High Dimensional Dynamic Copula Models

Implementation of Bootstrap Filter and Gibbs Sampler (CSMC)

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Presentation Outline

Introduction

Context:

- Modeling dependence between financial assets in high dimension
- Criticism of copula models after the 2007-2008 crisis: static parameters
- Need to capture **time-varying dependence**

Reference article:

- Creal & Tsay (2015), *Journal of Econometrics*
- Application: panel of 200 series (CDS + equities) for 100 US companies

Project objectives:

1. Implement a **Bootstrap Filter**
2. Implement the **Gibbs Sampler** with CSMC (Particle Gibbs)

What is a copula?

Sklar's Theorem: The joint distribution F can be decomposed as:

$$F(y_1, \ldots, y_n) = C\big(F_1(y_1), \ldots, F_n(y_n)\big)$$

where:

- F_i: univariate marginal distributions
- $C : [0,1]^n \to [0,1]$: **copula function**

Interpretation:

- The copula captures the **dependence structure** independently of the marginals
- $u_i = F_i(y_i) \in [0,1]$: probability integral transforms (PITs)

Examples of copulas:

- Gaussian, Student-t, Clayton, Gumbel...

Factor models for copulas

Key idea: Reduce dimensionality via a factor structure

General model:

$$u_{it} = P(x_{it}|\theta) \tag{1}$$
$$x_{it} = \tilde{\lambda}'_{it} z_t + \sigma_{it} \varepsilon_{it} \tag{2}$$

with:

- $z_t \sim p(z_t|\theta)$: latent common factors
- $\varepsilon_{it} \sim p(\varepsilon_{it}|\theta)$: idiosyncratic shocks
- $\tilde{\lambda}_{it}$: factor loadings (potentially time-varying)

Advantages in high dimension:

- Parsimony: $n \times p$ parameters instead of $n(n-1)/2$
- Efficient computation of R_t^{-1} and $|R_t|$ via Woodbury formula (see Annexe)

Stochastic State-Space Model

Factor model:

$$u_{it} = P(x_{it}|\theta) \tag{3}$$

$$x_{it} = \tilde{\lambda}'_{it} z_t + \sigma_{it}\varepsilon_{it} \tag{4}$$

with $z_t \sim \mathcal{N}(0, I_p)$, $\varepsilon_{it} \sim \mathcal{N}(0,1)$ independent.

Rescaling of loadings:

$$\tilde{\lambda}_{it} = \frac{\lambda_{it}}{\sqrt{1 + \|\lambda_{it}\|^2}}, \qquad \sigma_{it}^2 = \frac{1}{1 + \|\lambda_{it}\|^2}$$

$\Rightarrow$ The marginal $P(x_{it}|\theta)$ does not depend on λ_{it}

State dynamics (AR(1)):

$$\Lambda_{t+1} = \mu + \Phi_\lambda(\Lambda_t - \mu) + \eta_t, \quad \eta_t \sim \mathcal{N}(0, \Sigma)$$

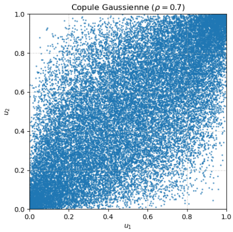
where $\Lambda_t = \text{vec}(\lambda_t)$ with $\lambda_t \in \mathbb{R}^{n \times p}$.

Challenge: Likelihood = integral over the latent path (no closed form).

Gaussian and Student-t Copulas

Conditional Gaussian copula:

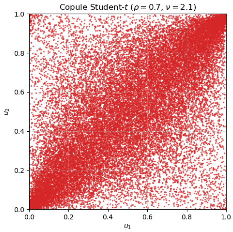
$$\begin{aligned} u_{it} &= \Phi(x_{it}) \\ x_{it} &= \tilde{\lambda}_{it}' z_t + \sigma_{it}\varepsilon_{it} \\ z_t &\sim \mathcal{N}(0, I_p), \quad \varepsilon_{it} \sim \mathcal{N}(0,1) \end{aligned}$$



Gaussian Dependence

Conditional Student-t copula:

$$\begin{aligned} u_{it} &= T_\nu(x_{it}) \\ x_{it} &= \sqrt{\zeta_t}\left(\tilde{\lambda}_{it}' \tilde{z}_t + \sigma_{it}\tilde{\varepsilon}_{it}\right) \\ \tilde{z}_t &\sim \mathcal{N}(0, I_p), \quad \tilde{\varepsilon}_{it} \sim \mathcal{N}(0,1) \\ \zeta_t &\sim \text{Inv-Gamma}\left(\frac{\nu}{2}, \frac{\nu}{2}\right) \end{aligned}$$



