Report for exercise 4 from group F

Tasks addressed:	5
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The work on tasks was divided in the following way:

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	Task 2	20%
	Task 3	20%
	Task 4	20%
	Task 5	20%
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	Task 2	20%
	Task 3	20%
	Task 4	20%
	Task 5	20%
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	Task 2	20%
	Task 3	20%
	Task 4	20%
	Task 5	20%
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	Task 2	20%
	Task 3	20%
	Task 4	20%
	Task 5	20%
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	Task 2	20%
	Task 3	20%
	Task 4	20%
	Task 5	20%

Report on task 1, Vector fields, orbits, and visualization

1 task 1, Vector fields, orbits, and visualization

To determine whether these systems are topologically equivalent, we need to analyze the characteristics of their phase diagrams. Topological equivalence means that the two systems exhibit qualitatively the same dynamical behavior, that is, their phase diagrams can be transformed into each other under continuous transformations, regardless of the specific numerical solutions. We need to consider aspects such as the type of fixed points, the direction of the trajectory, and the overall behavior.

we thoroughly analyzed the dynamical behavior of a linear system under different parameterized matrices by plotting and examining the phase portraits. Specifically, we selected three different parameter sets, generated corresponding phase portraits, and analyzed the system's eigenvalues and their impact on the system behavior.

1.1 Parameter Selection and Phase Portraits

1. **First Phase Portrait**:

- Parameters: $\alpha = -1$, $r = \frac{1}{4}$
- Matrix: $A_\alpha = \begin{pmatrix} -1 & -1 \\ \frac{1}{4} & 0 \end{pmatrix}$
- Eigenvalues: $\lambda_{1,2} = \frac{-1 \pm i}{2}$
- Results: The eigenvalues are complex with a negative real part and a non-zero imaginary part, indicating the system has a stable spiral point. The phase portrait shows trajectories spiraling inward, meaning the trajectories rotate around the fixed point and converge to it over time. The corresponding phase portrait is shown in the left panel of Figure 1.

2. **Second Phase Portrait**:

- Parameters: $\alpha = -1$, $r = -\frac{1}{4}$
- Matrix: $A_\alpha = \begin{pmatrix} -1 & -1 \\ -\frac{1}{4} & 0 \end{pmatrix}$
- Eigenvalues: $\lambda_{1,2} = \frac{-1 \pm \sqrt{5}}{2}$
- Results: The eigenvalues are real and of opposite signs, indicating the system has a saddle point. The phase portrait shows some trajectories moving away from the saddle point while others approach it, characteristic of saddle point behavior. The corresponding phase portrait is shown in the middle panel of Figure 1.

3. **Third Phase Portrait**:

- Parameters: $\alpha = 1$, $r = -\frac{1}{4}$
- Matrix: $A_\alpha = \begin{pmatrix} 1 & 1 \\ -\frac{1}{4} & 0 \end{pmatrix}$
- Eigenvalues: $\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}$
- Results: The eigenvalues are real and both positive, indicating the system has an unstable saddle point. The phase portrait shows all trajectories moving away from the saddle point, reflecting the system's divergent behavior. The corresponding phase portrait is shown in the right panel of Figure 1.

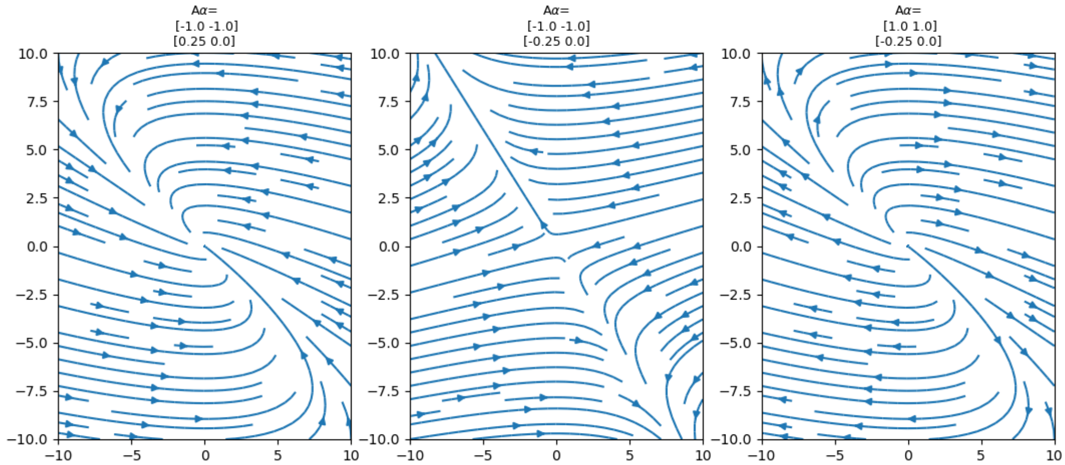


Figure 1: Phase portraits for different parameter values: (a) $\alpha = -1, r = \frac{1}{4}$, (b) $\alpha = -1, r = -\frac{1}{4}$, (c) $\alpha = 1, r = -\frac{1}{4}$.

1.2 Topological Equivalence Analysis

By analyzing the eigenvalues and phase portraits, we can draw the following conclusions:

1. **The first system** is not topologically equivalent to the other two systems. The first system has a spiral point, whereas the other two systems have saddle points, indicating significant qualitative differences in their phase portraits.

2. **The second and third systems** both have saddle points, but due to the different signs of α, their trajectory directions are opposite. The second system's saddle point is stable, while the third system's saddle point is unstable, thus they are not topologically equivalent.

1.3 Conclusion

From these analyses, we observe that systems exhibit different dynamical behaviors under different parameters. The type and sign of eigenvalues significantly influence the system's stability and trajectory shapes. Ultimately, these systems are not topologically equivalent, reflecting the profound impact of parameter variations on the system's properties. This analysis provides valuable insights and methods for understanding and designing complex dynamical systems.

Report on task 2, Common bifurcations in nonlinear systems

2 Task 2: Common Bifurcations in Nonlinear Systems

In this task, we analyze the bifurcations in nonlinear dynamical systems described by the following equations: 1. $\dot{x} = \alpha - x^2$ 2. $\dot{x} = \alpha - 2x^2 - 3$

We plot the bifurcation diagrams for different values of α and visually indicate the stability of the steady states. Additionally, we analyze the topological equivalence of the systems at specific values of α.

2.1 Bifurcation Analysis of $\dot{x} = \alpha - x^2$

For the first system $\dot{x} = \alpha - x^2$, the steady states are determined by solving $\alpha - x^2 = 0$:

$$x = \pm\sqrt{\alpha}$$

- For $\alpha > 0$, there are two steady states at $x = \pm\sqrt{\alpha}$. - For $\alpha < 0$, there are no real steady states. - At $\alpha = 0$, there is a bifurcation point.

The type of bifurcation at $\alpha = 0$ is a saddle-node bifurcation.

