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1 author:

Thomas B. Lynch
Oklahoma State University - Stillwater

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SIMULTANEOUS ESTIMATION OF SURFACE AREA, VOLUME AND FOREST PRODUCT YIELD FOR TREE STEMS WITH MONTE CARLO INTEGRATION

THOMAS B. LYNCH

Associate Professor, Department of Forestry, Room 008C Ag Hall, Oklahoma State University, Stillwater, OK 74078, USA. Email – tlynch@okstate.edu.[1]

ABSTRACT. Simultaneous estimation of tree stem surface area and volume with Monte Carlo integration is proposed. Using a common upper stem measurement, importance sampling is used to estimate volume while a control variate is used to estimate surface area. The surface area integral used for Monte Carlo integration contains the derivative of diameter with respect to height, which would be difficult to measure in the field. An analysis shows that this derivative should have a negligible effect on surface area estimates for many stem shapes. Simulation results indicated that a derivative calculated from a simple proxy taper function can be substituted for the measured derivative with little effect on the resulting tree stem estimate of surface area using Monte Carlo integration with control variates. Yield of forest products can be estimated using functional forms proposed by Grosenbaugh [8] for estimation of board-foot lumber yields from trees and logs. Cubic foot volume and surface area together with merchantable length are the independent variables used to predict product yields in Grosenbaugh's [8] equations. Thus, Monte Carlo integration for cubic volume and surface area could be used in Grosenbaugh's equation to obtain estimates of product yields for standing trees. Simulation results for surface area, volume, and board-foot estimation using Monte Carlo integration are presented.

KEYWORDS: importance sampling, Monte Carlo integration, board-foot, volume estimation, surface area estimation.

1 INTRODUCTION

If a tree stem is viewed as a solid of revolution, tree volume or surface area can be calculated by integral calculus techniques [10]. Monte Carlo integration can be used to find an unbiased estimate of the value of a definite integral by random sampling [16]. When tree stems are viewed as solids of rotation and cubic volume or surface area is computed using integral calculus, Monte Carlo integration can be used to estimate cubic volume or surface area by measurement of randomly selected upper-stem diameters or heights. If the tree stem volume is viewed as the sum of thin slices cut perpendicular to the central stem axis, upper-stem diameters are measured at randomly selected heights. When stem volume is the sum of thin cylinders centered on the central stem axis, heights to randomly selected upper-stem cross-sectional areas are measured. Since tree surface area can be expressed as an integral, it can also be estimated using Monte Carlo integration. Frequently, if it is desired to estimate tree surface area, an estimate of cubic volume will also be desired. Field work will be more efficient if the same upper-stem measurement of tree diameter can be used to simultaneously estimate surface area and volume. Grosenbaugh [8] suggested that board-foot volume estimates could be obtained from cubic volume,

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surface area, and length measurement. Thus, simultaneous estimation of cubic volume and surface area may lead to estimators for forest products yield. Lynch [12] proposed estimators for forest products yield which are reviewed and evaluated here.

Gregoire and others [4] suggested Monte Carlo integration with importance sampling for tree cubic foot volume estimation. Van Deusen and Lynch [21] used Monte Carlo integration with antithetic variates and importance sampling to estimate cubic foot stem volume. Van Deusen [20] suggested control variates for variance reduction in a cubic volume estimator for tree stem volume. Lynch and others [13] presented estimators using Monte Carlo integration with importance sampling and antithetic variates from the cylindrical shells integral. Recently, Robinson and others [15] introduced cut-off importance sampling for Monte Carlo estimation of bole volume, which can be used to eliminate selection of hard-to-measure sample heights in the crown.

Techniques for estimation of tree stem surface area by Monte Carlo integration have been proposed by Gregoire and others [6]. Lynch [11] discussed estimation of tree surface area in critical height sampling. Critical height sampling can be viewed as a type of Monte Carlo integration from the cylindrical shells volume integral for solids of rotation [11, 22].

Measurement of product yield from logs and trees has a long history, with lumber yield traditionally measured in board-feet in the United States [10]. Gregoire and others [4] have shown how cubic contents of trees can be estimated using Monte Carlo integration with variance reduction techniques including importance sampling. This technique provides unbiased estimates of cubic volume for tree stems when it can be assumed that stems are circular in cross-section. Unfortunately, there is no one constant that can be used to accurately convert cubic volume estimates obtained from importance sampling or other Monte Carlo integration methods to board-foot volume. Such conversion factors are affected by factors such as tree and log size [10]. Clark [2] originally suggested the concept of board-foot estimation based on log cubic volume, surface area, and length for his International $\frac{1}{8}^{\text{th}}$ inch log rule. Grosenbaugh [8] used the following function for estimation of board-foot volume for logs:

$$V_{BF} = b_1 V_C + b_2 S + b_3 L \tag{1}$$

where V_{BF} is board-foot volume, V_C is cubic volume, S is square surface area, L is length, and b_1, b_2, b_3 are constants. He presented constants that can be used to find board-foot volume according to the International ¼, Doyle, and Scribner log rules. The derivation of these constants is based on formulas presented by Grosenbaugh [7]. The function could also be used as a regression model in mill studies. Though the dependant variable in Eq. 1 has traditionally been measured in units of board-feet, alternate units of measure for lumber yield could be used. The model may be useful for predicting yields of other forest products such as plywood veneer.

