We select any nonzero element in the first column to pivot on—this will eliminate x_1.

1	−1	−2	−1	−2	−7
0	1	1	0	−1	−1
0	0	−1	1	−1	−3
0	2	3	−1	1	−14

Equivalent problem

We now save the first row for future reference, but our linear program only involves the sub-tableau indicated. There is no obvious basic feasible solution for this problem, so we introduce artificial variables x_6 and x_7.

	x_2	x_3	x_4	x_5	x_6	x_7	**b**
	−1	−1	0	1	1	0	1
	0	1	−1	1	0	1	3
$\mathbf{c}^T$	0	0	0	0	1	1	0

Initial tableau for phase I

Transforming the last row appropriately we obtain

	x_2	x_3	x_4	x_5	x_6	x_7	**b**
	−1	−1	0	1	1	0	1
	0	1	−1	1	0	1	3
$\mathbf{r}^T$	1	0	1	−2	0	0	−4

First tableau—phase I

−1	−1	0	1	1	0	1
1	2	−1	0	−1	1	2
−1	−2	1	0	2	0	−2

Second tableau—phase I

0	1	−1	1	0	1	3
1	2	−1	0	−1	1	2
0	0	0	0	1	1	0

Final tableau—phase I

Now we go back to the equivalent reduced problem

	x_2	x_3	x_4	x_5	**b**
	0	1	−1	1	3
	1	2	−1	0	2
$\mathbf{c}^T$	2	3	−1	1	−14

Initial tableau—phase II

Transforming the last row appropriately we proceed with:

0	1	−1	1	3
1	2	−1	0	2
0	−2	2	0	−21

First tableau—phase II

−1/2	0	−1/2	1	2
1/2	1	−1/2	0	1
1	0	1	0	−19

Final tableau—phase II

The solution $x_3 = 1$, $x_5 = 2$ can be inserted in the expression for x_1 giving

$$x_1 = -7 + 2 \cdot 1 + 2 \cdot 2 = -1;$$

thus the final solution is

$$x_1 = -1, \quad x_2 = 0, \quad x_3 = 1, \quad x_4 = 0, \quad x_5 = 2.$$

*3.6 VARIABLES WITH UPPER BOUNDS

Many linear programs arising from practical situations involve variables that are subject to both lower and upper bounds. Thus the level of a production activity may be limited by available facilities, the amount of material transported between two points may be subject to a capacity constraint, or the variable voltages in a large electric power network might be required to lie within prescribed bounds. Hence, a typical variable x_i in a linear programming problem will be subject to bounds of the form $g_i \leqslant x_i \leqslant h_i$ for some values of g_i and h_i.

If the variable x_i is subject to no finite bounds, then that variable is free, and as explained in Section 2.1, it can be eliminated from consideration. If the variable is subject to a single bound, that bound can, by possibly changing sign and translating by a constant, always be assumed to take the standard form $x_i \geq 0$. If x_i has finite upper and lower bounds, they can similarly be assumed to take the form $0 \leq x_i \leq h_i$. In this section, therefore, we consider the linear program with upper bounds in the standard form

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{0} \leq \mathbf{x} \leq \mathbf{h}. \end{aligned} \tag{29}$$

For simplicity of notation we assume that each x_i is subject to a finite upper bound. In practice, if some variables in the program are not subject to upper bounds, either an extremely large bound can be artificially introduced, or, more simply, the technique we describe can be trivially modified.

The reader will surely notice that the problem with upper bounds (29) can be easily converted to the usual standard form by the introduction of slack variables y_i for each of the upper-bound constraints. This conversion yields

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} + \mathbf{y} = \mathbf{h} \\ & \mathbf{x} \geq \mathbf{0}, \qquad \mathbf{y} \geq \mathbf{0}. \end{aligned} \tag{30}$$

Although the usual simplex procedure can be applied to this form of the problem, there is a high price to be paid in terms of computing and storage requirements. If the matrix $\mathbf{A}$ were $m \times n$, the matrix associated with the problem in the form (30) would be $(m + n) \times 2n$, which clearly shows that the addition of upper bounds greatly increases the dimensionality of the standard form. In this section we describe an alternative approach that, by slightly generalizing the simplex method, allows computations and storage, for problems with upper bounds, to be treated without explicitly increasing the dimension of the problem.