Student-t Dependence

$\Rightarrow$ Greater tail dependence

Bootstrap Filter: Principle

Objective: Approximate $p(\Lambda_t | u_{1:t}, \theta)$ by a set of particles

Bootstrap Filter Algorithm:

1. **Initialization** ($t = 1$): Draw $\Lambda_1^{(m)} \sim p(\Lambda_1 | \theta)$ for $m = 1, \ldots, M$
2. **For** $t = 1, \ldots, T$:
    - **Weights:** $w_t^{(m)} \propto p(u_t | \Lambda_t^{(m)}, \theta)$
    - **Normalize:** $\hat{w}_t^{(m)} = w_t^{(m)} / \sum_{m'} w_t^{(m')}$
    - **Resampling:** Draw with replacement according to $\hat{w}_t^{(m)}$
    - **Propagation:** $\Lambda_{t+1}^{(m)} \sim p(\Lambda_{t+1} | \Lambda_t^{(m)}, \theta)$

Likelihood estimator:

$$\hat{p}(u_{1:T} | \theta) = \prod_{t=1}^{T} \left(\frac{1}{M} \sum_{m=1}^{M} w_t^{(m)} \right)$$

Bootstrap Filter: Results on simulated data

Experimental protocol:

- Data simulation with known parameters
- Variation of the number of particles M
- Analysis of likelihood estimator variance

Expected results:

- Convergence to true log-likelihood as $M \to \infty$
- Estimator variance $\propto 1/M$
- Impact of dimension n on weight degeneracy

Bootstrap results

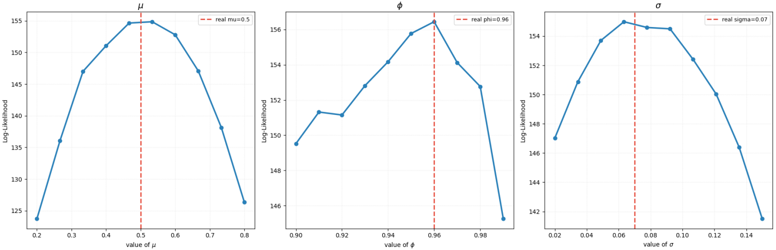


Figure 1: likelihood estimator for gaussian copula

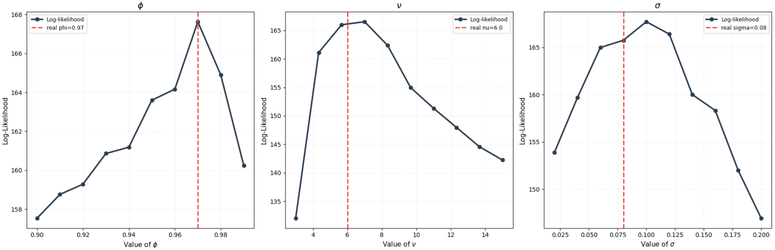
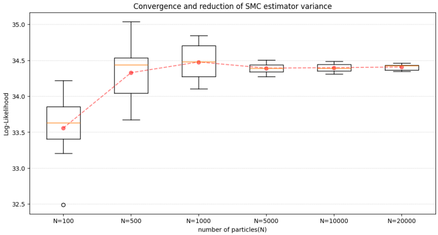
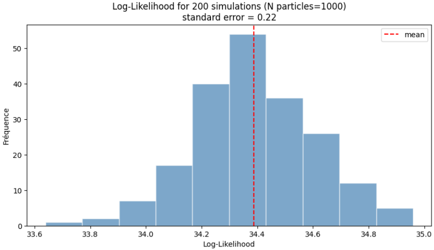
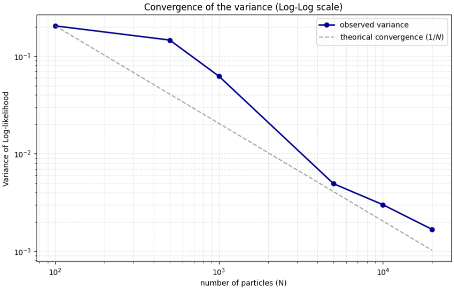


Figure 2: likelihood estimator for student copula

Bootstrap results







Gibbs Sampler: Structure

Objective: Sample from the joint posterior

$$p(\theta, \Lambda_{1:T}, z_{1:T}, \zeta_{1:T} | u_{1:T}, X_{1:T})$$

Gibbs Sampler iterations:

1. Draw $z_{1:T}^{(j)} \sim p(z_{1:T}|u_{1:T}, \Lambda_{1:T}^{(j-1)}, \zeta_{1:T}^{(j-1)}, \theta^{(j-1)})$
2. Draw $\Lambda_{1:T}^{(j)} \sim p(\Lambda_{1:T}|u_{1:T}, z_{1:T}^{(j)}, \zeta_{1:T}^{(j-1)}, \theta^{(j-1)})$ (via CSMC)
3. Draw $\zeta_{1:T}^{(j)} \sim p(\zeta_{1:T}|u_{1:T}, z_{1:T}^{(j)}, \Lambda_{1:T}^{(j)}, \theta^{(j-1)})$
4. Draw $\theta^{(j)} \sim p(\theta|u_{1:T}, z_{1:T}^{(j)}, \Lambda_{1:T}^{(j)}, \zeta_{1:T}^{(j)})$
   - Draw $\nu \sim p(\nu|u_{1:T}, \zeta_{1:T})$ (via MH)
   - Draw $\Sigma \sim p(\Sigma|\Lambda_{1:T}, \mu, \Phi_\lambda)$
   - Draw $\Phi_\lambda \sim p(\mu, \Phi_\lambda|\Lambda_{1:T}, \Sigma, \mu)$
   - Draw $\mu \sim p(\mu, \Phi_\lambda|\Lambda_{1:T}, \Sigma, \Phi_\lambda)$

Conditional SMC (Particle Gibbs)

Idea: Use a *conditional* particle filter to draw $\Lambda_{1:T}$

Difference from standard Bootstrap Filter:

- We **keep a reference trajectory** $\Lambda_{1:T}^{(1)}$ from the previous iteration
- This trajectory survives all resampling steps
- **Backward sampling**

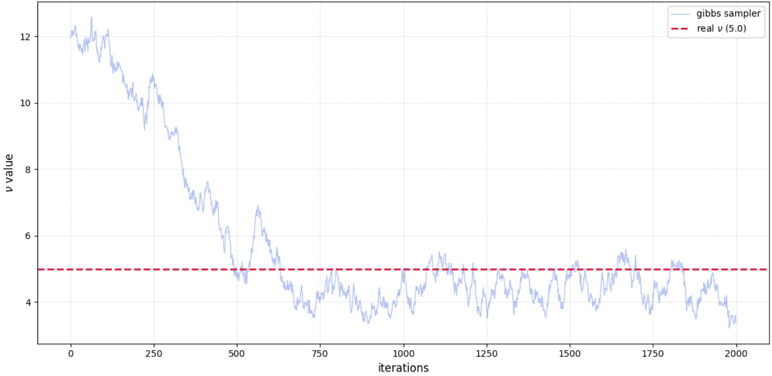
Parallelism

If Φ_λ and Σ are diagonal, the loading trajectories $\lambda_{i,1:T}$ are independent conditionally on $z_{1:T}$, allowing parallel computation.

Gibbs Algorithm: Detail of steps

1. **Mixing variables (ζ_t):** Drawing via independent Metropolis-Hastings (MH)
2. **Degrees of freedom (ν):** Adaptive random walk MH. Requires recalculating $x_{it} = T^{-1}(u_{it}|\nu_j^*)$ for each proposal.
3. **Common factors (z_t):** Direct draw from a **multivariate Gaussian** distribution.
4. **Loadings ($\Lambda_{1:T}$):** Use of **Particle Gibbs** with *backward sampling*..
5. **Transition parameters ($\mu, \Phi_\lambda, \Sigma$):**
    - Σ (diagonal): Direct draw via Inverse-Gamma distribution.
    - μ, Φ_λ: Truncated normal draw (to ensure stationarity).

Results



Gibbs sampler for ν estimation

Conclusion

Contributions:

- implementation of Bootstrap Filter for factor copulas
- Implementation of Gibbs Sampler with CSMC (Particle Gibbs)
- Validation on simulated data

Key points from the article:

- Stochastic factorized copulas enable high dimension
- Particle Gibbs with backward sampling is efficient
- Grouped Student-t copula outperforms alternatives

Appendix: Woodbury Formula

For efficient computation of R_t^{-1}:

$$R_t^{-1} = D_t^{-1} - D_t^{-1}\tilde{C}_t \left(I_{p+k} + \tilde{C}_t' D_t^{-1}\tilde{C}_t \right)^{-1} \tilde{C}_t' D_t^{-1}$$

Determinant:

$$|R_t| = \left| I_{p+k} + \tilde{C}_t' D_t^{-1}\tilde{C}_t \right| \cdot |D_t|$$

$\Rightarrow$ Complexity $\mathcal{O}(np^2 + p^3)$ instead of $\mathcal{O}(n^3)$