Eq. 1 could be used to estimate board-foot volume in forest inventories by using Monte Carlo integration to estimate cubic volume and surface area for the sawlog-merchantable portion of sample trees. Gregoire and others [5] described a method of board-foot volume estimation for standing trees based on selection of logs with probability proportional to volume predicted by a taper function. The selected logs would be scaled for board-foot volume. The inverse probability of log selection is then used to develop the board-foot volume estimate. This is similar in concept to importance sampling, which expands stem measurements by the probability density of selection. Estimation of board-foot volume using Eq. 1 provides an alternative approach. Use of Eq. 1 might be advantageous if parameters to Eq. 1 are fitted to data from a mill study. Also, the system of board-foot estimation with Eq. 1 and Monte Carlo integration includes estimates of cubic foot volume and surface area, which may be of independent interest.

Estimation of board-foot volume using Eq. 1 requires measurements of L, which can be determined by measuring the distance between the stump and the sawlog-merchantable height for the tree of interest. Monte Carlo integration techniques can then be used to estimate V_C and S for the sawlog-merchantable portion of the tree. The variance reduction techniques that have been proposed for tree volume and surface area estimation with Monte Carlo integration require what Gregoire and others [4] have termed a "proxy taper function." This proxy taper function approximates stem shape well enough for efficient random selection of stem dimensions. Often, a paraboloid or equivalent has been used. For the purpose of developing proxy taper and volume functions in this paper, consider rotation of the following curve about an x-axis representing the central stem axis:

$$y = \frac{Rx^{r/2}}{H^{r/2}} \tag{2}$$

where y is the stem radius, x is the distance from the stem tip to radius y, R is the radius at the stump, H is the distance from the stem tip to radius R, and r is a parameter determining the degree of curvature, r=1 is a standard paraboloid, r=2 is a cone, r=3 is a neiloid.

2 CUBIC FOOT VOLUME ESTIMATION

Gregoire and others [4] suggested importance sampling for variance reduction in tree stem volume estimation with Monte Carlo integration. Importance sampling locates randomly selected upper-stem diameters with probability density proportional to volume predicted by the proxy taper function. Therefore, in addition to providing variance reduction, importance sampling tends to locate upper-stem diameter measurements lower on the tree stem. Many of these measurements may be accessible with specially designed tree calipers [4]. Therefore, though there are several options available for variance reduction in cubic volume estimation, an importance sampling estimator will be presented here. It is desired to estimate cubic volume as expressed by the integral

$$V_C = k_1 \int_{H_s}^{H_U} d(h)^2 \, dh \tag{3}$$

where

$d(h)$ is stem diameter at upper-stem height h,
H_s is stump height,
H_U is sawlog merchantable height, and
$k_1=\pi/576$ for English units (dbh in inches, height in feet) or $k_1=\pi/40{,}000$ for metric units (dbh in centimeters, height in meters) to adjust for measurement of dbh and height in different units.

The importance sampling estimator is:

$$\hat{V}_C = \frac{1}{m}\sum_{i=1}^{m} \frac{k_1 \times d(h_i)^2}{f(h_i)} = \frac{1}{m}\sum_{i=1}^{m} \frac{d(h_i)^2}{d_p(h_i)^2} V_p \tag{4}$$

where

m is the number of upper-stem sample diameters,
$d(h_i)$ is measured diameter at randomly located height h_i,
$d_p(h_i)$ is diameter predicted by proxy taper function at randomly located height h_i,
$f(h_i)=\dfrac{k_1 \times d_p(h_i)^2}{V_p}$ probability density for randomly selected height h_i, and
V_p is cubic volume predicted by the proxy taper function for the stem section between H_s and H_U.

To understand estimator 4 from an intuitive point of view, note that when the proxy taper function mimics stem shape perfectly, the ratio between predicted and measured diameters is unity in Eq. 4, and the estimator reduces to V_p. In practice, the variance arising from the ratio of measured-to-predicted diameters should be relatively modest if the proxy taper function approximates stem shape reasonably well.

In a given situation there may be several possible choices for the proxy taper function. Gregoire and others [4] suggested a standard paraboloid constrained to pass through the tip of the tree stem at total height. A paraboloid with the latter constraint probably will not pass through the point determined by a measured sawlog merchantable height and diameter. Since it would be necessary to measure merchantable sawlog height to use Eq. 1, a proxy taper function constrained by merchantable sawlog height and one measured lower-stem diameter will be presented. This measured lower stem diameter would typically be either dbh or stump diameter. The function is:

$$d_p(h) = D_l\left[1-(h-H_l)\left(\frac{1-q_l^{2/r}}{H_U-H_l}\right)\right]^{r/2} \tag{5}$$

where

$d_p(h)$ = predicted diameter at height h,
D_l is measured lower-stem diameter (usually dbh or stump diameter),
d_U is diameter at sawlog merchantability limit height H_U,
$q_l = d_U/D_l$, and
H_l is stem height at which D_l is measured (usually stump height or breast height).

This proxy taper function is constrained to pass through the upper-stem diameter measured at the sawlog merchantable height. Thus, it should approximate tree shape for the sawtimber portion of the tree somewhat better than a proxy taper function constrained to pass through the tip of the tree at total tree height. Eq. (5) is a parabolid frustum when r=1, a cone frustum when r=2, Forslund's [3] paracone when r=4/3, and a neiloid frustum when r=3.

