Ejercicio1 parte A

Los ejercicios desde la obtención del vector de medias incondicional hasta descomposición de varianzas son desarrollados de forma genérica de forma tal que si se desea cambiar la restricción de identificación los cálculos sean inmediatos. Sumado a ello, se provee la resolución mediante el lenguaje R.

$$X_t = \begin{pmatrix} y_t \\ z_t \end{pmatrix} \quad A_0 = \begin{pmatrix} a_{10} \\ a_{20} \end{pmatrix} \quad A_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad e_t = \begin{pmatrix} e_1^y \\ e_2^z \end{pmatrix}$$

Definimos el siguiente sistema, al que nos referimos como un VAR(1) con dos ecuaciones.

$$X_t = A_0 + A_1 X_{t-1} + e_t$$

Estabilidad

Para analizar la estabilidad de un VAR(p) con p $\in Z$ podemos reexpresar el sistema de la siguiente forma haciendo uso del operador de rezagos:

$$(I - AL)X_t = A_0 + e_t$$

De esta forma, podemos analizar el comportamiento del sistema al calcular las raíces del determinante de la matriz $(I - AL)$:

$$\begin{aligned}
&\left|I - AL\right| = 0 \\
&\left|I - AL\right| = \left|\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - L\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right| = \left|\begin{pmatrix} 1 - a_{11}L & -a_{12}L \\ -a_{21}L & 1 - a_{22}L \end{pmatrix}\right| \\
&\qquad = (1 - a_{11}L)(1 - a_{22}L) - (-a_{21}L)(-a_{12}L) \\
&\qquad = 1 - (a_{22} + a_{11})L + (a_{11}a_{22} - a_{21}a_{12})L^2 \\
&\textit{resolviendo} \\
&\qquad 1 - (a_{22} + a_{11})L + (a_{11}a_{22} - a_{21}a_{12})L^2 = 0
\end{aligned}$$

Usando que los siguientes valores:

$$A_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0.5 & 0.1 \\ -0.4 & 0.5 \end{pmatrix}$$

Resolviendo la ecuación de segundo grado se obtienen raíces imaginarias:

$$\frac{1 \pm \sqrt{-0.16}}{0.58}$$

Por lo que podemos plantear $|I - AL| = 0$ como una ecuación en diferencias de segundo orden, omitiendo la parte no autoregresiva dandole valor 0:

$$y_t - y_{t-1} + 0.29y_{t-2} = 0 análogamente y_{t+2} - y_{t+1} + 0.29y_t = 0$$

La solución particular $y_p = k$ es:

$$\begin{aligned}
k - k + 0.29k &= 0 \\
0.29k &= -1 \\
k &= -3.448276 \\
y_p &= -3.448276
\end{aligned}$$

El polinomio característico de la forma $y_t = Ab^t$ es:

$$
\begin{aligned}
Ab^{t+2} - Ab^{t+1} + 0.29Ab^t &= 0 \\
b^2 - b + 0.29 &= 0 \\
&1 \pm \frac{\sqrt{-0.16}}{2} \\
&h \pm vi \\
h &= \frac{1}{2} \\
vi &= \frac{\sqrt{-0.16}}{2}
\end{aligned}
$$

Cuando las raíces son imaginarías podemos expresarlas de la siguiente forma y usar el teorema de De Moivre

$$
\begin{aligned}
y_c = A_1(h+vi)^t + A_2(h-vi)^t \\
\text{TM De Moivre:} \\
(h \pm vi)^t = R^t(cos\theta t \pm isen\theta t) \\
Siendo\ R: \\
R = \sqrt{h^2 + v2} \\
R = \sqrt{\frac{a_1 + 4a_2 - a_1^2}{4}} = \sqrt{a_2} \\
a1 = -1 \quad a_2 = 0.29 \\
cos\theta = \frac{h}{R} = \frac{-a_1}{2\sqrt{a_2}} \\
sen\theta = \frac{v}{R} = \sqrt{1 - \frac{a_1^2}{4a_2}}
\end{aligned}
$$

Siendo theta la medida en radianes del ángulo en el intervalo [0,2π] que satisface $cos\theta$ y $sen\theta$

Por lo tanto, dados nuestros valores tenemos que:

$$R = \sqrt{a_2} = \sqrt{0.29} = 0.5385165$$

Ya que, $R < 1$ las oscilaciones del sistema se atenúan con el tiempo y convergen, por lo cual el sistema es estable.