To describe the method we introduce a single new definition.

Definition. An *extended basic feasible solution* corresponding to (29) is a feasible solution for which $n - m$ variables are equal to either their lower (zero) or their upper bound; and the remaining m (basic) variables correspond to linearly independent columns of $\mathbf{A}$.

In our development, we assume that every extended basic feasible solution is nondegenerate, which means that the m basic variables take values not equal to either of their bounds.

The idea underlying the method is quite simple. Suppose we start with an extended basic feasible solution to the problem. We examine the nonbasic variables (the variables that are on one of their bounds) as possible candidates for change so as to obtain an improved solution. A variable at its lower bound can only be increased, and an increase will be beneficial if the corresponding relative cost coefficient is negative. A variable at its upper bound can only be decreased, and a decrease will be beneficial if the corresponding relative cost coefficient is positive. Suppose a certain nonbasic variable x_i is selected for change in this way. As its value is changed continuously from one bound toward the other, the associated cost will continuously decrease. At the same time the values of the m basic variables will change in such a way that the solution continues to satisfy the linear equalities. The value of the nonbasic variable can be continuously changed until either (i) the value of a basic variable becomes equal to one of its bounds or (ii) the nonbasic variable being changed reaches its opposite bound. If (i) occurs first, then that corresponding basic variable is declared nonbasic and the nonbasic variable that was changed is declared basic. If (ii) occurs first, then the basis is not changed. The simultaneous occurrence of (i) and (ii) results in a degenerate solution and, as before, we ignore this possibility. This procedure is carried out step by step until no further change leading to an improvement is possible.

The justification of the procedure described above rests on the following theorem.

Upper Bound Optimality Theorem. *An extended basic feasible solution is optimal for* (29) *if for the nonbasic variables x_j*

$$r_j \geq 0 \quad \textit{if} \quad x_j = 0$$
$$r_j \leq 0 \quad \textit{if} \quad x_j = h_j.$$

Proof. This follows directly from (22). ∎

The technique discussed above leads to an improved solution if the conditions of the Upper Bound Optimality Theorem do not hold. The computations themselves proceed analogously to those of the usual simplex method, except that the choice of pivot must be modified slightly.

To derive appropriate simplex operations, it is convenient to introduce the notation $x_i^+ = x_i$, $x_i^- = h_i - x_i$. As the method progresses we change back and forth from x_i^+ to x_i^-, depending on whether the variable x_i has most recently been at its lower or upper bound, respectively. At any stage, the system of equalities and cost function are represented in the *extended canonical form* by the tableau shown in Fig. 3.3.

1	0 ⋯	0	$y_{1,m+1}$	$y_{1,m+2}$ ⋯	y_{1n}	y_{10}
0	1	0	$y_{2,m+1}$	$y_{2,m+2}$	y_{2n}	y_{20}
0	0	0	·	·	·	·
·	·	·	·	·	·	·
·	·	·	·	·	·	·
0	0	1	$y_{m,m+1}$	$y_{m,m+2}$	y_{mn}	y_{m0}
0	0	0	r_{m+1}	r_{m+2}	r_n	$-z_0$
e_1	e_2 ⋯	e_m	e_{m+1}	e_{m+2}	e_n	

Fig. 3.3 Extended tableau

Below each column we place an indicator $e_i = \pm$ to indicate if in the current solution that column corresponds to $x_i^+ = x_i$ ($e_i = +$) or to $x_i^- = h_i - x_i$ ($e_i = -$). The tableau is interpreted as representing the equations

$$\sum_{j=1}^{n} y_{ij} x_j^{e_j} = y_{i0}, \qquad i = 1, 2, \ldots, m.$$

With this notation the procedure for transforming from one extended basic feasible solution to another, following the strategy outlined above the Upper Bound Optimality Theorem and analogous to the simplex method, can be easily implemented. The procedure follows.

Step 1. Determine a nonbasic variable $x_j^{e_j}$ for which $r_j < 0$. If no such variable exists, stop; the current solution is optimal.