Bruce [1] noted that solids formed by rotating a power function about an axis are called Fermat paraboloids. For stem profiles described by Eq. 5, the corresponding volume can be found by using Smalian's [18] formula for the volume of the frustum of a Fermat paraboloid (see Bruce [1]):

$$V_p = \frac{k_1 D_s^2 L}{r+1}\left[\frac{1-q_s^{2/r+2}}{1-q_s^{2/r}}\right] \tag{6}$$

where

$D_s = d_p(H_s)$ the stump diameter predicted by Eq. 5, if $D_l \neq D_s$,
H_s is stump height,
$L = H_U - H_s$,sawlog merchantable length, and
$q_s = d_U/D_s$.

Note that if in development of Eq. 5, $D_l \neq D_s$, then the value of D_s used in Eq. 6 should be predicted using Eq. 5. Otherwise there will be an inconsistency in the probability density such that the integral of the density over its range will probably not equal unity. This is due to the fact that when a diameter other than stump diameter (such as dbh) is used for D_l in Eq. 5, it is unlikely that measured stump diameter will be equal to the stump diameter predicted by Eq. 5.

Randomly located heights can be obtained by the inverse transform method based on the probability density (see Appendix Eq. $A1$)

$$f(h) = \frac{k_1 d_p(h)^2}{V_p}. \tag{7}$$

3 SURFACE AREA

Gregoire and others [6] have developed Monte Carlo integration approaches to the estimation of tree stem surface area and bark volume. They present estimators using a variety of variance reduction techniques including importance sampling, antithetic variates, and control variates. It should be noted that one would probably not want to use importance sampling to estimate both cubic foot volume (V_C) and surface area (S) if the objective is to use Eq. 1 to estimate board-foot volume. The appropriate probability density for volume estimation will be based on accumulation of volume with changing upper-stem heights. The appropriate probability density for surface area estimation will be based on the accumulation of square surface with changing upper-stem height, and will be quite different than the density function for volume estimation. Thus, the distribution of sample heights for volume estimation with importance sampling will be quite different than the distribution of sample heights for surface area estimation with importance sampling.

Surface area integrals for solids of rotation are presented in standard mathematics texts such as Thomas [19]. Husch and others [10] indicate how the surface area integral for solids of rotation can be used to calculate tree stem surface area It is desired to use Monte Carlo integration to estimate the value of the following definite integral for surface area of a tree stem:

$$S = 2\pi k_2 \int_{H_s}^{H_U} d(h)\sqrt{1+\left(\frac{k_2 \partial d(h)}{\partial h}\right)^2}\, dh \tag{8}$$

where

k_2 = 1/24 with measurements in English units or k_2 =1/200 with measurements in metric units.

It is desirable to use the diameter measurement that has already been obtained through importance sampling for cubic volume estimation. However, the density used for volume estimation is probably not the best for variance reduction in the surface area estimate. Therefore it is desirable to use an additional variance reduction technique for surface area estimation. Gregoire and others [6] discuss use of control variates for surface area estimation. Use of a control variate is proposed here because it can be used with the diameter measurement already selected by importance sampling for cubic volume estimation. The resulting surface area estimator is:

$$\hat{S} = \frac{1}{m}\sum_{i=1}^{m} \frac{2\pi k_2 \times \left[d(h_i)\sqrt{1+\left(\frac{k_2 \partial d(h_i)}{\partial h_i}\right)^2} - d_p(h_i)\sqrt{1+\left(\frac{k_2 \partial d_p(h_i)}{\partial h_i}\right)^2}\right]}{f(h_i)} + S_p \tag{9}$$

where

$\frac{k_2 \partial d(h_i)}{\partial h_i}$ = estimate of rate of change of diameter with respect to change in height, based on measurements made at and near h_i,

$d_p(h_i)$ is the diameter at h_i predicted by the proxy taper function (Eq. 5),

$$\frac{\partial d_p(h)}{\partial h} = -\frac{r}{2}\left(\frac{1-q^{2/r}}{H_U - H_l}\right) D_l \left[1-(h-H_l)\left(\frac{1-q^{2/r}}{H_U - H_l}\right)\right]^{r/2-1}$$

S_p is the square surface area predicted by the proxy taper function for the portion of the tree stem between the stump and the sawlog-merchantable height.

$f(h_i)$ is the probability density used for importance sampling in cubic volume estimation, defined above for Eq. 4.

To gain an intuitive understanding of Eq. 9, note that if the proxy taper function were a perfect representation of stem shape, then $d(h_i)= d_p(h_i)$ and the estimator would reduce to the predicted square surface area S_p. Although the diameters predicted by the proxy taper function are not perfect, they are close enough to actual measurements so that the variance of the difference between measured and predicted diameters is much less than the variance among the raw stem diameter measurements. Thus, variance in surface error estimation using Monte Carlo integration is reduced. The well-known fact that Monte Carlo integration with control variates is unbiased [16] as easily confirmed by finding the expected value of Eq. 9 with respect to the sampling density for stem heights, $f(h)$.