La solución general de la ecuación en diferencia tiene la siguiente forma y es estable:

$$\begin{aligned}
y_t &= y_p + y_c = \\
&= -3.45 + A_1(h+vi)^t + A_2(h-vi)^t \\
&= -3.45 + A_1 R^t(\cos\theta t + i\sin\theta t) + A_2 R^t(\cos\theta t - i\sin\theta t) \\
&= 1 + R^t[(A_1+A_2)\cos\theta t + (A_1+A_2)i\sin\theta t] \\
&= 1 + R^t[]A_5\cos\theta t + A_6\sin\theta t] \\
Usando\ que: & \\
R &= 0.54 \\
\cos\theta &= 0.93 \\
\sin\theta &= 0.37 \\
t &= 0 \\
y_0 &= 0 \\
Tenemos: & \\
0 &= -3.45 + A_5 \\
3.45 &= A_5 \\
Tomando: & \\
t &= 1 \\
y_1 &= 1 \\
1 &= -3.45 + 0.54[3.45 * 0.93 + A_6 * 0.37] \\
A_6 &= 13.6 \\
Por\ lo\ tanto: & \\
y_t &= y_p + y_c = 0.54^t[3.45 * 0.93t + 13.6 * 0.37t] - 3.45
\end{aligned}$$

Vector de medias

Ya que la serie es estable se cumple que $E(X_t) = E(X_{t-1}) = \mu$ y además $E(e_t) = 0$ entonces:

$$\begin{aligned}
E(X_t) &= E(A_0 + A_1 X_{t-1} + e_t) \\
E(X_t) &= A_0 + A_1 E(X_{t-1}) \\
(I - A_1)\mu &= A_0 \\
\mu &= (I - A_1)^{-1} A_0 \\
\mu &= \begin{pmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{pmatrix}^{-1} A_0 = J^{-1} A_0 \\
\mu &= \frac{1}{det(J)} Adj(J) A_0 \\
\frac{1}{det(J)} &= \frac{1}{j_{11}j_{22} - j_{21}j_{12}} = \frac{1}{(1-a_{11})(1-a_{22}) - a_{21}a_{12}} = \frac{1}{0.29} \\
Adj(J) &= \begin{pmatrix} j_{22} & -j_{12} \\ -j_{21} & j_{11} \end{pmatrix} \\
\mu &= \frac{1}{det(J)} \begin{pmatrix} (1-a_{22})a_{10} + a_{12}a_{20} \\ a_{21}a_{10} + (1-a_{11})a_{20} \end{pmatrix} \\
\mu &= \begin{pmatrix} \frac{4}{29} \\ \frac{71}{145} \end{pmatrix}
\end{aligned}$$

Función Impulso Respuesta, FIR

El desarrollo es inductivo, remplazando los valores razagados:

$$
\begin{aligned}
&X_t = A_0 + AX_{t-1} + e_t \\
&X_t = A_0 + e_t + A[A_0 + AX_{t-2} + e_{t-1}] \\
&X_t = A_0 + e_t + Ac + Ae_{t-1} + A^2 X_{t-2} \\
&X_t = AA_0 + e_t + Ae_{t-1} + A^2[A_0 + AX_{t-3} + e_{t-2}] \\
&X_t = A_0 + AA_0 + A^2A_0 + e_t + Ae_{t-1} + A^2e_{t-2} + A^3X_{t-3} \\
&X_t = A_0 + AA_0 + A^2A_0 + A^3A_0 + e_t + Ae_{t-1} + A^2e_{t-2} + A^3e_{t-3} + A^4e_{t-4} \\
&\quad ... \\
&X_t = A_0 + AA_0 + A^2A_0 + ... + e_t + Ae_{t-1} + A^2e_{t-2} + ... + A^nX_{t-n} \\
&X_t = [I - A + A^2 + ...]A_0 + [I + AL + (AL)^2 + (AL)^3 + ...]e_t + A^nX_{t-n} \\
&X_t = \sum_{i=0}^{s} A^iA_0 + \sum_{i=0}^{s} A^ie_{t-i} + A^{s+1}X_{t-(s+1)}
\end{aligned}
$$