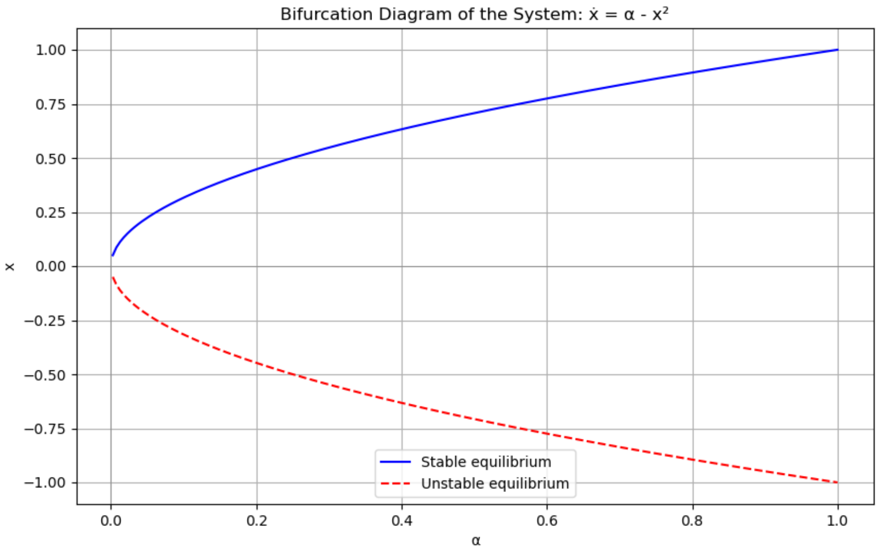


Figure 2: Bifurcation diagram for $\dot{x} = \alpha - x^2$. The stability of the steady states is indicated by solid lines (stable) and dashed lines (unstable).

2.2 Bifurcation Analysis of $\dot{x} = \alpha - 2x^2 - 3$

For the second system $\dot{x} = \alpha - 2x^2 - 3$, the steady states are determined by solving $\alpha - 2x^2 - 3 = 0$:

$$x = \pm\sqrt{\frac{\alpha - 3}{2}}$$

- For $\alpha > 3$, there are two steady states at $x = \pm\sqrt{\frac{\alpha-3}{2}}$. - For $\alpha \leq 3$, there are no real steady states. - At $\alpha = 3$, there is a bifurcation point.

The type of bifurcation at $\alpha = 3$ is also a saddle-node bifurcation.

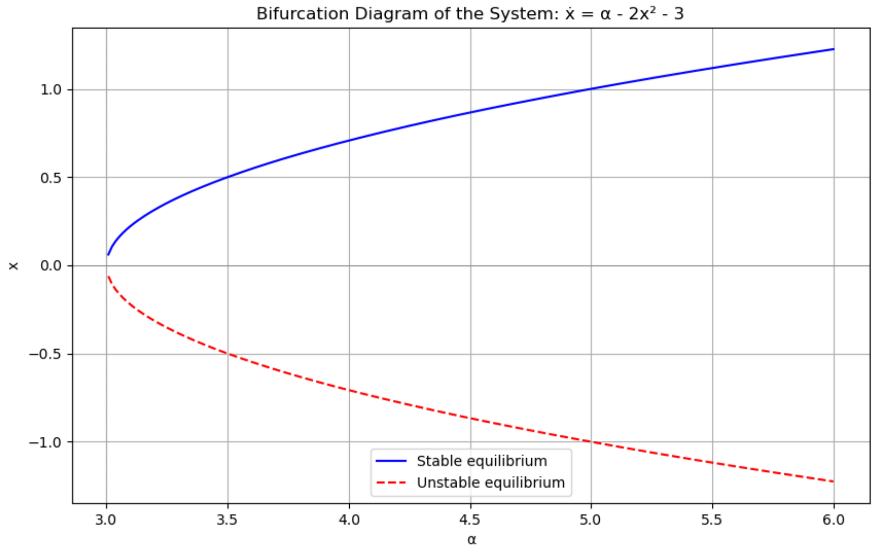


Figure 3: Bifurcation diagram for $\dot{x} = \alpha - 2x^2 - 3$. The stability of the steady states is indicated by solid lines (stable) and dashed lines (unstable).

2.3 Topological Equivalence Analysis

- At $\alpha = 1$: - System (6) has two steady states. - System (7) has no steady states since $1 - 3 = -2 < 0$.
  Since one system has steady states and the other does not, they are not topologically equivalent at $\alpha = 1$.
  - At $\alpha = -1$: - Both systems (6) and (7) have no real steady states.
  Thus, they are topologically equivalent at $\alpha = -1$ as both systems behave similarly with no steady states.

2.4 Normal Form Argument

Both systems can be transformed to have the same normal form around their bifurcation points, showing that their qualitative behavior near the bifurcation is similar.

2.5 Conclusion

From the bifurcation diagrams and the analysis of the steady states' stability, we observe that both systems exhibit saddle-node bifurcations. However, their topological equivalence depends on the specific value of α. This analysis helps in understanding the qualitative behavior of nonlinear dynamical systems under parameter variations.

Report on task 3 Bifurcations in higher dimensions

3 Task 3: Bifurcations in Higher Dimensions

In this task, we analyze the bifurcations in nonlinear dynamical systems described by the following equations:

1. The Andronov-Hopf bifurcation system:

$$\begin{cases} \dot{x}_1 = \alpha x_1 - x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 = x_1 + \alpha x_2 - x_2(x_1^2 + x_2^2) \end{cases}$$

2. The cusp bifurcation system:

$$\dot{x} = \alpha_1 + \alpha_2 x - x^3$$

We visualize the system's bifurcation by plotting three phase diagrams at representative values of α, numerically compute and visualize two orbits of the system, and visualize the bifurcation surface of the cusp bifurcation in a 3D plot.

3.1 Phase Diagrams for Different Values of Alpha

For the Andronov-Hopf bifurcation system, we plotted phase diagrams for $\alpha = -0.5, 0, 0.5$. The code for generating the phase diagrams is as follows:

```python
import numpy as np
import matplotlib.pyplot as plt

def vector_field(x, y, alpha):
    dx = alpha * x - y - x * (x ** 2 + y ** 2)
    dy = x + alpha * y - y * (x ** 2 + y ** 2)
    return dx, dy

def plot_phase_diagram(alpha, ax):
    x = np.linspace(-3, 3, 400)
    y = np.linspace(-3, 3, 400)
    X, Y = np.meshgrid(x, y)
    DX, DY = vector_field(X, Y, alpha)
    ax.streamplot(X, Y, DX, DY, color=np.sqrt(DX ** 2 + DY ** 2), linewidth=1, cmap='autumn')
    ax.set_title(f'Alpha = {alpha}')
    ax.set_xlabel('$x_1$')
    ax.set_ylabel('$x_2$')
    ax.grid(True)

alphas = [-0.5, 0, 0.5]
fig, axs = plt.subplots(1, 3, figsize=(15, 5))
for i, alpha in enumerate(alphas):
    plot_phase_diagram(alpha, axs[i])
plt.show()
```

The phase diagrams are shown in Figure 4. We can observe different dynamical behaviors as α changes.