A possible difficulty with use of the Monte Carlo surface area estimator is the estimation of the rate of change in diameter with respect to change in height at the randomly selected sample heights. The best way to estimate this quantity is probably to make another upper stem measurement in the vicinity of the randomly selected height so that a taper rate can be used to approximate the rate of change in dbh with respect to height at that point. Unfortunately, this may be time consuming. In order to investigate the potential influence of the diameter change rate on surface area estimation, consider a stem shaped as a cone frustum with Clark's [2] taper rate of 1 inch per 8 feet of log length. The change rate will be:

$$\frac{k_2 \partial d(h_i)}{\partial h_i} = \frac{1}{24 \times 8} = \frac{1}{192}$$

Now consider the quantity:

$$\sqrt{1 + \left(\frac{k_2 \partial d(h_i)}{\partial h_i}\right)^2} \approx 1$$

This indicates that because of their long, thin shape, for trees shaped as cones the term:

$$d(h_i)\sqrt{1 + \left(\frac{k_2 \partial d(h_i)}{\partial h_i}\right)^2} = d(h_i)\sqrt{1 + \left(\frac{1}{192}\right)^2} = d(h_i) \times 1.0000136$$

which would greatly simplify Monte Carlo estimation of surface area, because estimation of the diameter change rate would not be necessary. It is likely that the change rate would be small for shapes other than cone frustums. Another possibility in the control variate Eq. 9 would be to use the proxy taper estimate also for $\partial d(h_i)/\partial h_i$, which would otherwise be measured. The difference between measured and proxy taper estimates for this quantity is not likely to greatly influence surface area estimates for trees when this estimator is used. If the latter suggestion were used in importance sampling, the term containing the derivative would "cancel out" because importance sampling for surface area would lead to a ratio of these quantities. However, there is no such cancellation in the control variate estimator of surface area.

If Monte Carlo estimation for surface area is done using critical height sampling [11] or integration with respect to tree radius (rather than with respect to height) the corresponding derivative will be $\partial h_i / \partial d(h_i)$. Since this quantity will be quite large, it could not be ignored in the estimation process.

The total surface area of the sawtimber-merchantable portion of the tree is required according the proxy taper function. Applying the basic form of the surface area integral the following is obtained:

$$S_p = 2\pi k_2 \int_{H_s}^{H_U} d_p(h)\sqrt{1 + \left(\frac{k_2 \partial d_p(h)}{\partial h}\right)^2}\, dh \tag{10}$$

With appropriate choice of the shape parameter r (which must be consistent with Eq. 5) this could be used in the control variate estimator for square surface area 9. Based on a formula reported in Husch and others [10] for the frustum of a cone, for r =2:

$$S_p = \pi k_2 (D_s + D_U)\sqrt{k_2^2 (D_s - D_U)^2 + L^2} \tag{11}$$

For a paraboloid with r =1 a formula reported by Husch and others [10] can be used to obtain:

$$S_p = \frac{2\pi k_2 D_s}{12}\left(\frac{1-q^2}{L}\right)^2 \left[\left[(k_2 D_s)^2 + 4\left(\frac{L}{1-q^2}\right)^2\right]^{3/2} - (k_2 D_s)^3\right]$$

$$- \frac{2\pi k_2 D_U}{12}\left(\frac{1-q^2}{Lq^2}\right)^2 \left[\left[(k_2 D_U)^2 + 4\left(\frac{Lq^2}{1-q^2}\right)^2\right]^{3/2} - (k_2 D_U)^3\right] \qquad (12)$$

4 ADJUSTMENTS FOR BARK THICKNESS

The coefficients presented by Grosenbaugh for calculation of board-foot volumes with Eq. 1 were based on inside-bark cubic volume and surface area. This would imply that inside bark diameter measurements be used for Monte Carlo integration estimates of cubic volume (Eq. 4) and square surface area (Eq. 9). Gregoire and others [6] suggested the use of Meyer's [14] bark ratio for bark volume estimation with Monte Carlo integration. Meyer [14] postulated that for a given site and species, the ratio between diameter inside bark and diameter outside bark would often be roughly constant at various points on the tree stem:

$$\rho = \frac{\text{diameter inside bark}}{\text{diameter outside bark}} \qquad (13)$$

Using this assumption one could obtain inside bark diameters:

$$d_{ib}(h_i) = \rho d_{ob}(h_i) \qquad (14)$$

where

$d_{ib}(h_i)$ = diameter inside bark at randomly selected height h_i, and

$d_{ob}(h_i)$ = diameter outside bark at randomly selected height h_i.

More accurate estimation of diameter inside bark at various stem heights may be obtained in some situations by using one of the other options proposed by Grosenbaugh [9] (also see discussion by Schreuder and others [17]).

