Understanding contact mechanics with Π theorem

BRAUN-DELVOYE Baptiste, CAN Erdi, CARTERON Augustin
CMI (Formation Cursus de Master en Ingénierie), Mechanical Engineering, Sorbonne Université

November 2022

Introduction

Dimensional analysis is a method used to verify the homogeneity of a given formula: any equation that has n parameters, must have the same dimensions on both sides. A parameter n has a unit but has one only k dimensions. The Buckingham Π theorem states that a relationship between n dimensioned parameters involving k independent dimensions can always be rewritten as a relationship between $(n-k)$ dimensionless parameters, also written as Π numbers.
To understand the importance of the Buckingham Π theorem, we decided to take interest in the contact of two surfaces: a sphere and a plane. In our study, we're interested in how the sphere is "squished" when we apply a certain force on it. Theoretically, both the sphere and the plane, are linked by one point in space when they are in equilibrium. But when the plane applies a certain force on the sphere, a contact surface of diameter d_s now defines the intersection between both. So to summarize physically, we'll observe how the diameter of this surface changes when the sphere is "squished". Depending on our n parameters, we will get an equation that would answer our question. Here are our n parameters:

- Diameter of the circle: d_s.
- Gravity: g.
- Weight: m.
- Young's modulus: E.
- Radius of the sphere: r.

We have $n = 5$ dimensioned parameters, which could be linked by:

$$d_s = f(g, m, E, r) \tag{1}$$

We can note that all of our dimensioned parameters have one or more of the following $k = 3$ dimensions: L(length), T(time) and M(mass).

By applying the Π theorem, we get $n - k = 2$ Π numbers. These dimensionless parameters are the following:

$$\Pi_1 = \frac{d_s}{r} \qquad and \qquad \Pi_2 = \frac{E}{g \cdot m \cdot r^{-2}} \tag{2}$$

By following the theorem, we get the following equation :

$$\Pi_1 = F(\Pi_2)$$

$$\Rightarrow \frac{d_s}{r} = F\left(\frac{E}{g \cdot m \cdot r^{-2}}\right) \tag{3}$$

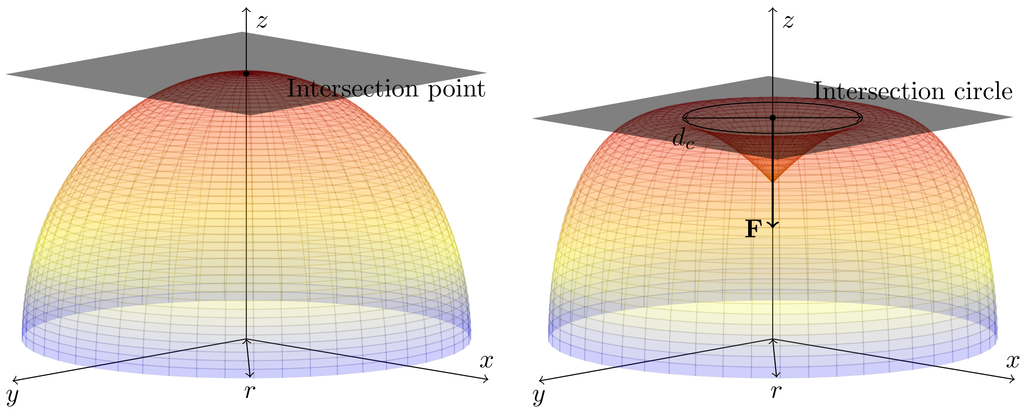

Figure 1: Sphere being squished

Experimental measurements and results

The most difficult task in our study is the use of Young's modulus E. It's defined as the following, with F the applied force, S the surface, and ϵ the strain deformation:

$$E = \frac{F}{S \cdot \epsilon} \tag{4}$$

By its ratio defined above, we can easily determine the Young's modulus for multiple spheres. In our case, we have a total of three with different material proprieties, depicted by their colors: pink, blue and green. We have regrouped the important information in the following table:

Parameters	F $\pm$ 0.15 (N)	S $\pm$1.5 (mm^2)	$\varepsilon \pm 0.01$	Y (GPa)
Pink	9.81	185.28	0.37	$1.4 \cdot 10^{-4}$
Blue	9.81	127.5	0.22	$3.5 \cdot 10^{-4}$
Green	9.81	209	0.04	$1.17 \cdot 10^{-3}$

Table 1: Young's modulus E for different spheres

Concerning the diameter of the contact surface d_s, we fixed one sphere and changed the force applied by the plane by changing the weight of the mass M. For the following graphs, we'll represent our three different sizes of our spheres with their colors respected.

