Some Exercises on ARMA Processes

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Abstract

These exercises do not represent outright study material but simply an attempt to encourage a deeper understanding of the subject.

1. **Exercise 1**

Consider the following $MA(2)$ process:

$$x_t = \varepsilon_t + 2.4\varepsilon_{t-1} + 0.8\varepsilon_{t-2}, \quad \varepsilon_t \sim IID\ N(0, \sigma_\varepsilon^2). \tag{1}$$

1a. Re-write this process using the lag operator.

1b. Is the MA process invertible? Explain why it is (or it is not) invertible.

1c. Derive the first two unconditional moments of the process.

Consider now the following $AR(2)$ process:

$$x_t = 1.1x_{t-1} - 0.18x_{t-2} + \varepsilon_t, \quad \varepsilon_t \sim IID\ N(0, \sigma_\varepsilon^2). \tag{2}$$

1d. Re-write this process using the lag operator.

1e. Is it possible to rewrite the process as an $MA(\infty)$ process? Why? If possible, re-write it as an $MA(\infty)$ process.

1f. Is it possible to apply Wold's decomposition theorem? Why is this (or is this not) possible?

1g. Compute $E(x_t \mid x_{t-1})$, $E(x_t \mid x_{t-2})$, $Var(x_t \mid x_{t-1})$, and $Var(x_t \mid x_{t-2})$.

2. **Exercise 2**

In order to illustrate a practical application of an MA process, suppose you win 1€ if by throwing a fair coin, it shows head and you lose 1€ if it shows tail. Denote the outcome on toss t by ε_t. The sequence $\{\varepsilon_t\}$ is a white noise process. However, for each coin toss t, your average payoff on the last *four* tosses is given by:

$$x_t = 0.25\varepsilon_t + 0.25\varepsilon_{t-1} + 0.25\varepsilon_{t-2} + 0.25\varepsilon_{t-3},\ \varepsilon_t \sim IID\ N(0, \sigma_\varepsilon^2).$$

2a. Can you recognize any of the models shown in class in the process for x_t?

2b. Find the expected value of your winnings (i.e., $E(x_t)$). Find the expected value of x_t given that $\varepsilon_{t-2} = \varepsilon_{t-3} = 1$ (i.e., $E(x_t \mid \varepsilon_{t-2} = \varepsilon_{t-3} = 1)$).

2c. Find $Var(x_t)$ and $Var(x_t \mid \varepsilon_{t-2} = \varepsilon_{t-3} = 1)$.

2d. Find $Cov(x_t, x_{t-1})$, $Cov(x_t, x_{t-2})$, and $Cov(x_t, x_{t-3})$.

Solutions

1a. Using the lag operator, (1) can be re-written as:

$$x_t = (1 + 2.4L + 0.8L^2)\varepsilon_t, \quad \varepsilon_t \sim IID\ N(0, \sigma_\varepsilon^2).$$

1b. An MA process is said to be invertible if all the roots of the characteristic equation lie outside the unit circle. Starting from the expression derived under 1a,

$$x_t = (1 + 2.4L + 0.8L^2)\varepsilon_t,$$

we can verify whether the roots of the characteristic equation lie outside the unit circle. The characteristic equation is given by:

$$(1 + 2.4z + 0.8z^2) = 0,$$

whose roots are:

$$z_1 = -2.5 \text{ and } z_2 = -0.5.$$

Because $|z_2| < 1$ we conclude that the process is not invertible.

1c.

$$\begin{aligned} E(x_t) &= E(\varepsilon_t + 2.4\varepsilon_{t-1} + 0.8\varepsilon_{t-2}) \\ &= E(\varepsilon_t) + 2.4E(\varepsilon_{t-1}) + 0.8E(\varepsilon_{t-2}). \end{aligned}$$

Because

$$E(\varepsilon_t) = 0, \forall t,$$

it follows that

$$E(x_t) = E(\varepsilon_t) + 2.4E(\varepsilon_{t-1}) + 0.8E(\varepsilon_{t-2}) = 0.$$

$Var(x_t)$ is equal to:

$$Var(x_t) = Var(\varepsilon_t + 2.4\varepsilon_{t-1} + 0.8\varepsilon_{t-2}),$$

then, because $\varepsilon_t \sim IID$ (which implies that all cross product expectations are zero), we can write:

$$\begin{aligned} Var(x_t) &= Var(\varepsilon_t + 2.4\varepsilon_{t-1} + 0.8\varepsilon_{t-2}) \\ &= Var(\varepsilon_t) + 2.4^2 Var(\varepsilon_{t-1}) + 0.8^2 Var(\varepsilon_{t-2}). \end{aligned}$$

Moreover, we know that

$$Var(\varepsilon_t) = \sigma_\varepsilon^2, \forall t$$

so that

$$\begin{aligned} Var(x_t) &= Var(\varepsilon_t) + (2.4)^2 Var(\varepsilon_{t-1}) + (0.8)^2 Var(\varepsilon_{t-2}) \\ &= \sigma_\varepsilon^2 + 5.76\sigma_\varepsilon^2 + 0.64\sigma_\varepsilon^2 = 7.40\sigma_\varepsilon^2. \end{aligned}$$