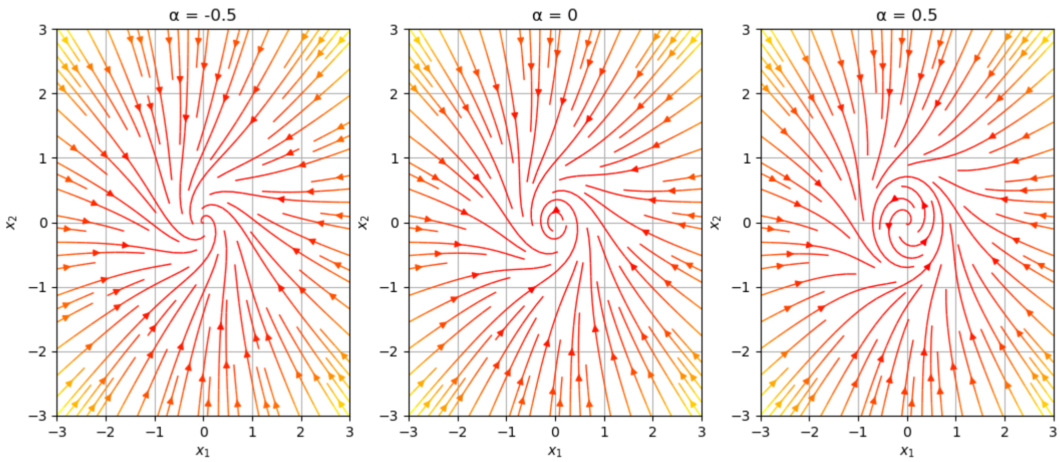


Figure 4: Phase diagrams for the system at $\alpha = -0.5, 0, 0.5$.

3.2 Numerical Computation and Visualization of Two Orbits at Alpha = 1

We numerically computed and visualized two orbits of the system starting at the points (2,0) and (0.5,0) using Euler's method. The code for this computation is as follows:

```python
def euler_method(x0, y0, alpha, dt, steps):
    x = np.zeros(steps)
    y = np.zeros(steps)
    x[0], y[0] = x0, y0

    for i in range(1, steps):
        dx, dy = vector_field(x[i - 1], y[i - 1], alpha)
        x[i] = x[i - 1] + dx * dt
        y[i] = y[i - 1] + dy * dt

    return x, y

alpha = 1
dt = 0.01
steps = 5000

initial_conditions = [(2, 0), (0.5, 0)]

plt.figure(figsize=(10, 8))
for x0, y0 in initial_conditions:
    x, y = euler_method(x0, y0, alpha, dt, steps)
    plt.plot(x, y, label=f'Start at ({x0}, {y0})')

plt.title('Trajectories at Alpha = 1')
plt.xlabel('$x_1$')
plt.ylabel('$x_2$')
plt.legend()
plt.grid(True)
plt.show()
```

The resulting trajectories are shown in Figure 5. These trajectories illustrate the system's behavior for different initial conditions at $\alpha = 1$.

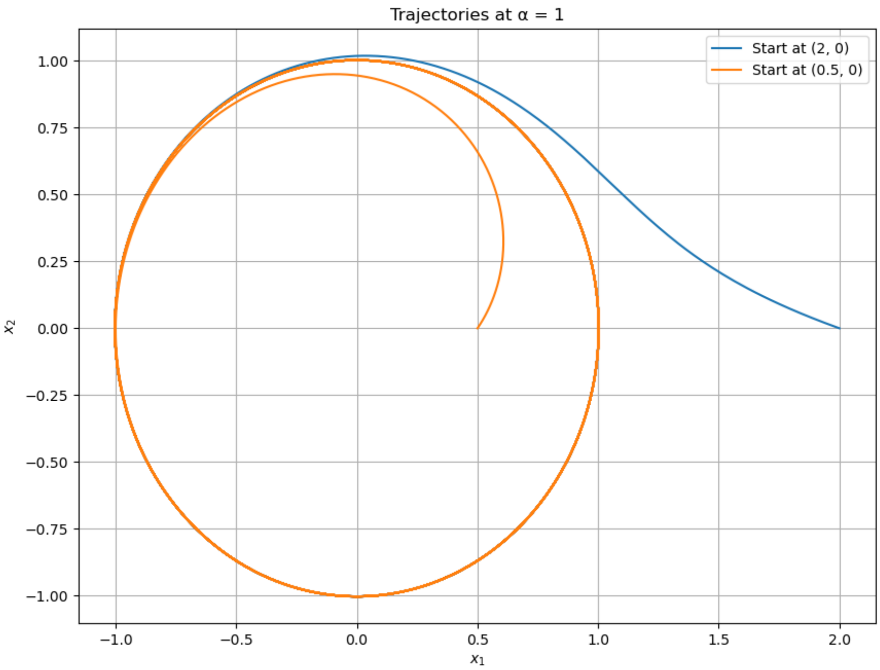


Figure 5: Trajectories at $\alpha = 1$, starting at (2,0) and (0.5,0).

3.3 Visualization of the Cusp Bifurcation Surface

We visualized the bifurcation surface of the cusp bifurcation by sampling points (x, α_1, α_2) and plotting the surface where $\dot{x} = 0$. The code for generating the 3D plot is as follows:

```
alpha1_values = np.linspace(-1, 1, 50)
alpha2_values = np.linspace(-1, 1, 50)
X = np.linspace(-2, 2, 200)

alpha1, alpha2, x = np.meshgrid(alpha1_values, alpha2_values, X)

F = alpha1 + alpha2 * x - x**3

indices = np.abs(F) < 0.05

alpha1_surface = alpha1[indices]
alpha2_surface = alpha2[indices]
x_surface = x[indices]

fig = plt.figure(figsize=(12, 8))
ax = fig.add_subplot(111, projection='3d')
ax.scatter(alpha1_surface, alpha2_surface, x_surface, c='r', marker='o')

ax.set_xlabel('$\\alpha_1$')
ax.set_ylabel('$\\alpha_2$')
ax.set_zlabel('$x$')
ax.set_title('Cusp Bifurcation Surface')

plt.show()
```

The cusp bifurcation surface is shown in Figures 6, 7, and 8. These figures illustrate the complex structure of the bifurcation surface.

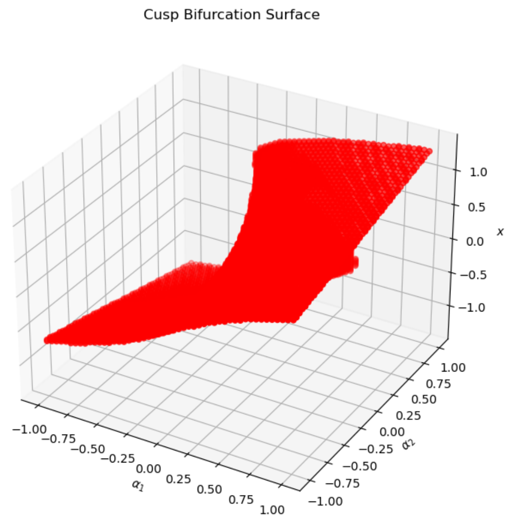


Figure 6: Cusp bifurcation surface. Points satisfying $\dot{x} = 0$.

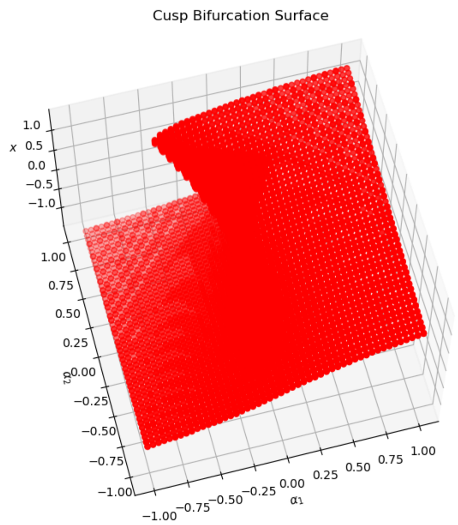


Figure 7: Different view of the cusp bifurcation surface.