5 VARIANCE FOR BOARD-FOOT ESTIMATION

The variance for board-foot estimation with Eq. 1 can be related to the variance of cubic volume and square surface estimates from Monte Carlo integration by a well-known expression for the variance of the sum of two random variables:

$$\text{var}(\hat{V}_{BF}) = b_1^2 \text{ var}(\hat{V}_C) + b_2^2 \text{ var}(\hat{S}) + 2b_1 b_2 \text{ cov}(\hat{V}_C, \hat{S}) \qquad (15)$$

The covariance between the Monte Carlo integration estimates of cubic volume and square surface area is expected to be positive, since the estimators Eq. 4 and Eq. 9 are designed to use common stem diameter measurements. Thus it would be expected that "larger than average" cubic volume estimates should be associated with "larger than average" surface area measurements. Inspection of variance expression Eq. 15 indicates that the signs and magnitudes of the coefficients b_1, b_2, and b_3 influence the variance of the board-foot volume estimator. Grosenbaugh [8] obtained b_1>0, b_2<0 and b_1b_2<0 for International ¼, Doyle, and Scribner board-foot volume. This is consistent with Judson Clark's [2] proposal that certain losses in lumber production are proportional to a ring-shaped region outside the log. Thus, for these log rules, the covariance term in Eq. 15 would have a negative coefficient, so that the term would tend to reduce the total variance in the board-foot volume estimator. The formulas presented by Grosenbaugh [8] for the International ¼ (V_{INT}), Scribner (V_{SCR}) and Doyle (V_{DOY}) rules with 5-meter logs (16-foot, English units, the International ¼ coefficients would not change with log length) are:

$$V_{INT} = 9.1236 V_C/0.3048^3 \;\; - 0.70846 S/0.3048^2 + 0.04222 L/0.3048 \qquad (16)$$

$$V_{SCR} = 9.057 V_C/0.3048^3 \;\; - 0.852 S/0.3048^2 + 0.112 L/0.3048 \qquad (17)$$

$$V_{DOY} = 11.459 V_C/0.3048^3 \;\; - 2.387 S/0.3048^2 + 1.542 L/0.3048 \qquad (18)$$

where V_C is cubic meter volume, S is square meter of surface area, and L is sawlog-merchantable length in meters. Since Grosenbaugh originally presented these formulas for English units of measure, conversion factors 0.3048 (feet-to-meters), 0.3048^2 (square feet-to-square meters) and 0.3048^3 (cubic feet-to-cubic meters) have been indicated above. Applying Eq. 15 the variance function for International ¼ board-foot estimation would be:

$$\mathrm{var}(\hat{V}_{BF}) = 103{,}810.742\,\mathrm{var}(\hat{V}_C) + 5.4026\,\mathrm{var}(\hat{S}) - 4{,}913.8618\,\mathrm{cov}(\hat{V}_C, \hat{S}) \qquad (19)$$

Although conversion factors for converting cubic foot volume to board-foot volume are problematic, approximate conversion factors may be between 180 and 210 board-feet per cubic meter (corresponding roughly to English units values of from 5 to 6 board-feet per cubic feet). Taking 200 board-feet per cubic meter as an example, one could consider Eq. 19 in the light of:

$$\begin{aligned} \hat{V}_{BF} &= 200\hat{V}_C \\ \mathrm{var}(\hat{V}_{BF}) &= 40{,}000\,\mathrm{var}(\hat{V}_C) \end{aligned} \qquad (20)$$

From this analysis it appears that an estimate of board-foot volume using Eq. 1 with Monte Carlo estimates of cubic volume and surface area may not have a smaller variance than multiplication of a conversion factor by the cubic foot volume estimate from Monte Carlo integration. However, a board-foot estimator using Eq. 1 should be more nearly unbiased for true board-foot volume (or board-foot volume according to a specified log rules), due to uncertainty in specification of a conversion factor, and the fact that no single conversion factor will suffice for all tree or log sizes.

6 Simulation Results

Computer simulation was used to evaluate the simultaneous Monte Carlo integration estimators of surface area, volume and forest products yield proposed above. Four sets of stem section dimensions were selected for simulated Monte Carlo estimation. Merchantable lengths ranged from 10 to 25 meters, lower-stem diameters (at the bottom of merchantable length) ranged from 33 to 48 centimeters, while upper-stem diameters were 18 centimeters. Three models for actual stem shape were used, the frustum of a cone (r=2 in Eq. 2) the frustum of a neiloid (r=1) and the frustum of Forslund's paracone (r=4/3). Of these, Forslund's paracone is probably the best representation of actual tree stem sections since it was developed by Forslund [3] as a simple model representative of many tree stem shapes, which often tend to lie somewhere between a paraboliod (r=1) and a cone (r=2), as does the Forslund paracone (r=4/3). The frustum of a cone is essentially the shape assumed by Clark's [2] taper rate for sawlogs, and may be reasonably accurate for some stem sections. The lower tree bole is often compared to a neiloid (r=3) [10], though it is not usually accurate for the upper bole. However, it was of interest to consider a shape more distant from the proxy taper function than a cone. A standard paraboloid (r=1) was used as the proxy taper function for both the importance sampling cubic volume estimator (Eq. 4) and the control variate surface area estimator (Eq. 9).

Due to the difficulties involved in estimation of the derivative of diameter with respect to height in surface area estimator Eq. 9, the following alternate surface area estimator was also tested using computer simulation:

$$\hat{S}_{ALT} = \frac{1}{m}\sum_{i=1}^{m} \frac{2\pi k_2 \times \left[d(h_i)\sqrt{1+\left(\frac{k_2 \partial d_p(h_i)}{\partial h_i}\right)^2} - d_p(h_i)\sqrt{1+\left(\frac{k_2 \partial d_p(h_i)}{\partial h_i}\right)^2}\right]}{f(h_i)} + S_p \qquad (21)$$

Careful inspection of Eq. 21 indicates that the proxy derivative $\partial d_p(h)/\partial h$ has been substituted for the actual derivative $\partial d(h)/\partial h$ in Eq. 9. The analysis above showed that the derivative was not likely to have a significant influence on the surface area estimate. The proxy derivative (which does not need to be measured in the field) should frequently serve as an adequate substitute for field measurements to estimate the derivative of diameter with respect to height.