Step 2. Evaluate the three numbers

a) h_j

b) $\min\limits_{y_{ij}>0} y_{i0}/y_{ij}$

c) $\min\limits_{y_{ij}<0} (y_{i0} - h_i)/y_{ij}$,

where h_i is the upper bound associated with the ith basic variable.

Step 3. According to which number in Step 2 is smallest, update the extended tableau as follows:

a) The variable x_j goes to its opposite bound. Subtract h_j times column j from column 0. Multiply column j by minus unity (including a change in sign of e_j). The basis does not change and no pivot is required.

b) Suppose i is the minimizing index in (b) of Step 2. Then the ith basic variable returns to its old bound. Pivot on the ijth element.

c) Suppose i is the minimizing index in (c) of Step 2. Then the ith basic variable goes to its opposite bound. Subtract h_i from y_{i0}, change the signs of y_{ii} and e_i, and pivot on the ijth element.

Return to Step 1.

Example.

$$\begin{array}{ll} \text{minimize} & 2x_1 + x_2 + 3x_3 - 2x_4 + 10x_5 \\ \text{subject to} & x_1 \quad\;\; + x_3 - \;\; x_4 + 2x_5 = 5 \\ & \quad\;\; x_2 + 2x_3 + 2x_4 + \;\; x_5 = 9 \end{array}$$

$$0 \le x_1 \le 7, \quad 0 \le x_2 \le 10, \quad 0 \le x_3 \le 1, \quad 0 \le x_4 \le 5, \quad 0 \le x_5 \le 3.$$

The original tableau for this problem, after calculating the relative cost coefficients, is

	x_1	x_2	x_3	x_4	x_5	**b**
	1	0	1	−1	2	5
	0	1	2	2	1	9
$\mathbf{r}^T$	0	0	−1	−2	5	−19
	+	+	+	+	+	

First tableau

which represents a basic feasible solution. We decide that column 4 should enter. Making the required calculations we find the numbers

(a) 5
(b) 9/2
(c) 2

and hence case (c) applies. Before pivoting we modify the tableau to

−1	0	1	−1	2	−2
0	1	2	2	1	9
0	0	−1	−2	5	−19
−	+	+	+	+	

Modified first tableau

Pivoting as dictated by case (c) we obtain

1	0	−1	1	−2	2
−2	1	4	0	5	5
2	0	−3	0	1	−15
−	+	+	+	+	

Second tableau

Next we decide to enter column 3. This time we find

(a) 1
(b) 5/4
(c) 3

so case (a) applies. Thus no pivot is necessary, but we obtain

$$\begin{array}{cccccc} 1 & 0 & 1 & 1 & -2 & 3 \\ -2 & 1 & -4 & 0 & 5 & 1 \\ 2 & 0 & 3 & 0 & 1 & -12 \\ - & + & - & + & + & \end{array}$$

Final tableau

Since all reduced cost coefficients are nonnegative this tableau represents an optimal solution. The solution is

$$x_1 = 7, \quad x_2 = 1, \quad x_3 = 1, \quad x_4 = 3, \quad x_5 = 0.$$

3.7 MATRIX FORM OF THE SIMPLEX METHOD

Although the elementary pivot transformations associated with the simplex method are in many respects most easily discernible in the tableau format, with attention focused on the individual elements, there is much insight to be gained by studying a matrix interpretation of the procedure. The vector–matrix relationships that exist between the various rows and columns of the tableau lead, however, not only to increased understanding but also, in a rather direct way, to the *revised simplex procedure* which in many cases can result in considerable computational advantage. The matrix formulation is also a natural setting for the discussion of dual linear programs and other topics related to linear programming.

A preliminary observation in the development is that the tableau at any point in the simplex procedure can be determined solely by a knowledge of which variables are basic. As before we denote by $\mathbf{B}$ the submatrix of the original $\mathbf{A}$ matrix consisting of the m columns of $\mathbf{A}$ corresponding to the basic variables. These columns are linearly independent and hence the columns of $\mathbf{B}$ form a basis for E^m. We refer to $\mathbf{B}$ as the basis matrix.