The first graph shows us how Π_1 varies as a function of Π_2: by doing so, we will be able to determine the function f that links our both Π numbers. Or second graph gives us confirmation of our function by seeing if our equation does follow the same line of $y = x$. Also, our first graph is plotted in a log scale to easily determine the slope, and by using simple relationships between logarithm and reel numbers, we are able to determine our function.
If we make an average of the slope, we get $a = -0.32$ and $b = 1.3$. By using logarithm proprieties, we get the following:

$$\Pi_1 = 1.3 \cdot \Pi_2^{-0.32} \approx 1.3 \cdot \sqrt[3]{(\Pi_2)^{-1}} \tag{5}$$

Finally, we get the following function:

$$d_c = 1.38 \cdot \sqrt[3]{\frac{g \cdot m \cdot r}{E}} \tag{6}$$

If we take a look at specific studies about this experiment in literature, we get following true equation:

$$a \approx \sqrt[3]{\frac{F \cdot R}{E}} \tag{7}$$

In conclusion, we realize that with the Buckingham Π theorem, we can get a really close approximation of the theoretical equation.

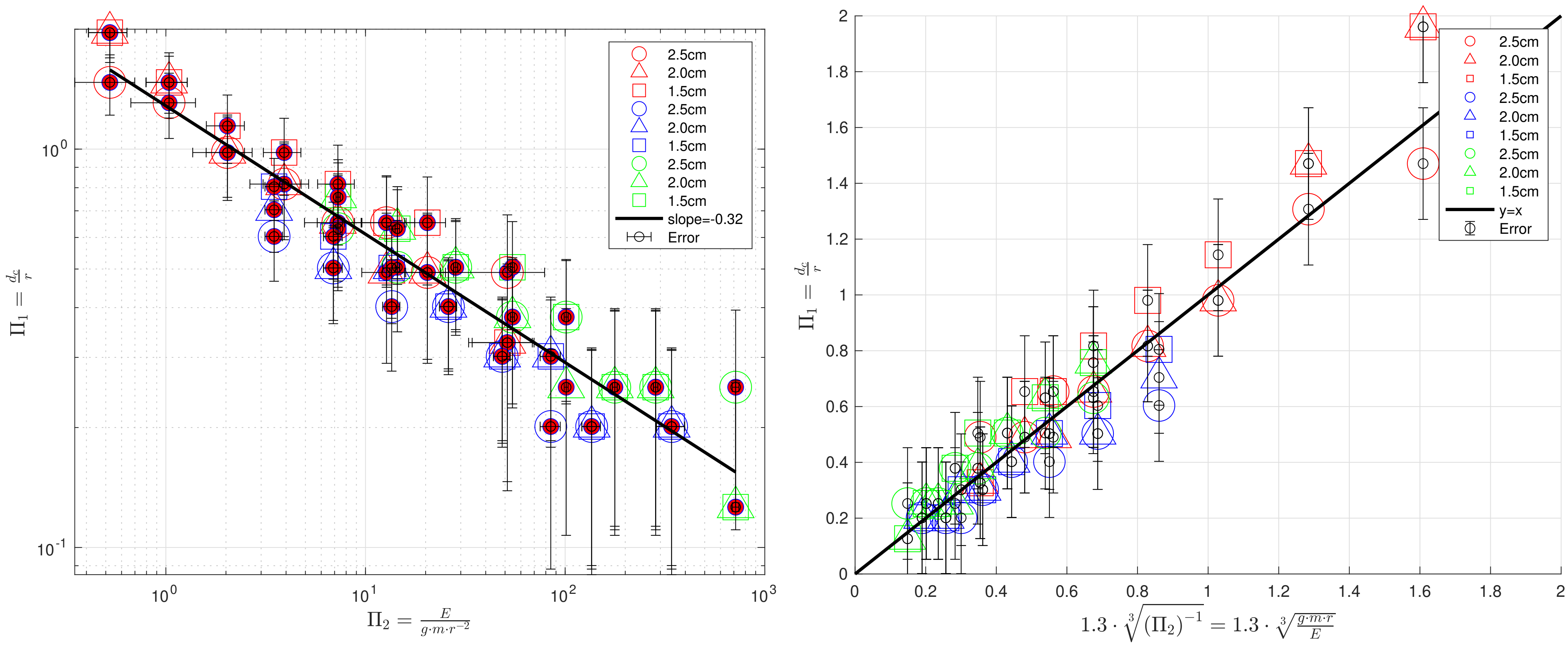

Figure 2: Determination of the relationship between our Π numbers