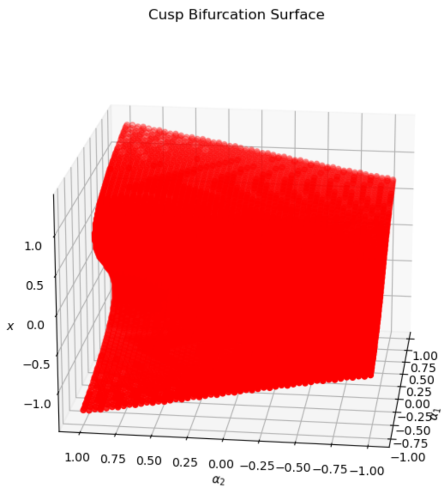


Figure 8: Another view of the cusp bifurcation surface.

3.4 Why It Is Called Cusp Bifurcation

The cusp bifurcation is named because the bifurcation surface forms a cusp shape in the parameter space. This cusp structure indicates that small changes in the parameters can lead to significant qualitative changes in the system's behavior. This complexity is evident in the 3D bifurcation surface plots.

3.5 Conclusion

By analyzing these two systems' bifurcations, we observe rich dynamical behaviors under different parameters. The Andronov-Hopf bifurcation demonstrates periodic behavior in the system, while the cusp bifurcation reveals complex bifurcation structures in the parameter space. These analyses provide valuable insights into understanding and predicting the behavior of nonlinear dynamical systems.

Report on task 4 Chaotic dynamics

4 Task 4: Chaotic Dynamics

Part 1: Discrete Maps

Consider the discrete map:

$$x_{n+1} = rx_n(1 - x_n), \quad n \in \mathbb{N},$$

with the parameter $r \in (0, 4]$ and $x \in [0, 1]$. We perform the following bifurcation analyses:

4.1 Vary r from 0 to 2

We varied r from 0 to 2 to observe the bifurcations. The fixed points of the system are found by solving the equation $x = rx(1 - x)$. The bifurcation points are identified, and the steady states are determined. As r increases from 0 to 2, we observe the system transitioning from a single steady state to periodic doubling bifurcations.

4.2 Vary r from 2 to 4

Next, we varied r from 2 to 4. The behavior of the system changes drastically, showing periodic doubling bifurcations leading to chaos. In this range, the system exhibits chaotic dynamics with periodic windows. The detailed behavior includes the identification of the periodic windows within the chaotic regime.

4.3 Bifurcation Diagram

We plotted the bifurcation diagram for r between 0 and 4, and x between 0 and 1, roughly indicating the positions of steady states and limit cycles.

```
# Parameters for bifurcation diagram
r_values = np.linspace(0, 4, 1000)
iterations = 1000
last = 100

# Initialize array to store results
x = 1e-5 * np.ones(len(r_values))

# Iterate and plot
for i in range(iterations):
    x = r_values * x * (1 - x)
    if i >= (iterations - last):
        ax.plot(r_values, x, ',b', alpha=0.25)
```

The resulting bifurcation diagram is shown in Figure 9.

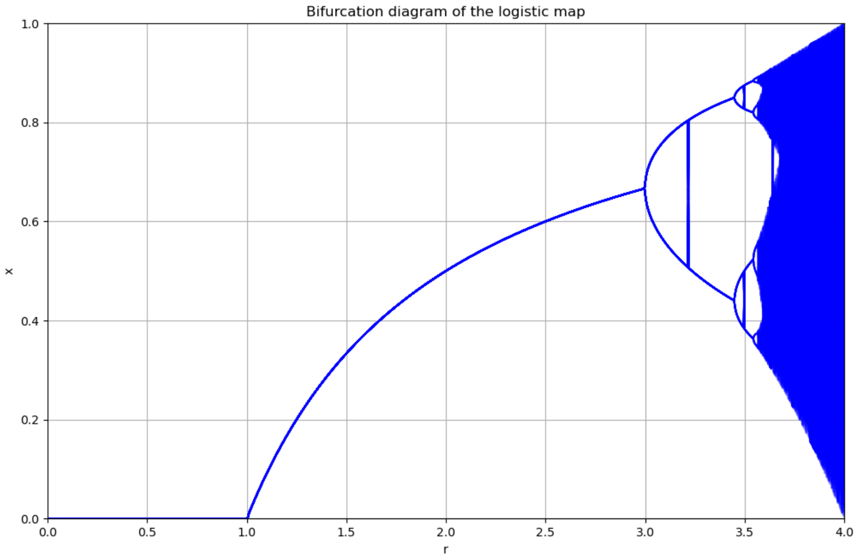


Figure 9: Bifurcation diagram of the logistic map for r between 0 and 4.

Part 2: Lorenz System
The Lorenz system is defined by the following equations:

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = x(\rho - z) - y \\ \dot{z} = xy - \beta z \end{cases}$$

with the parameters $\sigma = 10$, $\beta = 8/3$, and $\rho = 28$.

4.4 Visualize a Single Trajectory

We visualized a single trajectory of the Lorenz system starting at $x_0 = (10, 10, 10)$ for $\rho = 28$.

```
# Define Lorenz system
def lorenz(t, state, sigma, beta, rho):
    x, y, z = state
    dx = sigma * (y - x)
    dy = x * (rho - z) - y
    dz = x * y - beta * z
    return [dx, dy, dz]

# Parameters and initial state
sigma, beta, rho = 10, 8/3, 28
initial_state = [10, 10, 10]

# Solve ODE
sol = solve_ivp(lorenz, [0, 1000], initial_state, args=(sigma, beta, rho), t_eval=np.
    linspace(0, 1000, 10000))

# Plot trajectory
fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')
ax.plot(sol.y[0], sol.y[1], sol.y[2])
```

The resulting trajectory is shown in Figure 10.

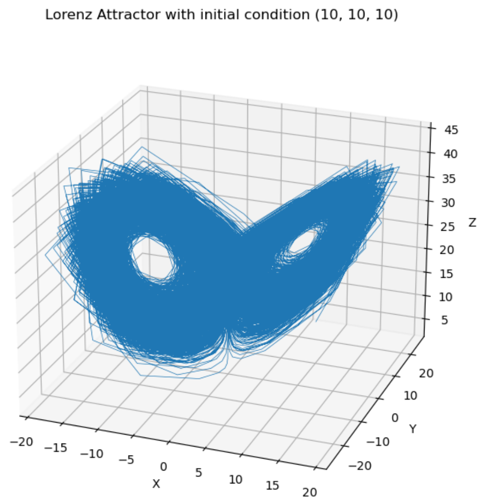


Figure 10: Lorenz attractor with initial condition (10, 10, 10) and $\rho = 28$.