Computer simulation was used to obtain 100,000 random stem samples from each stem section with each of the three taper equations. The inverse transform method was used (Appendix Eq. *A*1) with a distribution function developed from the parabolic proxy taper function to randomly locate stem samples. At each randomly located point on the stem, the actual stem function (either Forslund's paracone, a cone, or a neiloid) was used to obtain an actual diameter measurement. For each of the 100,000 samples, the actual measurement, together with information from the parabolic proxy taper function was used to estimate cubic volume using importance sampling estimator Eq. 4, the surface area estimator Eq. 9, and the alternate surface area estimator Eq. 21. Eqs. 16-18 were used with the cubic volume and surface area estimates for each stem sample to estimate International, Scribner and Doyle board-foot volumes. The alternate surface area estimate from Eq. 21 was also tested in Eqs. 16-18 for board-foot volume estimation.

Simulation results for four merchantable stem sections are given in Table 1 for stems shaped as frusta of Forslund's paracone, in Table 2 for stems shaped as frusta of cones, and in Table 3 for stems shaped as frusta of neiloids (r=3). The average values of 100,000 Monte Carlo volume and surface area estimates are extremely close to the actual volumes and surface areas given in Tables 1-3 for each of the four stem sections. This is expected since the Monte Carlo estimate with importance sampling or control variates has been shown to be mathematically unbiased [4, 6, 16]. Differences between actual and average estimated volume and surface area are due to sampling error. Actual stem section volumes were computed using Eq. 6 and actual surfaces areas by Eq. 11 for cone frusta or by Eq. 8 using Simpson's rule for neiloid and Forslund paracone frusta. Differences between actual and average estimated board-foot volumes were similar in magnitude though not reported in the table. The alternate surface area estimator (Eq. 21) is not mathematically unbiased because the proxy derivative is substituted for the actual derivative. However, inspection of Tables 1-3 indicates that the magnitude of differences between the average alternative surface area estimator and the actual stem segment surface area were extremely small and similar in magnitude to average differences between the conventional control variate estimator (Eq. 9) and actual surface area. The same pattern of extremely small differences were observed when the alternate surface area estimator was used in Eqs. 16-18 to predict board-foot volumes. The ratio of bias to standard error based on 100,000 Monte Carlo estimates was computed for the cubic volume estimator, surface area estimator, alternate surface area estimator, and all board-foot volume estimators. For the stem sections shaped as frusta of paracones, cones or neiloids this ratio was less than 1.8 in all cases, failing to demonstrate significant bias.

Standard deviations and coefficients of variation were also reported for surface area and volume estimators in Tables 1-3. Standard deviations and coefficients of variation tended to increase for all estimators under all three shape assumptions with increasing stem length. With the minimum merchantable diameter fixed at 18cm, and the increase in length being accompanied by an increase in lower diameter, longer lengths are associated with a greater difference in end diameters. This essentially makes the population being sampled more diverse in that there is a greater range of diameters to sample. For Forsuld's paracone frusta, probably the most realistic tree shape assumption considered, Table 1 indicates that coefficients of variation for cubic volume range from 1.4 to 3.6 percent while coefficients of variation for surface area estimation range from 0.8 to 2.4 percent.

Table 1: Results from 100,000 samples for estimation of volume, surface area, and board-foot content using Monte Carlo integration for actual stems shaped as Forslund's paracones with paraboloid frusta as proxy taper function.

Merchantable length (m)	10	15	20	25
Lower diameter (cm)	33	38	43	48
Upper diameter (cm)	18	18	18	18
Actual volume (m^3)	0.5403	1.0026	1.6261	2.4355
Avg. Monte Carlo volume estimate(m^3)	0.5402	1.0026	1.6263	2.4352
Standard deviation volume estimate (m^3)	0.0077	0.0216	0.0472	0.0883
Coefficient of variation vol. est. %	1.4225	2.1541	2.9005	3.6258
Actual surface area (m^2)	8.1280	13.4798	19.7117	26.8341
Avg. Monte Carlo surface area estimate (m^2)	8.1278	13.4799	19.7133	26.8322
Standard deviation surface area estimate (m^2)	0.0674	0.1789	0.3695	0.6564
Coefficient of variation surf. est. %	0.8293	1.3275	1.8742	2.4462
Covariance, surface and volume estimates	0.0005	0.0038	0.0168	0.0558
Correlation, surface and volume estimates	0.9775	0.9711	0.9661	0.9623
Avg. alternate[a] Monte Carlo surf. est. (m^2)	8.1278	13.4799	19.7133	26.8322
Standard deviation alternate[a] surf. est. (m^2)	0.0674	0.1789	0.3694	0.6563
Coefficient of var. alternate[a] surface est. %	0.8291	1.3273	1.8740	2.4459
Covariance, alt. [a] surface and volume estimate	0.0005	0.0038	0.0168	0.0558
Correlation, alt. [a] surface and volume estimate	0.9776	0.9712	0.9661	0.9623
Coef. of variation, est. international bd.-ft.%	1.7419	2.5381	3.3200	4.0570
Coef. of variation, est. Scribner bd.-ft. %	1.9647	2.7817	3.5678	4.2985
Coef. of variation, est. Doyle bd.-ft. %	2.4234	3.2624	4.0412	4.7419
Coef. var. alt. [a] est. international bd.-ft. %	1.7419	2.5381	3.3201	4.0571
Coef. var. alt. [a] est. Scribner bd.-ft. %	1.9648	2.7818	3.5679	4.2986
Coef. var. alt. [a] est. Doyle bd.-ft. %	2.4234	3.2625	4.0413	4.7420