The autocovariances of x_t are given by:

$$\begin{aligned}
Cov(x_t, x_{t-1}) &= Cov(\varepsilon_t + 2.4\varepsilon_{t-1} + 0.8\varepsilon_{t-2}, \varepsilon_{t-1} + 2.4\varepsilon_{t-2} + 0.8\varepsilon_{t-3}) = 4.32\sigma_\varepsilon^2; \\
Cov(x_t, x_{t-2}) &= Cov(\varepsilon_t + 2.4\varepsilon_{t-1} + 0.8\varepsilon_{t-2}, \varepsilon_{t-2} + 2.4\varepsilon_{t-3} + 0.8\varepsilon_{t-4}) = 0.8\sigma_\varepsilon^2; \\
Cov(x_t, x_{t-j}) &= 0 \text{ for } j > 2.
\end{aligned}$$

1d. Using the lag operator, (2) can be re-written as:

$$(1 - 1.1L + 0.18L^2)x_t = \varepsilon_t, \quad \varepsilon_t \sim IID\ N(0, \sigma_\varepsilon^2).$$

1e . Wold's decomposition theorem states that any stationary process can be expressed as the sum of two processes, a deterministic part and a stochastic part, which will be a $MA(\infty)$ process. A simpler way of stating this in the context of AR modeling is that any stationary $AR(p)$ process with no constant and no other terms can be expressed as a $MA(\infty)$ process. In order to answer the question it is necessary to determine whether (2) is stationary (i.e., to verify whether all the roots of the characteristic equation lie outside the unit circle). Starting from the equation derived in 1d,

$$(1 - 1.1L + 0.18L^2)x_t = \varepsilon_t,$$

the characteristic equation is given by:

$$1 - 1.1z + 0.18z^2 = 0,$$

whose roots are:

$$z_1 = 5 \text{ and } z_2 = 1.11.$$

Because both $|z_1| > 1$ and $|z_2| > 1$, we can conclude that (2) is stationary and it can be written as a $MA(\infty)$ processes.

In order to express (2) as an $MA(\infty)$ process, first, we re-write:

$$(1 - 1.1L + 0.18L^2)x_t = \varepsilon_t$$

as

$$\Phi(L)x_t = \varepsilon_t$$

then, the Wold's decomposition is:

$$x_t = \Psi(L)\varepsilon_t$$

where $\Psi(L) = \Phi(L)^{-1}$.

1f. See answer 1e.

1g. $E(x_t|x_{t-1})$ is given by:

$$\begin{aligned}
E(x_t|x_{t-1}) &= E(1.1x_{t-1} - 0.18x_{t-2} + \varepsilon_t \mid x_{t-1}) \\
&= 1.1E(x_{t-1}|x_{t-1}) - 0.18E(x_{t-2}|x_{t-1}) + E(\varepsilon_t|x_{t-1}) \\
&= 1.1x_{t-1} - 0.18x_{t-2}.
\end{aligned}$$

In order to find $E(x_t|x_{t-2})$ we need to express x_t as function of x_{t-2}. First, we express x_{t-1} as function of x_{t-2} :

$$x_{t-1} = 1.1x_{t-2} - 0.18x_{t-3} + \varepsilon_{t-1}.$$

Now we can express x_t as function of x_{t-2} :

$$\begin{aligned} x_t &= 1.1(1.1x_{t-2} - 0.18x_{t-3} + \varepsilon_{t-1}) - 0.18x_{t-2} + \varepsilon_t \\ &= 1.03x_{t-2} - 0.20x_{t-3} + \varepsilon_t + 1.1\varepsilon_{t-1}. \end{aligned} \tag{3}$$

Taking expectations of both sides of (3) gives:

$$E(x_t|x_{t-2}) = E(1.03x_{t-2} - 0.20x_{t-3} + \varepsilon_t + 1.1\varepsilon_{t-1} \mid x_{t-2}) = 1.03x_{t-2} - 0.20x_{t-3}.$$

$Var(x_t|x_{t-1})$ and $Var(x_t|x_{t-2})$ are given by:

$$\begin{aligned} Var(x_t|x_{t-1}) &= Var(1.1x_{t-1} - 0.18x_{t-2} + \varepsilon_t \mid x_{t-1}) = \sigma_\varepsilon^2 \\ Var(x_t|x_{t-2}) &= Var(1.03x_{t-2} - 0.20x_{t-3} + \varepsilon_t + 1.1\varepsilon_{t-1} \mid x_{t-1}) = (1^2 + 1.1^2)\sigma_\varepsilon^2. \end{aligned}$$

In order to find $Var(x_t|x_{t-2})$, we haved used the expression of x_t as function of x_{t-2} found in (3).