4.5 Initial Condition Dependence

To test the sensitivity to initial conditions, we plotted another trajectory starting from $x_0 = (10 + 10^{-8}, 10, 10)$ and computed the difference between the two trajectories over time.

```
# Perturbed initial state
initial_state_perturbed = [10 + 1e-8, 10, 10]

# Solve ODE for perturbed state
sol_perturbed = solve_ivp(lorenz, [0, 1000], initial_state_perturbed, args=(sigma,
    beta, rho), t_eval=np.linspace(0, 1000, 10000))

# Compute difference
diff = np.sqrt((sol.y[0] - sol_perturbed.y[0])**2 + (sol.y[1] - sol_perturbed.y[1])
    **2 + (sol.y[2] - sol_perturbed.y[2])**2)

# Plot difference
plt.plot(np.linspace(0, 1000, 10000), diff)
plt.yscale('log')
plt.xlabel('Time')
plt.ylabel('Difference')
plt.title('Difference between trajectories with initial conditions (10, 10, 10) and
    (10 + 1e-8, 10, 10)')
```

The resulting plot showing the difference between trajectories is in Figure 11.

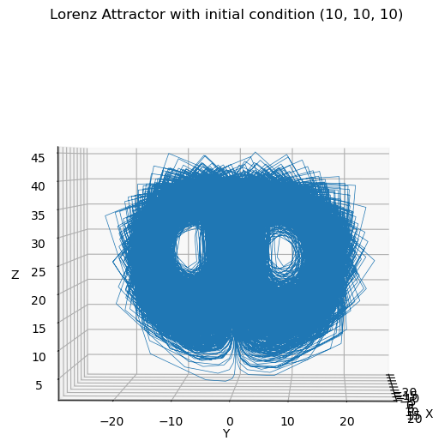


Figure 11: Difference between trajectories with initial conditions (10, 10, 10) and (10 + 1e-8, 10, 10).

4.6 Changing Parameter ρ to 0.5

We changed the parameter ρ to 0.5 and computed and plotted the two trajectories.

```
# New rho value
rho_new = 0.5

# Solve ODE for new rho
sol_new = solve_ivp(lorenz, [0, 1000], initial_state, args=(sigma, beta, rho_new),
    t_eval=np.linspace(0, 1000, 10000))
sol_perturbed_new = solve_ivp(lorenz, [0, 1000], initial_state_perturbed, args=(sigma
    , beta, rho_new), t_eval=np.linspace(0, 1000, 10000))

# Compute new difference
diff_new = np.sqrt((sol_new.y[0] - sol_perturbed_new.y[0])**2 + (sol_new.y[1] -
    sol_perturbed_new.y[1])**2 + (sol_new.y[2] - sol_perturbed_new.y[2])**2)

# Plot new trajectory
fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')
ax.plot(sol_new.y[0], sol_new.y[1], sol_new.y[2], color='r')
```

The resulting plots are shown in Figures 12 and 13.

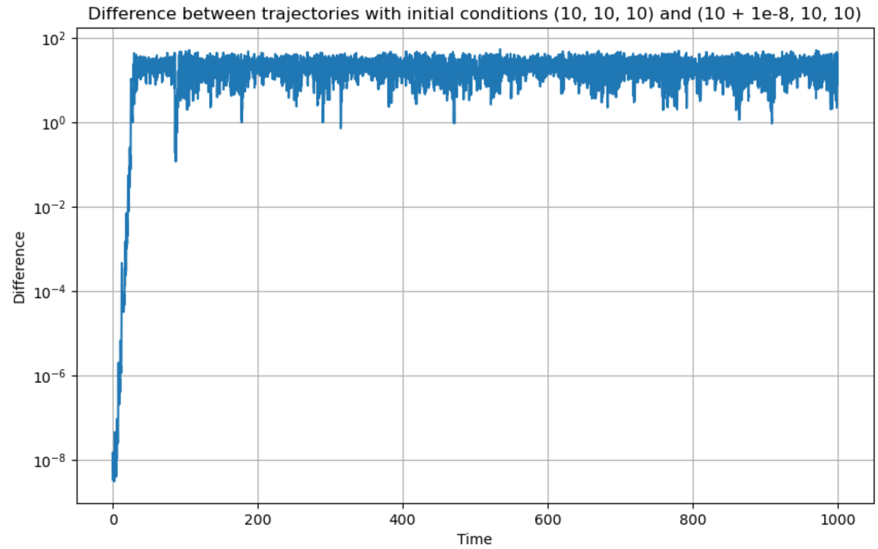


Figure 12: Lorenz attractor with initial condition (10, 10, 10) and $\rho = 0.5$.

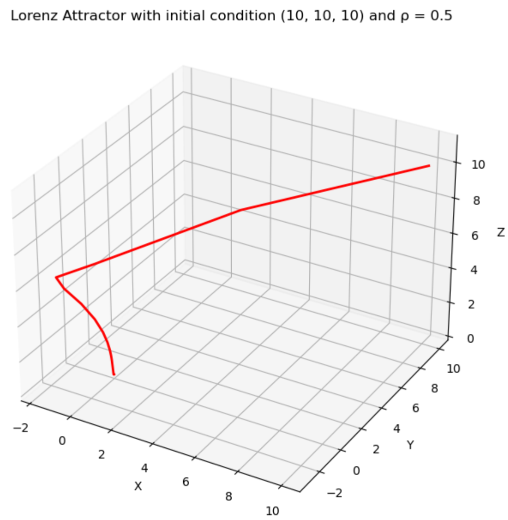


Figure 13: Difference between trajectories with initial conditions (10, 10, 10) and (10 + 1e-8, 10, 10) with $\rho = 0.5$.

4.7 Visualization of the Lorenz Attractor in the New Regime

We also visualized the Lorenz attractor for the new parameter $\rho = 0.5$ to observe its structure. The trajectory shows a non-chaotic behavior as indicated by the less complex structure of the attractor.

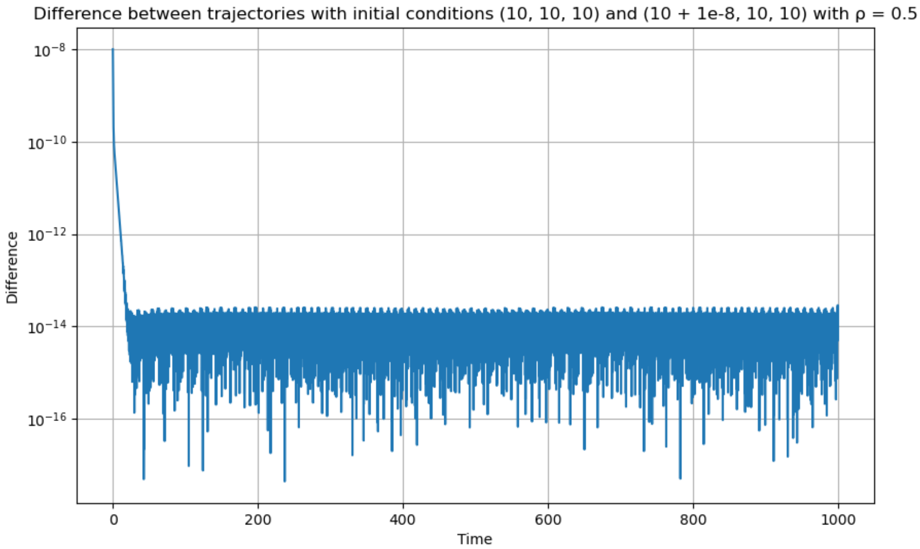


Figure 14: Lorenz attractor with initial condition (10, 10, 10) and $\rho = 0.5$, different angle.

4.8 Additional Visualization of the Lorenz Attractor

To further explore the impact of changing the parameter ρ to 0.5, we visualized the attractor from another perspective. This helps to confirm the non-chaotic behavior observed earlier.

