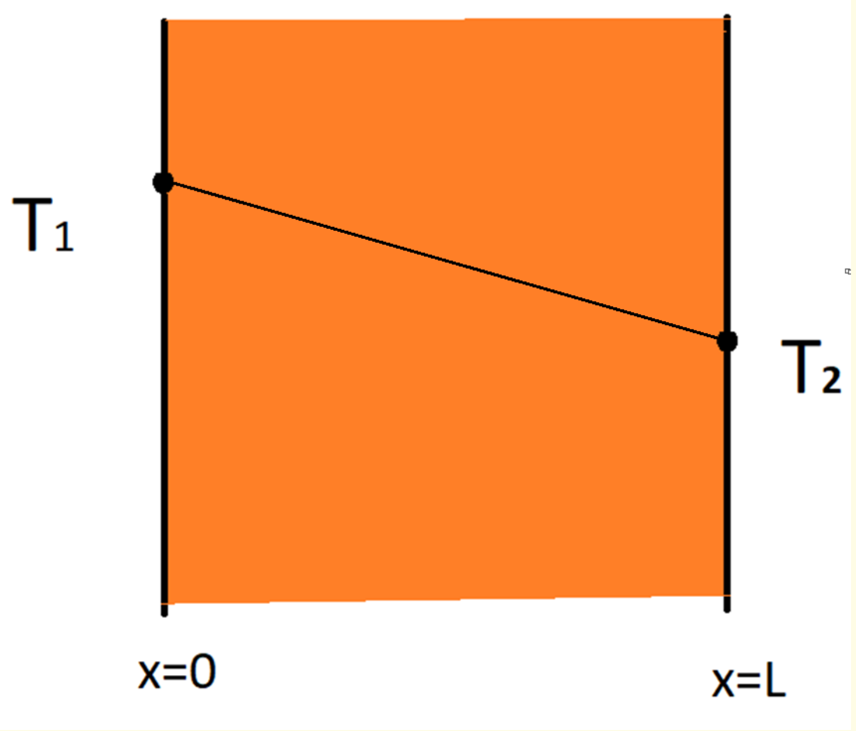
So that solves for one unknown constant. We also know that $T = T_2 \; at \; x = L$:

$$T = C_1 x + T_1 \qquad \rightarrow \qquad T_2 = C_1 L + T_1$$

$$C_1 = \frac{T_2 - T_1}{L}$$

To find the particular, unique solution, we sub our constants into the general solution:

$$T = \frac{(T_2 - T_1)}{L} x + T_1$$



Heat Equation Example

$$T = \frac{(T_2 - T_1)}{L} x + T_1$$

With this solution, we can determine the heat flux by putting the temperature field into Fourier's Law:

$$q''_x = -k \frac{dT}{dx} = \quad -k \frac{d\left(\frac{(T_2 - T_1)}{L} x + T_1\right)}{\mathrm{dx}}$$

If we separate the terms and pull the constants out of the derivative terms:

$$q''_x = -k\left(\left(\frac{T_2 - T_1}{L}\right)\left(\frac{dx}{dx}\right) + \frac{dT_1}{dx}\right) = -k\frac{T_2 - T_1}{L}$$

- What direction is this flux in?
- What is the flux at x=0? x=L?   Does this result make sense?