2a. It is an $MA(3)$ process.

2b.

$$\begin{aligned} E(x_t) &= E(0.25\varepsilon_t + 0.25\varepsilon_{t-1} + 0.25\varepsilon_{t-2} + 0.25\varepsilon_{t-3}) = 0; \\ E(x_t &\mid \varepsilon_{t-2} = \varepsilon_{t-3} = 1) = E(0.25\varepsilon_t + 0.25\varepsilon_{t-1} + 0.25\varepsilon_{t-2} + 0.25\varepsilon_{t-3} \mid \varepsilon_{t-2} = \varepsilon_{t-3} = 1) \\ &= 0.25(0 + 0 + 1 + 1) = 0.50. \end{aligned}$$

For a proof of the first result, see answer 1c. The second result depends on the fact that while $E(\varepsilon_{t-2} \mid \varepsilon_{t-2} = \varepsilon_{t-3} = 1) = 1$ and $E(\varepsilon_{t-2} \mid \varepsilon_{t-2} = \varepsilon_{t-3} = 1) = 1$, $E(\varepsilon_t \mid \varepsilon_{t-2} = \varepsilon_{t-3} = 1) = E(\varepsilon_t) = 0$ and $E(\varepsilon_{t-1} \mid \varepsilon_{t-2} = \varepsilon_{t-3} = 1) = E(\varepsilon_{t-1}) = 0$.

2c. $Var(x_t)$ and $Var(x_t \mid \varepsilon_{t-2} = \varepsilon_{t-3} = 1)$ are equal to:

$$\begin{aligned} Var(x_t) &= Var(0.25\varepsilon_t + 0.25\varepsilon_{t-1} + 0.25\varepsilon_{t-2} + 0.25\varepsilon_{t-3}) \\ &= 0.25^2 Var(\varepsilon_t) + 0.25^2 Var(\varepsilon_{t-1}) + 0.25^2 Var(\varepsilon_{t-2}) + 0.25^2 Var(\varepsilon_{t-3}) = 0.25\sigma_\varepsilon^2 \\ Var(x_t &\mid \varepsilon_{t-2} = \varepsilon_{t-3} = 1) = Var(0.25\varepsilon_t + 0.25\varepsilon_{t-1} + 0.25\varepsilon_{t-2} + 0.25\varepsilon_{t-3} \mid \varepsilon_{t-2} = \varepsilon_{t-3} = 1) = 0.13\sigma_\varepsilon^2. \end{aligned}$$

For a demonstration of the first result, see answer 1c. The second result depends on the fact that $Var(\varepsilon_{t-2} \mid \varepsilon_{t-2} = \varepsilon_{t-3} = 1) = 0$ and $Var(\varepsilon_{t-3} \mid \varepsilon_{t-2} = \varepsilon_{t-3} = 1) = 0$, while $Var(\varepsilon_t \mid \varepsilon_{t-2} = \varepsilon_{t-3} = 1) = Var(\varepsilon_t) = \sigma_\varepsilon^2$ and $Var(\varepsilon_{t-1} \mid \varepsilon_{t-2} = \varepsilon_{t-3} = 1) = Var(\varepsilon_{t-1}) = \sigma_\varepsilon^2$.

2d. The autocovariances of x_t are given by:

$$\begin{aligned}
Cov(x_t, x_{t-1}) &= Cov(0.25\varepsilon_t + 0.25\varepsilon_{t-1} + 0.25\varepsilon_{t-2} + 0.25\varepsilon_{t-3}, 0.25\varepsilon_{t-1} + 0.25\varepsilon_{t-2} + 0.25\varepsilon_{t-3} + 0.25\varepsilon_{t-4}) \\
&= 3(0.25^2)\sigma_\varepsilon^2 \\
Cov(x_t, x_{t-2}) &= Cov(0.25\varepsilon_t + 0.25\varepsilon_{t-1} + 0.25\varepsilon_{t-2} + 0.25\varepsilon_{t-3}, 0.25\varepsilon_{t-2} + 0.25\varepsilon_{t-3} + 0.25\varepsilon_{t-4} + 0.25\varepsilon_{t-5}) \\
&= 2(0.25^2)\sigma_\varepsilon^2 \\
Cov(x_t, x_{t-3}) &= Cov(0.25\varepsilon_t + 0.25\varepsilon_{t-1} + 0.25\varepsilon_{t-2} + 0.25\varepsilon_{t-3}, 0.25\varepsilon_{t-3} + 0.25\varepsilon_{t-4} + 0.25\varepsilon_{t-5} + 0.25\varepsilon_{t-6}) \\
&= (0.25^2)\sigma_\varepsilon^2.
\end{aligned}$$

All these results depend on the fact that $Cov(\varepsilon_i, \varepsilon j) = \begin{cases} \sigma_\varepsilon^2 \text{ when } i = j \\ 0 \text{ when } i \neq j \end{cases}$.