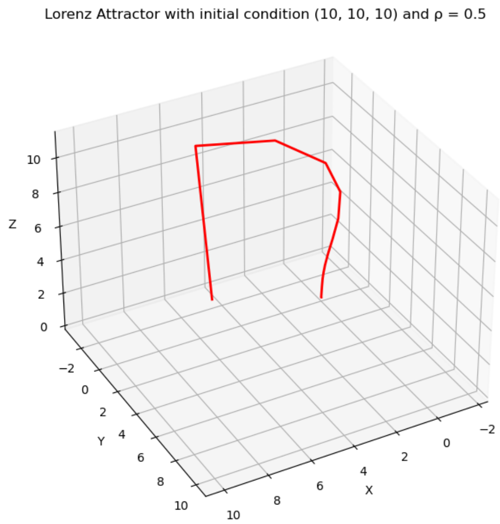


Figure 15: Lorenz attractor with initial condition (10, 10, 10) and $\rho = 0.5$, yet another angle.

4.9 Discussion

The results show that when $\rho = 28$, the system exhibits chaotic behavior, and small perturbations in the initial conditions grow exponentially, as seen in Figure 11. When ρ is reduced to 0.5, the system no longer exhibits chaotic behavior, and the difference between the trajectories remains very small, as shown in Figure 13. This indicates that the system is highly sensitive to initial conditions in the chaotic regime but not in the non-chaotic regime.

4.10 Summary

In summary, we have analyzed the bifurcations in the logistic map and the Lorenz system. The logistic map exhibits a range of behaviors from steady states to chaos as the parameter r varies. The Lorenz system, with its famous strange attractor, shows sensitivity to initial conditions in the chaotic regime. The bifurcation diagrams and trajectory plots provide a comprehensive understanding of these complex dynamical systems.

Report on task 5 Bifurcations in crowd dynamics

5 task 5 Bifurcations in crowd dynamics

The SIR model is a widely used epidemiological model that describes the spread of a contagious disease in a population. The population is divided into three compartments:

- **S (Susceptible)**: Individuals who are susceptible to the infection.
- **I (Infective)**: Individuals who are currently infected and can transmit the disease to susceptible individuals.
- **R (Removed)**: Individuals who have recovered from the disease and are now immune, or have died.

The dynamics of the SIR model are governed by the following set of differential equations:

$$\frac{dS}{dt} = A - \delta S - \frac{\beta SI}{S+I+R} \tag{1}$$

$$\frac{dI}{dt} = -(\delta + \nu)I - \mu(b,I)I + \frac{\beta SI}{S+I+R} \tag{2}$$

$$\frac{dR}{dt} = \mu(b,I)I - \delta R \tag{3}$$

where:

- A: Recruitment rate (birth rate) of susceptible population.
- δ: Per capita natural death rate.
- ν: Per capita disease-induced death rate.
- β: Transmission rate.
- $\mu(b,I)$: Per capita recovery rate, which depends on the number of infective individuals and the availability of healthcare resources (beds b).

The recovery rate $\mu(b,I)$ is modeled as:

$$\mu(b,I) = \mu_0 + (\mu_1 - \mu_0)\frac{b}{b+I} \tag{4}$$

where μ_0 and μ_1 are the minimum and maximum recovery rates, respectively.

In this task, we explore the bifurcation phenomena in the SIR model by varying the parameter b (number of beds per 10,000 persons) and observing how it affects the dynamics of the system.

5.1 Implement the SIR Model Equations that are Missing in the Model Method

In this task, we need to implement the missing differential equations in the SIR model. The differential equations for the SIR model are as follows:

$$\frac{dS}{dt} = A - \delta S - \frac{\beta SI}{S+I+R} \tag{5}$$

$$\frac{dI}{dt} = -(\delta + \nu)I - \mu(b,I)I + \frac{\beta SI}{S+I+R} \tag{6}$$

$$\frac{dR}{dt} = \mu(b,I)I - \delta R \tag{7}$$

where the recovery rate $\mu(b,I)$ is defined as:

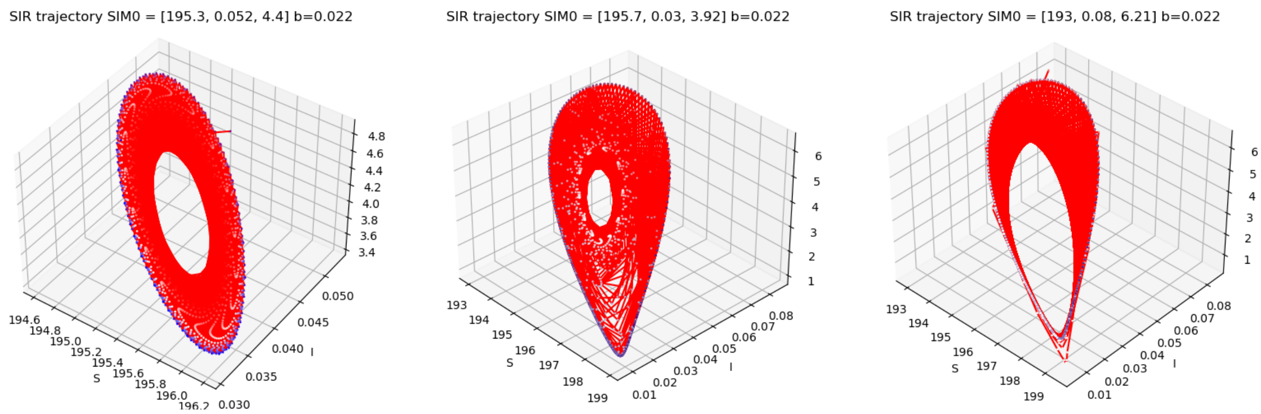
$$\mu(b,I) = \mu_0 + (\mu_1 - \mu_0)\frac{b}{b+I} \tag{8}$$

To implement these equations in the model method, we define the function `model` as follows:

```python
def model(t, y, mu0, mu1, beta, A, d, nu, b):
    """
    SIR model including hospitalization and natural death.

    Parameters:
    -----------
    mu0 : float
        Minimum recovery rate
    mu1 : float
        Maximum recovery rate
    beta : float
        Average number of adequate contacts per unit time with infectious individuals
    A : float
        Recruitment rate of susceptibles (e.g. birth rate)
    d : float
        Natural death rate
    nu : float
        Disease induced death rate
    b : float
        Hospital beds per 10,000 persons
    """
    S, I, R = y[:]
    m = mu(b, I, mu0, mu1)

    dSdt = A - d*S - (beta*S*I)/(S+I+R)
    dIdt = -(d+nu)*I - m*I + (beta*S*I)/(S+I+R)
    dRdt = m*I - d*R

    return [dSdt, dIdt, dRdt]
```

5.2 Analysis of SIR Model with Varying Initial Conditions

To understand the dynamics of the SIR model under different initial conditions, we simulate the system with a fixed parameter $b = 0.022$ and vary the initial conditions. The following figures illustrate the trajectories of the SIR model for three different initial conditions.



(a) Trajectory with initial condition [195.3, 0.052, 4.4]

(b) Trajectory with initial condition [195.7, 0.03, 3.92]

(c) Trajectory with initial condition [193, 0.08, 6.21]

Figure 16: Comparison of SIR trajectories with different initial conditions and fixed $b = 0.022$. The trajectories exhibit different dynamic behaviors due to the sensitivity of the system to initial conditions.