[a]The alternate surface area estimator (Eq. 21) substitutes the proxy derivative for a measured estimate

Coefficients of variation for the alternate surface area estimator (Eq. 21) were very similar to those for the standard control variate estimator (Eq. 9). Coefficients of variation for board-foot estimation ranged from 1.7 to 4.7 percent for paracone frusta (Table 1). While these are higher than those reported for cubic volume estimation, they are small enough to be useable. Due to cancellation of units, the coefficient of variation for board-foot estimation with a simple conversion factor would be the same as that for cubic foot volume estimation. However, there is no one satisfactory board-foot to cubic conversion factor. Indeed, the lack of such a factor lead Grosenbaugh to develop Eqs. 16-18. Thus, use of Eqs. 16-18 with Monte Carlo estimation of cubic volume and surface area appears to be a usable way to estimate board-foot volume. Tables 2 and 3 indicate that standard deviations and coefficients of variation tend to be higher for stem sections shaped as cone or neiloid frusta than for stem sections with the same dimensions shaped as paracone frusta. Standard deviations and coefficients of variation increase progressively as one changes stem shape from paracone (r=4/3) to cone(r=2) and finally neiliod (r=3). This might be expected since cones and neiloids are more dissimilar to the proxy taper function (a paraboloid r=1). The largest coefficients of variation were 13.4 percent for Scribner and Doyle board-foot estimation with 25 meter stem sections shaped as neiloids. Tables 1-3 indicate that there is a high positive correlation between the surface area and cubic volume estimates, with correlation coefficients above 0.95 for all stem lengths and all three shape assumptions.

Table 2: Results from 100,000 samples for estimation of volume, surface area, and board-foot content using Monte Carlo integration for actual stems shaped as cones with paraboloid frusta as proxy taper functions.

Merchantable length (m)	10	15	20	25
Lower diameter (cm)	33	38	43	48
Upper diameter (cm)	18	18	18	18
Actual volume (m^3)	0.5254	0.9629	1.5430	2.2855
Avg. Monte Carlo volume estimate(m^3)	0.5254	0.9629	1.5435	2.2848
Standard deviation volume estimate (m^3)	0.0149	0.0415	0.0900	0.1672
Coefficient of variation vol. est. %	2.8386	4.3098	5.8287	7.3180
Actual surface area (m^2)	8.0113	13.1950	19.1641	25.9186
Avg. Monte Carlo surface area estimate (m^2)	8.0109	13.1951	19.1675	25.9145
Standard deviation surface area estimate (m^2)	0.1309	0.3449	0.7088	1.2547
Coefficient of variation surf. est. %	1.6340	2.6142	3.6981	4.8418
Covariance, surface and volume estimates	0.0019	0.0139	0.0614	0.2009
Correlation, surface and volume estimates	0.9757	0.9682	0.9623	0.9576
Avg. alternate[a] Monte Carlo surf. est. (m^2)	8.0109	13.1951	19.1674	25.9145
Standard deviation alternate[a] surf. est. (m^2)	0.1309	0.3449	0.7087	1.2546
Coefficient of var. alternate[a] surface est. %	1.6336	2.6137	3.6976	4.8412
Covariance, alt.[a] surface and volume estimate	0.0019	0.0139	0.0614	0.2009
Correlation, alt.[a] surface and volume estimate	0.9758	0.9683	0.9623	0.9576
Coef. of variation, est. international bd.-ft.%	3.5021	5.1224	6.7334	8.2646
Coef. of variation, est. Scribner bd.-ft. %	3.9696	5.6437	7.2738	8.8004
Coef. of variation, est. Doyle bd.-ft. %	4.9687	6.7231	8.3663	9.8498
Coef. var. alt.[a] est. international bd.-ft. %	3.5023	5.1225	6.7335	8.2647
Coef. var. alt.[a] est. Scribner bd.-ft. %	3.9698	5.6439	7.2739	8.8006
Coef. var. alt.[a] est. Doyle bd.-ft. %	4.9689	6.7232	8.3665	9.8500

[a]The alternate surface area estimator (Eq. 21) substitutes the proxy derivative for a measured estimate

This is expected since large surface area estimates tend to be associated with large cubic volume estimates. This is significant in the light of Eqs. 15 and 19 because the covariance term in the board-foot variance expression has a negative sign due to the signs of Grosenbaugh's coefficients. Therefore, a high correlation between volume and surface area estimates helps to reduce the variation in board-foot estimation. This, in addition to faster field work, is another reason to estimate volume and surface area with common upper-stem diameter measurements. If the theory espoused by Grosenbaugh [8] and Clark [2] is correct, the sign of the surface area term in Eq. 1 may be negative for equations fitted to mill yields by regression techniques for lumber or other forest products produced by a mechanical conversion process. This would make positive correlations between surface area and volume estimation advantageous.

