Understanding contact mechanics with Π theorem

BRAUN-DELVOYE Baptiste, CAN Erdi, CARTERON Augustin
CMI (Formation Cursus de Master en Ingénierie), Mechanical Engineering, Sorbonne Université

November 2022

Introduction - Buckingham Π theorem

Dimensional analysis is a method used to verify the homogeneity of a given formula: any equation that has n parameters, must have the same dimensions on both sides. A parameter n has a unit but has one only k dimensions. The Buckingham Π theorem states that a relationship between n dimensioned parameters involving k independent dimensions can always be rewritten as a relationship between $(n-k)$ dimensionless parameters, also written as Π numbers.
To understand the importance of the Buckingham Π theorem, we decided to take interest in the contact of two surfaces: a sphere and a plane. In our study, we're interested in how the sphere is "squished" when we apply a certain force on it. Theoretically, both the sphere and the plane, are linked by one point in space when they are in equilibrium. But when the plane applies a certain force on the sphere, a contact surface of diameter d_s now defines the intersection between both. So to summarise physically, we'll observe how the diameter of this surface changes when the sphere is "squished". Depending on our n parameters, we will get an equation that would answer our question. Here are our n parameters:

- Diameter of the circle d_s.
- Gravity g.
- Weight m.
- Young's modulus E.
- Radius of the sphere r.

We have $n = 5$ dimensioned parameters, which could be linked by:

$$d_s = f(g, m, E, r) \tag{1}$$

We can note that all of our dimensioned parameters have one or more of the following $k = 3$ dimensions: L(length), T(time) and M(mass). By applying the Π theorem, we get $n - k = 2$ Π numbers. These dimensionless parameters are the following:

$$\Pi_1 = \frac{d_s}{r} \qquad \textit{and} \qquad \Pi_2 = \frac{E}{g \cdot m \cdot r^{-2}} \tag{2}$$

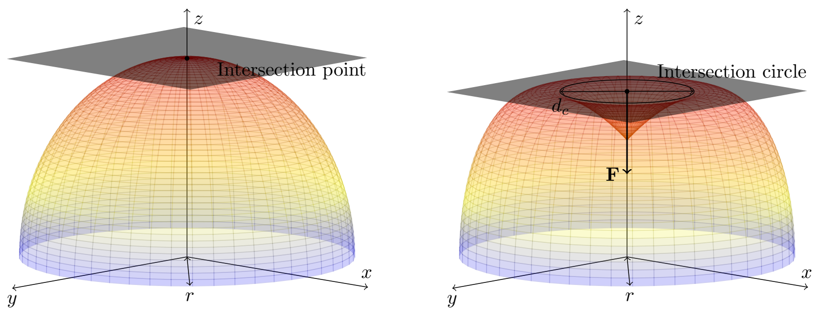
We can easily check if our Π numbers are dimensionless numbers:

$$[\Pi_1] = \frac{L}{L} = 1 \qquad\qquad [\Pi_2] = \frac{ML^{-1}T^{-2}}{LT^{-2} \cdot M \cdot L^{-2}} = 1$$

By following the theorem, we get the following equation :

$$\Pi_1 = F(\Pi_2)$$

$$\Rightarrow \frac{d_s}{r} = F\left(\frac{E}{g \cdot m \cdot r^{-2}}\right) \tag{3}$$



(a) Sphere not squished (b) Sphere quished

Figure 1: Evolution of the sphere when squished

Experimental measurements

The most difficult task in our study is the use of Young's modulus E. It's defined as the following, with σ the stress (force pet unit area), and ϵ the strain (deformation):

$$E = \frac{\sigma}{\epsilon} \quad | \quad \sigma = \frac{F}{S} \quad | \quad \epsilon = \frac{\delta l}{l_0} \tag{4}$$

$$\Rightarrow E = \frac{F}{S \cdot \epsilon} \tag{5}$$

By its new ration defined above, we can easily determine the Young's modulus for multiple spheres. In our case, we have a total of three with different colors: pink, blue and green. We have regrouped the important information in the following table:

Parameters	F ± 0.15 (N)	S ± 1.5 (mm^2)	$\varepsilon \pm 0.01$	Y (GPa)
Pink	9.81	185.28	0.37	$1.4 \cdot 10^{-4}$
Blue	9.81	127.5	0.22	$3.5 \cdot 10^{-4}$
Green	9.81	209	0.04	$1.17 \cdot 10^{-3}$

Table 1: Young's modulus E for different spheres

Concerning the diameter of the contact surface d_s, we fixed one sphere and changed the force applied by the plane by changing the weight of the mass M.

Results

For the following graphs, we'll represent our three different sizes of our spheres with their colours respected. The first graph shows us how Π_1 varies as a function of Π_2: by doing so, we will be able to determine the function f that links our both Π numbers. Or second graph gives us confirmation of our function by seeing if our equation does follow the same line of $y = x$. Also, our first graph is plotted in a log scale to easily determine the slope of our function. And by using simple relationships between logarithm and reel numbers, we will be able to determine our function.