The next figure shows the overall dynamics of the model with the parameter $b = 0.022$.

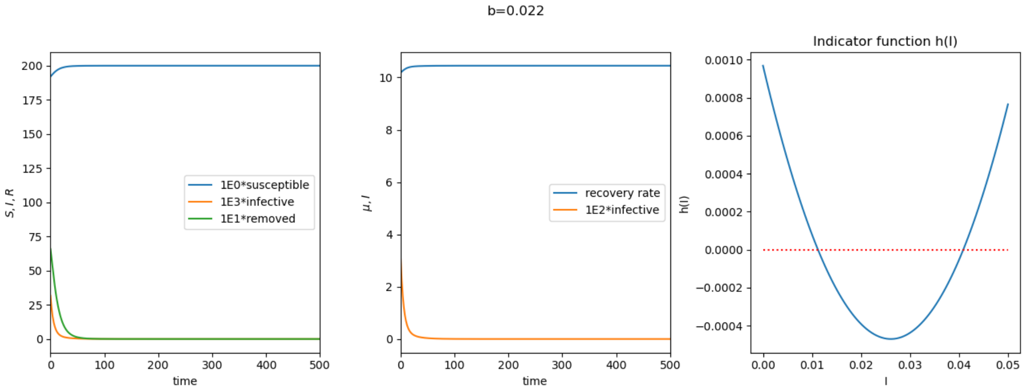


Figure 17: Model dynamics with $b = 0.022$. The plots display the susceptible, infective, and removed populations over time, as well as the recovery rate and the indicator function $h(I)$.

These visualizations highlight the complexity and sensitivity of the SIR model to initial conditions, illustrating the varied trajectories and long-term behaviors possible within the same parameter space.

5.3 Analysis of SIR Model with Varying b

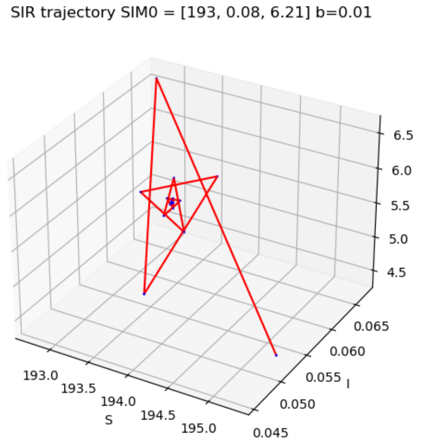


Figure 18: SIR trajectory for $b = 0.01$

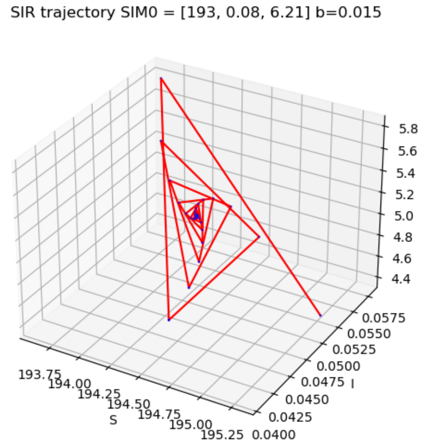


Figure 19: SIR trajectory for $b = 0.015$

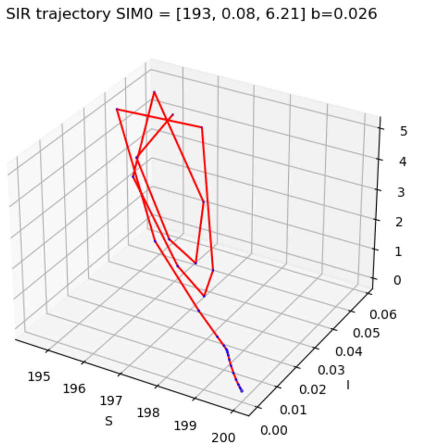


Figure 20: SIR trajectory for $b = 0.026$

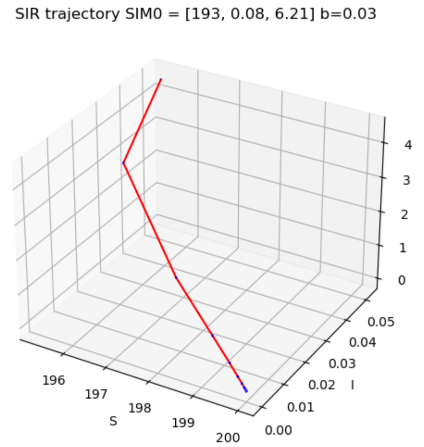


Figure 21: SIR trajectory for $b = 0.03$

Figure 22: Comparison of SIR trajectories with different b values: 0.01, 0.015, 0.026, and 0.03.

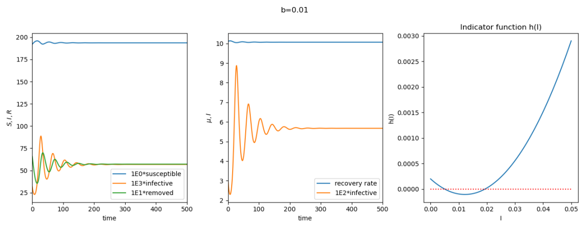


Figure 23: Model dynamics for $b = 0.01$

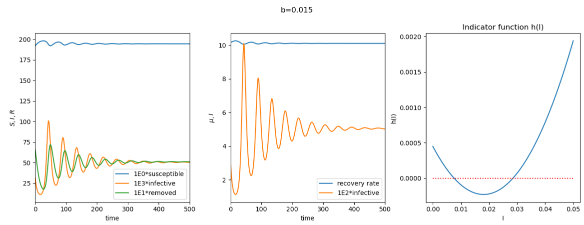


Figure 24: Model dynamics for $b = 0.015$

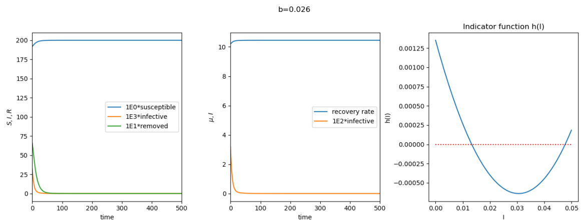


Figure 25: Model dynamics for $b = 0.026$

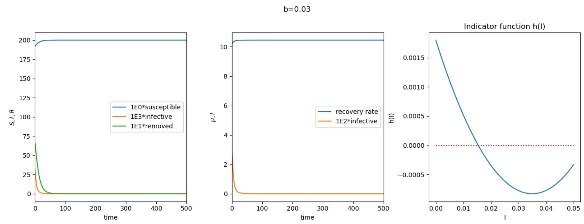


Figure 26: Model dynamics for $b = 0.03$

Figure 27: Comparison of model dynamics with different b values: 0.01, 0.015, 0.026, and 0.03.

As we change the parameter b from 0.01 to 0.03 in small increments, the trajectories of the SIR model show significant variations. Each trajectory corresponds to different initial conditions and illustrates how the system evolves over time. At $b = 0.01$, the system displays complex dynamics with noticeable oscillations. As b increases to 0.015, the oscillations become more pronounced, indicating higher sensitivity to initial conditions. When b is set to 0.026, the trajectory shows a transition towards more stable behavior, and at $b = 0.03$, the system appears to stabilize further, reducing the oscillatory nature observed at lower b values. This demonstrates the impact of the parameter b on the stability and behavior of the SIR model. In this task, we explore the effect of varying the parameter b from 0.01 to 0.03 in small increments on the SIR model. The parameter b represents the number of hospital beds per 10,000 persons and influences the recovery rate in the model.