Table 3: Results from 100,000 samples for estimation of volume, surface area, and board-foot content using Monte Carlo integration for actual stems shaped as neiloids with paraboloid frusta as proxy taper functions.

Merchantable length (m)	10	15	20	25
Lower diameter (cm)	33	38	43	48
Upper diameter (cm)	18	18	18	18
Actual volume (m^3)	0.5155	0.9362	1.4870	2.1840
Avg. Monte Carlo volume estimate(m^3)	0.5154	0.9362	1.4876	2.1831
Standard deviation volume estimate (m^3)	0.0194	0.0537	0.1156	0.2134
Coefficient of variation vol. est. %	3.7722	5.7321	7.7684	9.7740
Actual surface area (m^2)	7.9326	13.0020	18.7909	25.2912
Avg. Monte Carlo surface area estimate (m^2)	7.9321	13.0022	18.7954	25.2856
Standard deviation surface area estimate (m^2)	0.1707	0.4469	0.9139	1.6112
Coefficient of variation surf. est. %	2.1518	3.4368	4.8626	6.3720
Covariance, surface and volume estimates	0.0032	0.0232	0.1013	0.3280
Correlation, surface and volume estimates	0.9744	0.9661	0.9594	0.9539
Avg. alternate[a] Monte Carlo surf. est. (m^2)	7.9321	13.0021	18.7954	25.2856
Standard deviation alternate[a] surf. est. (m^2)	0.1706	0.4468	0.9138	1.6110
Coefficient of var. alternate[a] surface est. %	2.1512	3.4361	4.8618	6.3712
Covariance, alt.[a] surface and volume estimate	0.0032	0.0232	0.1013	0.3279
Correlation, alt.[a] surface and volume estimate	0.9745	0.9662	0.9595	0.9540
Coef. of variation, est. international bd.-ft.%	4.6789	6.8563	9.0365	11.1175
Coef. of variation, est. Scribner bd.-ft. %	5.3221	7.5834	9.8003	11.8844
Coef. of variation, est. Doyle bd.-ft. %	6.7333	9.1405	11.407	13.4563
Coef. var. alt.[a] est. international bd.-ft. %	4.6791	6.8565	9.0367	11.1177
Coef. var. alt.[a] est. Scribner bd.-ft. %	5.3223	7.5836	9.8006	11.8846
Coef. var. alt.[a] est. Doyle bd.-ft. %	6.7336	9.1409	11.4074	13.4566

[a]The alternate surface area estimator (Eq. 21) substitutes the proxy derivative for a measured estimate

7 Conclusions

The same randomly located upper-stem diameter measurements can be used to simultaneously obtain unbiased Monte Carlo integration estimates of cubic volume with importance sampling and surface area with control variate estimation. These Monte Carlo integration estimators can be used to estimate board-foot volumes by making use of a linear relationship between board-foot volume, cubic foot volume, square surface area, and sawlog-merchantable length. Results from computer simulation indicate that these estimators are sufficiently precise for practical use. Though the estimating equations may appear to be somewhat involved they can easily be programmed into field computers commonly used for field data collection. Since importance sampling based on the cubic foot volume estimator is proposed for selecting diameter measurement points on the stem, many of the measurement points should be on the lower bole. Since common stem diameter measurements would be used for volume and surface area estimation, the procedure would require little more effort in the field than previously proposed estimators for cubic volume. Eq. 1 could be applied with regression estimates of the coefficients for lumber yields from improved technology or to other forest products. Inventory data bases which contain volume, surface area, and merchantable length could then be used to assess the value of product yield with coefficients in Eq. 1 changing as technology improves product yields.

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8 APPENDIX: THE INVERSE TRANSFORM

Using importance sampling, randomly selected upper-stem heights can be chosen by the inverse transform method. This method equates the variable U randomly selected from a uniform distribution on the interval (1,0) to the importance sampling distribution function:

$$U = F(h_m) = \int_{H_l}^{h_m} f(h)dh$$

The randomly chosen height is found by solving for the upper-stem height h_m:

$$h_m = F^{-1}(U)$$

For this application the distribution function can be found by dividing the volume up to h_m by the total volume from H_l to H_U for the proxy taper function (using taper Eq. 5):

$$F(h_m) = \frac{\int_{H_l}^{h_m} d_p(h)^2\, dh}{V_p} = \frac{k_1 \int_{H_l}^{h_m} D_l^2 \{1 - c(h - H_l)\}^r}{V_p} = \frac{k_1 D_l^2}{V_p c(r+1)} \left\{ - [1 - c(h_m - H_l)]^{(r+1)} \right\}$$

where

$$c = \frac{1 - q^{2/r}}{(H_U - H_l)}$$

By equating the distribution function above to the uniform random variable U and solving for the upper-stem height h_m the following is obtained:

$$h_m = H_l + \frac{1}{c} - \frac{1}{c}\left[1 - \frac{U \times V_p \times c(r+1)}{k_1 D_l^2}\right]^{\frac{1}{r+1}} \qquad (A1)$$

Eq. $A1$ above can be used to select random upper-stem heights for Monte Carlo integration with importance sampling.