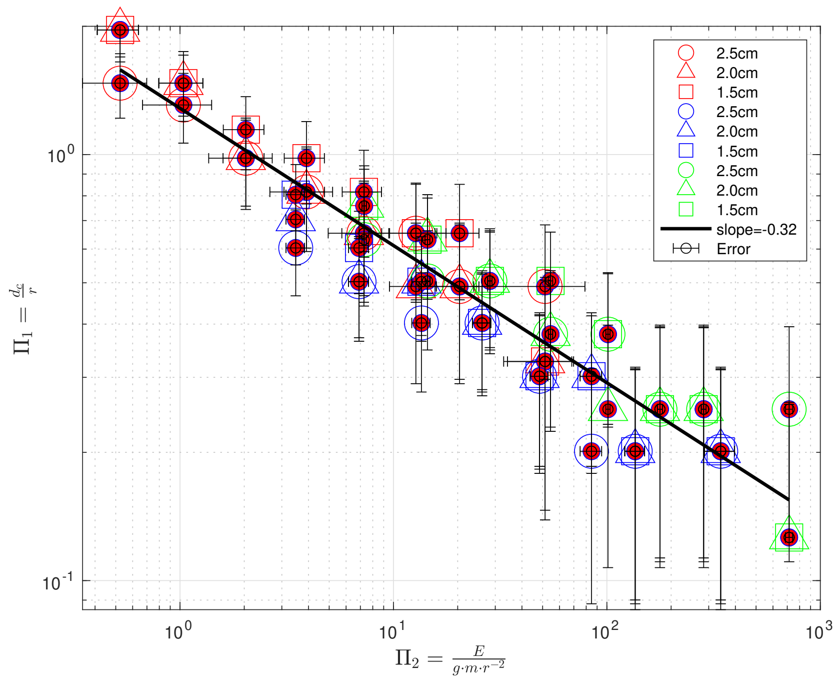


Figure 2: Determination of the relationship between our Π numbers

If we make an average of the slope of our line, we get $a = -0.32$ and $b = 1.3$. By using logarithm proprieties, we get the following:

$$\log(\Pi_1) = a\log(\Pi_2) + \log(b) \Rightarrow \log(a \cdot (\Pi_2)^b)$$

$$\Pi_1 = 1.3 \cdot \Pi_2^{-0.32} \approx 1.3 \cdot \sqrt[3]{(\Pi_2)^{-1}} \tag{6}$$

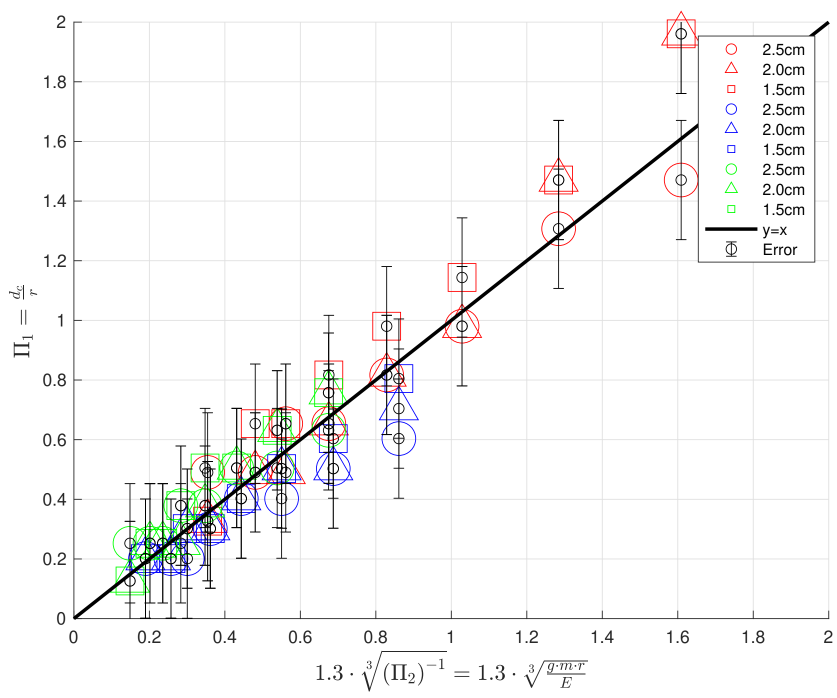


Figure 3: Verification of the relationship between our Π numbers

Conclusion

With our experiments, we get the following function:

$$d_c = 1.38 \cdot \sqrt[3]{\frac{g \cdot m \cdot r}{E}} \tag{7}$$

If we take a look at specific studies about this experiment, we get following true equation :

$$a \approx \sqrt[3]{\frac{F \cdot R}{E}} \tag{8}$$

In conclusion, we realize that with the Buckingham Π theorem, we can get a really close approximation.