Dado que hemos probado que el sistema es estable cuando $s \to \infty$ $A^{s+1}X_{t-(s+1)} \to 0$, por lo tanto:

$$X_t = [I - A]^{-1}A_0 + \sum_{i=0}^{\infty} A^ie_{t-i}$$

con

$$[I - A]^{-1}A_0 = E[X_t] = \mu$$

tenemos:

$$X_t = \mu + \sum_{i=0}^{\infty} A^ie_{t-i} \qquad (VMA(\infty))$$

El paso siguiente es pasar de usar errores de forma reducida a errores de forma estructural, mediante $e_t = B^{-1}\epsilon_t$:

$$e_t = B^{-1}\epsilon_t = \frac{1}{1-\beta_{12}\beta_{21}} \begin{pmatrix} 1 & -\beta_{12} \\ -\beta_{12} & 1 \end{pmatrix} \begin{pmatrix} \epsilon_t^y \\ \epsilon_t^z \end{pmatrix}$$

Entonces renombrando a $A^ie_{t-i} = \psi(i)$ tenemos:

$$\psi(i) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^i \frac{1}{1-\beta_{21}\beta_{12}} \begin{pmatrix} 1 & -\beta_{12} \\ -\beta_{12} & 1 \end{pmatrix} \begin{pmatrix} \epsilon_{t-i}^y \\ \epsilon_{t-i}^z \end{pmatrix}$$

$$X_t = \mu + \sum_{i=0}^{\infty} \psi(i)\epsilon_{t-i}$$

Los casos que nos interesan son $i = [1, 3]$ con $i \in Z$, es decir, $\psi(0)$, $\psi(1)$, $\psi(2)$. Para el caso $\psi(0)$ tenemos:

$$\psi(0) = \frac{1}{1-\beta_{21}\beta_{12}} \begin{pmatrix} 1 & -\beta_{12} \\ -\beta_{12} & 1 \end{pmatrix} \begin{pmatrix} \epsilon_{t-i}^y \\ \epsilon_{t-i}^z \end{pmatrix}$$

renombrando:

$$\alpha = det(B) = 1 - \beta_{21}\beta_{12}$$

tenemos:

$$\psi(0)_{11} = \frac{1}{\alpha} \quad \psi(0)_{12} = \frac{-\beta_{12}}{\alpha} \quad \psi(0)_{21} = \frac{-\beta_{21}}{\alpha} \quad \psi(0)_{22} = \frac{1}{\alpha}$$

Para el obtener $\psi 1$ calculamos:

$$\psi(1) = AB^{-1} = \frac{1}{det(B)} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & -\beta_{12} \\ -\beta_{12} & 1 \end{pmatrix}$$

$$\psi(1) = \frac{1}{det(B)} \begin{pmatrix} a_{11} - a_{12}\beta_{21} & -a_{11}\beta_{12} + a_{12} \\ a_{21} - a_{22}\beta_{21} & -a_{21}\beta_{12} + a_{22} \end{pmatrix}$$

renombrando:

$$\alpha = det(B) = 1 - \beta_{21}\beta_{12}$$

tenemos:

$$\psi(1)_{11} = \frac{a_{11} - a_{12}\beta_{21}}{\alpha} \quad \psi(1)_{12} = \frac{-a_{11}\beta_{12} + a_{12}}{\alpha} \quad \psi(1)_{21} = \frac{a_{21} - a_{22}\beta_{21}}{\alpha} \quad \psi(1)_{22} = \frac{-a_{21}\beta_{12} + a_{22}}{\alpha}$$

Finalmente $\psi(2)$ proviene de:

$$\psi(2) = A^2B^{-1} = AAB^{-1} = A\psi(1)$$

$$\psi(2) = A\frac{1}{det(B)} \begin{pmatrix} a_{11} - a_{12}\beta_{21} & -a_{11}\beta_{12} + a_{12} \\ a_{21} - a_{22}\beta_{21} & -a_{21}\beta_{12} + a_{22} \end{pmatrix}$$

operando:

$$\psi(2)_{11} = \frac{a_{11}^2 + a_{12}a_{21} - \beta_{21}(a_{11}a_{12} + a_{12}a_{22})}{\alpha} \quad \psi(2)_{12} = \frac{-\beta_{12}(a_{11}^2 + a_{12}a_{21}) + a_{11}a_{12} + a_{22}a_{12}}{\alpha}$$

$$\psi(2)_{21} = \frac{a_{21}a_{11} + a_{22}a_{21} - \beta_{21}(a_{21}a_{12} + a_{22}^2)}{\alpha} \quad \psi(2)_{22} = \frac{-\beta_{12}(a_{21}a_{11} + a_{22}a_{21}) + a_{21}a_{12} + a_{22}^2}{\alpha}$$