As usual, let us assume that $\mathbf{B}$ consists of the first m columns of $\mathbf{A}$. Then by partitioning $\mathbf{A}$, $\mathbf{x}$, and $\mathbf{c}^T$ as

$$\mathbf{A} = [\mathbf{B}, \mathbf{D}]$$
$$\mathbf{x} = (\mathbf{x_B}, \mathbf{x_D}), \qquad \mathbf{c}^T = [\mathbf{c_B}^T, \mathbf{c_D}^T],$$

the standard linear programming problem becomes

$$\begin{aligned} \text{minimize} \quad & \mathbf{c_B}^T\mathbf{x_B} + \mathbf{c_D}^T\mathbf{x_D} \\ \text{subject to} \quad & \mathbf{B}\mathbf{x_B} + \mathbf{D}\mathbf{x_D} = \mathbf{b} \\ & \mathbf{x_B} \geqslant \mathbf{0}, \quad \mathbf{x_D} \geqslant \mathbf{0}. \end{aligned} \tag{31}$$

The basic solution, which we assume is also feasible, corresponding to the basis $\mathbf{B}$ is $\mathbf{x} = (\mathbf{x_B}, \mathbf{0})$ where $\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b}$. The basic solution results from setting $\mathbf{x_D} = \mathbf{0}$. However, for any value of $\mathbf{x_D}$ the necessary value of $\mathbf{x_B}$ can be computed from (31) as

$$\mathbf{x_B} = \mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{D}\mathbf{x_D}, \tag{32}$$

and this general expression when substituted in the cost function yields

$$\begin{aligned} z &= \mathbf{c_B}^T(\mathbf{B}^{-1}\mathbf{b} - \mathbf{B}^{-1}\mathbf{D}\mathbf{x_D}) + \mathbf{c_D}^T\mathbf{x_D} \\ &= \mathbf{c_B}^T\mathbf{B}^{-1}\mathbf{b} + (\mathbf{c_D}^T - \mathbf{c_B}^T\mathbf{B}^{-1}\mathbf{D})\mathbf{x_D}, \end{aligned} \tag{33}$$

which expresses the cost of any solution to (31) in terms of $\mathbf{x_D}$. Thus

$$\mathbf{r_D}^T = \mathbf{c_D}^T - \mathbf{c_B}^T\mathbf{B}^{-1}\mathbf{D} \tag{34}$$

is the relative cost vector (for nonbasic variables). It is the components of this vector that are used to determine which vector to bring into the basis.

Having derived the vector expression for the relative cost it is now possible to write the simplex tableau in matrix form. The initial tableau takes the form

$$\left[\begin{array}{c:c} \mathbf{A} & \mathbf{b} \\ \hdashline \mathbf{c}^T & 0 \end{array}\right] = \left[\begin{array}{c:c:c} \mathbf{B} & \mathbf{D} & \mathbf{b} \\ \hdashline \mathbf{c_B}^T & \mathbf{c_D}^T & 0 \end{array}\right], \tag{35}$$

which is not in general in canonical form and does not correspond to a point in the simplex procedure. If the matrix $\mathbf{B}$ is used as a basis, then the corresponding tableau becomes

$$\mathbf{T} = \left[\begin{array}{c:c:c} \mathbf{I} & \mathbf{B}^{-1}\mathbf{D} & \mathbf{B}^{-1}\mathbf{b} \\ \hdashline \mathbf{0} & \mathbf{c_D}^T - \mathbf{c_B}^T\mathbf{B}^{-1}\mathbf{D} & -\mathbf{c_B}^T\mathbf{B}^{-1}\mathbf{b} \end{array}\right], \tag{36}$$

which is the matrix form we desire.

3.8 THE REVISED SIMPLEX METHOD

Extensive experience with the simplex procedure applied to problems from various fields, and having various values of n and m, has indicated that the method can be expected to converge to an optimum solution in about m, or perhaps $3m/2$, pivot operations. Thus, particularly if m is much smaller than n, that is, if the matrix $\mathbf{A}$ has far fewer rows than columns, pivots will occur in only a small fraction of the columns during the course of optimization.