The four trajectory plots demonstrate how the system's behavior changes with different values of b:

1. **For** $b = 0.01$, the system shows oscillatory behavior in the beginning before stabilizing.
2. **For** $b = 0.015$, the oscillations persist but gradually reduce in amplitude.
3. **For** $b = 0.026$, the system stabilizes more quickly without significant oscillations.
4. **For** $b = 0.03$, the system exhibits a stable behavior almost immediately.

The corresponding model dynamics plots show the time evolution of susceptible, infective, and removed populations for the same b values. As b increases, the system stabilizes faster, indicating a higher recovery rate due to increased availability of hospital beds.

The indicator function $h(I)$ in each case provides a measure of the system's stability. As b increases, the indicator function suggests improved system stability, aligning with the observed trajectories and model dynamics.

5.4 Answers to some questions

1. **What bifurcation is that?**

   The bifurcation observed is a saddle-node bifurcation. In a saddle-node bifurcation, as the parameter value changes, two fixed points in the system (one attractor and one repeller) collide and annihilate each other, causing a qualitative change in system behavior. Specifically, as the value of b varies between 0.02 and 0.03, the system exhibits typical characteristics of a saddle-node bifurcation.

2. **Describe which variables are used to compute the reproduction rate...**

   The basic reproduction number R_0 is calculated using the following variables:

- β: Transmission rate, indicating how many susceptible individuals an infected person contacts and can infect per unit time.
- d: Natural death rate.
- ν: Disease-induced death rate.
- μ_1: Maximum recovery rate.

3. **... and what it means for the number of infective persons.**

   The basic reproduction number R_0 is a crucial parameter for measuring the spread of an infectious disease. When $R_0 > 1$, the disease will spread within the population, and the number of infective persons will increase. When $R_0 < 1$, the disease will gradually die out, and the number of infective persons will decrease.

4. **What does E_0 being an attractive node mean?**

   When E_0 is an attractive node, it means that the system reaches a stable state at E_0. Regardless of the initial conditions, as long as they are within a certain range, the system will eventually converge to this stable point. This implies that the disease will gradually disappear, with no new infective persons in the population, and the numbers of infective and removed individuals stabilize.

5. **What happens for values of (S, I, R) close to E_0?**

   For values of (S, I, R) close to E_0, the system behavior will gradually stabilize and approach the attractor node E_0. This means that the spread of the disease will weaken, the number of infective persons will decrease, and the system will eventually stabilize in a disease-free state.

5.5 Bonus:Discussion of Another Type of Bifurcation in the SIR Model

In this section, we explore another type of bifurcation in the SIR model by changing the parameters and observing the resulting trajectories. We have altered the parameters β and ν to investigate how the system's behavior changes. Specifically, we set $\beta = 15$ and $\nu = 0.5$. The numerical integration is performed using these new parameters, and the results are visualized in the following figures.

The first figure shows the SIR model dynamics with the new parameters. We observe the populations of Susceptible (S), Infective (I), and Removed (R) individuals over time.

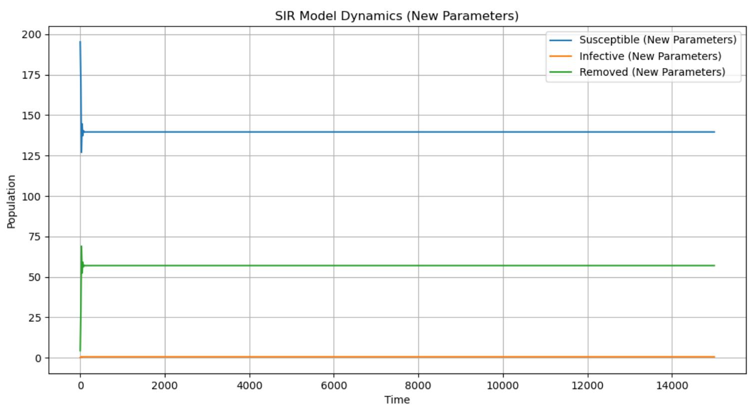


Figure 28: SIR Model Dynamics with New Parameters ($\beta = 15$, $\nu = 0.5$)

The second figure provides a comparison between the infective population with the original parameters and the infective population with the new parameters. This comparison helps us understand the impact of the parameter changes on the infection dynamics.

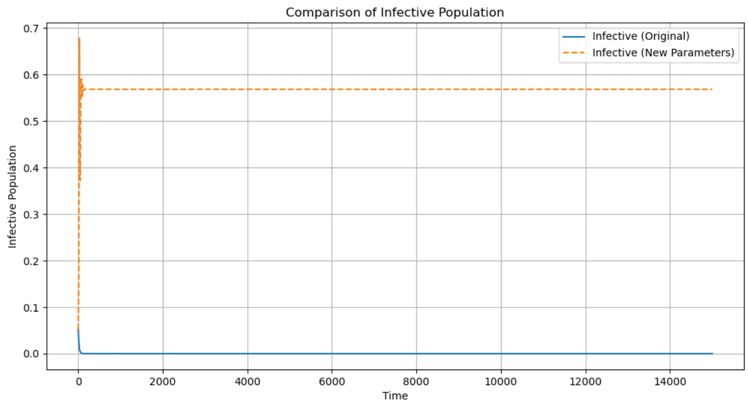


Figure 29: Comparison of Infective Population with Original and New Parameters

5.6 Bonus:Discussion of the Observed Bifurcation

By changing β and ν, we can see a significant alteration in the behavior of the SIR model. The increased β value indicates a higher rate of adequate contacts per unit time with infectious individuals, which intensifies the spread of the infection. On the other hand, increasing ν represents a higher per capita disease-induced death rate, which can balance the infection rate to some extent.

The visualizations illustrate that the infective population stabilizes at a higher value compared to the original parameters. This suggests that the system undergoes a bifurcation, leading to a new equilibrium state where the infection persists at a higher level. The bifurcation observed here is likely a transcritical bifurcation, where the stable equilibrium point shifts due to changes in the parameter values.

This analysis demonstrates how sensitive the SIR model is to parameter changes, which can lead to different types of bifurcations. Identifying and understanding these bifurcations is crucial for predicting the behavior of infectious diseases and implementing effective control measures.

5.7 Conclusion

In this task, we analyzed the bifurcation phenomena in an SIR model by varying the parameter b (number of beds per 10,000 persons) and observing its impact on the dynamics of the system. The numerical simulations and trajectories in the (S, I, R) space revealed that the system exhibits different behaviors for different values of b. When b is small, the system shows regular oscillations before stabilizing, while for larger values of b, the system displays slower progression towards equilibrium, indicating the presence of bifurcations.

We also analyzed the impact of changing the parameters β (transmission rate) and ν (disease-induced death rate) on the dynamics of the SIR model. The results showed that increasing β and decreasing ν significantly increases the infective population, leading to a higher number of infections.

These analyses highlight the importance of healthcare resources (beds) and the transmission rate in controlling the spread of infectious diseases. By understanding the bifurcation phenomena and the impact of different parameters, we can better predict and manage disease outbreaks.