Descomposición varianza del error de predicción

El error de predicción a 1, $ep_{t+1|t}$ es:

$$ep_{t+1|t} = X_{t+1} - X_{t+1|t} = X_{t+1} - E(X_{t+1}|X_t)$$
$$ep_{t+1|t} = A_0 + A_1X_t + e_{t+1} - E(A_0 + A_1X_t + e_{t+1}|t)$$

Usando que $E(A) = A \quad E(X_t|t) = X_t$ y que $E(e_{t+1}) = 0$

$$ep_{t+1|t} = e_{t+1}$$

El error de predicción a 2 pasos es:

$$ep_{t+2|t} = X_{t+2} - X_{t+2|t} = X_{t+2} - E(X_{t+2}|X_t)$$
$$ep_{t+2|t} = A_0 + A_1X_{t+1} + e_{t+2} - E(A_0 + A_1X_{t+1} + e_{t+2}|t)$$
$$ep_{t+2|t} = A_1X_{t+1} + e_{t+2} - A_1E(X_{t+1}|t)$$

No conocemos X_{t+1}pero si X_t remplazamos

$$ep_{t+2|t} = A_1(A_0 + A_1X_t + e_{t+1}) - e_{t+2} - A_1E(A_0 + A_1X_t + e_{t+1})$$
$$ep_{t+2|t} = Ae_{t+1} + e_{t+2}$$

El error de predicción a 3 pasos es:

$$ep_{t+3|t} = X_{t+3} - X_{t+3|t} = X_{t+3} - E(X_{t+3}|X_t)$$
$$ep_{t+3|t} = A_0 + A_1X_{t+2} + e_{t+3} - E(A_0 + A_1X_{t+1} + e_{t+2}|t)$$
$$ep_{t+3|t} = A_1X_{t+2} + e_{t+3} - A_1E(X_{t+2}|t)$$
$$ep_{t+3|t} = e_{t+3} - A_1(X_{t+2} - E(X_{t+2}|t))$$
$$ep_{t+3|t} = e_{t+3} - A_1(ep_{t+2|t})$$
$$ep_{t+3|t} = e_{t+3} + A_1e_{t+2} + A_1^2e_{t+1}$$

El error de predicción a n pasos es:

$$ep_{t+s|t} = \sum_{i=0}^{s-1} A_1^i e_{t+s-i}$$

Reescribiendolo en su forma estructural y usando las FIR:

$$\begin{aligned}
ep_{t+s|t} &= \sum_{i=0}^{s-1} A_1^i e_{t+s-i} = \sum_{i=0}^{s-1} A_1^i B^{-1} \epsilon_{t+s-i} \\
&= \begin{pmatrix} ep^y_{t+s|t} \\ ep^z_{t+s|t} \end{pmatrix} = \sum_{i=0}^{s-1} \psi(i) \begin{pmatrix} \epsilon^y_{t+s-i} \\ \epsilon^z_{t+s-i} \end{pmatrix} \\
&= \sum_{i=0}^{s-1} \begin{pmatrix} \psi(i)_{11} & \psi(i)_{12} \\ \psi(i)_{21} & \psi(i)_{22} \end{pmatrix} \begin{pmatrix} \epsilon^y_{t+s-i} \\ \epsilon^z_{t+s-i} \end{pmatrix} \\
&= \sum_{i=0}^{s-1} \begin{pmatrix} \psi(i)_{11} \epsilon^y_{t+s-i} + \psi(i)_{12} \epsilon^z_{t+s-i} \\ \psi(i)_{21} \epsilon^y_{t+s-i} + \psi(i)_{22} \epsilon^z_{t+s-i} \end{pmatrix}
\end{aligned}$$

La varianza del error de predicción a t+s pasos es:

$$\begin{aligned}
\mathrm{Var}(ep_{t+s|t}) &= \mathrm{E}[(ep_{t+s|t} - \mathrm{E}(ep_{t+s|t}))(ep_{t+s|t} - \mathrm{E}(ep_{t+s|t}))'] \\
&= \mathrm{E}[(ep_{t+s|t})(ep_{t+s|t})'] \\
&= \mathrm{E} \sum_{i=0}^{s-1} \begin{pmatrix} \psi(i)_{11} \epsilon^y_{t+s-i} + \psi(i)_{12} \epsilon^z_{t+s-i} \\ \psi(i)_{21} \epsilon^y_{t+s-i} + \psi(i)_{22} \epsilon^z_{t+s-i} \end{pmatrix} \sum_{i=0}^{s-1} \begin{pmatrix} \psi(i)_{11} \epsilon^y_{t+s-i} + \psi(i)_{12} \epsilon^z_{t+s-i} & \psi(i)_{21} \epsilon^y_{t+s-i} + \psi(i)_{22} \epsilon^z_{t+s-i} \end{pmatrix} \\
&= \mathrm{E} \sum_{i=0}^{s-1} \sum_{j=0}^{s-1} \begin{pmatrix} (\psi(i)_{11} \epsilon^y_{t+s-i} + \psi(i)_{12} \epsilon^z_{t+s-i})(\psi(i)_{11} \epsilon^y_{t+s-j} + \psi(i)_{12} \epsilon^z_{t+s-j}) & (\psi(i)_{11} \epsilon^y_{t+s-i} + \psi(i)_{12} \epsilon^z_{t+s-i})(\psi(i)_{21} \epsilon^y_{t} \\ (\psi(i)_{21} \epsilon^y_{t+s-i} + \psi(i)_{22} \epsilon^z_{t+s-i})(\psi(i)_{11} \epsilon^y_{t+s-j} + \psi(i)_{12} \epsilon^z_{t+s-j}) & (\psi(i)_{21} \epsilon^y_{t+s-i} + \psi(i)_{22} \epsilon^z_{t+s-i})(\psi(i)_{21} \epsilon^y_{t} \end{pmatrix} \\
&\text{se cumple que } \epsilon^y_t \perp \epsilon^z_t \quad \forall t \text{ entonces:} \\
&= \mathrm{E} \sum_{i=0}^{s-1} \begin{pmatrix} \psi(i)^2_{11} \epsilon^{2_y}_{t+s-i} + \psi(i)^2_{12} \epsilon^{2_z}_{t+s-i} & \psi(i)_{11}(\psi(i)_{21} \epsilon^{2_y}_{t+s-i} + \psi(i)_{12}(\psi(i)_{22} \epsilon^{2_z}_{t+s-i} \\ \psi(i)_{11}(\psi(i)_{21} \epsilon^{2_y}_{t+s-i} + \psi(i)_{12}(\psi(i)_{22} \epsilon^{2_z}_{t+s-i} & \psi(i)^2_{21} \epsilon^{2_y}_{t+s-i} + \psi(i)^2_{22} \epsilon^{2_z}_{t+s-i} \end{pmatrix} \\
&\text{bajo varianza homógenea:} \quad \sigma^2_{y,t} = \sigma^2_{y,t+1} \cdot\cdot \\
\sigma^2_y(s) &= \sigma^2_y(\psi(0)^2_{11} + \psi(1)^2_{11} + ...\psi(s-1)^2_{11}) + \sigma^2_z(\psi(0)^2_{12} + \psi(1)^2_{12} + ...\psi(s-1)^2_{12}) \\
\sigma^2_z(s) &= \sigma^2_y(\psi(0)^2_{21} + \psi(1)^2_{21} + ...\psi(s-1)^2_{21}) + \sigma^2_z(\psi(0)^2_{22} + \psi(1)^2_{22} + ...\psi(s-1)^2_{22})
\end{aligned}$$

Por lo cual las correspondientes varianzas de los errores de predicción son:

$$\begin{aligned}
VD^{yy}(s) &= \frac{\sigma^2_y \sum_{i=0}^{s-1} \psi^2_{11}(i)}{\sigma^2_y(s)} * 100 \quad VD^{yz}(s) = \frac{\sigma^2_z \sum_{i=0}^{s-1} \psi^2_{12}(i)}{\sigma^2_y(s)} * 100 \\
VD^{zz}(s) &= \frac{\sigma^2_z \sum_{i=0}^{s-1} \psi^2_{22}(i)}{\sigma^2_z(s)} * 100 \quad VD^{zy}(s) = \frac{\sigma^2_y \sum_{i=0}^{s-1} \psi^2_{21}(i)}{\sigma^2_z(s)} * 100
\end{aligned}$$

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