# Time-evolution Method to Calculate Neutrino Oscillations in Different Mediums

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1 author:

Xiangjun Tan
UNSW Sydney

# Time-evolution Method to Calculate Neutrino Oscillations in Different Mediums

Xiangjun Tan, xiangjun.tan@student.unsw.edu.au

[1]School of EE&T, The University of New South Wales, Sydney, New South Wales 2052, Australia

## Abstract

Recent years the neutrino physics has a great improvement, especially for the theoretical part. For a better and deeper understanding for the beginner, this paper does not include any Quantum Field Theory requirements, the main thing is to connect the oscillation between the changing basis by using unitary operator. PMNS matrix can act the unitary transformation operator in neutrino oscillation, which equivalent to the $3 \times 3$ rotate matrix in mathematics. Also, the universe and the trajectory for neutrino propagation is not an absolutely vacuum, so the oscillation in matter with matter potential $V$, and the interaction with dark matter are included by changing to the effective Hamiltonian with two parameter matrixes $a(\hat{p})$ and $b(\hat{p})$. This paper aims to give a brief introduction to the neutrino oscillation in some scenarios may beyond the standard model, also evaluated how did the parameter matrix which defined in dark matter background influence the oscillation and let particles turns to anti-particles. Not only for some figures generated by Python, but also show the future potential of Quantum Computing Technology applications, which is connected to the oscillation simulating as a new and interesting method.

Key words: Neutrino Oscillation, dark matter, flavour physics

# Introduction

In the late 1970s, Masatoshi Koshiba of Japan proposed to conduct the Kamiokande experiment to find proton decay. In the existing theory, the proton is stable. If a more fundamental grand unified theory existed, protons would decay. The Kamiokande experiment, located in an abandoned arsenic mine 1 km underground in Gifu Prefecture, uses 3,000 tons of pure water and 1,000 extremely sensitive photomultiplier tubes capable of detecting individual photons. Construction of the experiment began in 1982 and was completed in 1983.[2]

The Kamiokande experiment did not find proton decay, but it did discover a strange phenomenon. High-energy cosmic rays from space produce large numbers of neutrinos in the Earth's atmosphere, called atmospheric neutrinos, including electron neutrinos, muon neutrinos and their antiparticles.[3] In 1988, Masatoshi Koshiba's student Takashi Kajita, analyzed the data and found that fewer neutrinos were measured than expected, which became known as the "atmospheric neutrino anomaly".[1]

The theoretical prediction of the observed number of neutrinos does not vary with the zenith angle, but is a constant value. However, the Super-Kamiokande probe found in 1998 that muon neutrinos coming in from below the detector (produced on the other side of the Earth) were observed at half the number of muon neutrinos coming in from above the detector. [2]This result can be interpreted as a shift or oscillation of neutrinos to other types of undetected neutrinos, a phenomenon known as neutrino oscillation.[5] This finding indicates that neutrinos have a finite mass and suggests that the Standard Model needs to be extended. Neutrinos oscillate between three flavour eigenstates, and each has its own rest mass. Further analysis in 2004 showed that the event rate is a function of length divided by energy and has a sinusoidal correspondence, confirming the neutrino oscillation theory.

The transformation of one type of neutrino into another, also called the mixing of different types of neutrinos. Electron neutrinos, muon neutrinos and tauon neutrinos, they can also be seen as in superposition of three different masses of neutrinos. The difference in their mixing ratio determines their type. Neutrinos that are initially emitted in a certain ratio of mass states, change in the mixing ratio during propagate due to the different intrinsic frequencies of the different mass states, resulting in a transition between different neutrinos, the so-called neutrino oscillation phenomenon. The neutrino mixing parameter describes the pattern of mutual transitions between neutrinos. The neutrino oscillations are indirect evidence that neutrinos have tiny masses and therefore also have some errors with the predictions of the Standard Model.

# Background

Observations from solar and atmospheric neutrino experiments suggest that neutrino oscillations are rooted in the fact that their weak interaction flavour eigenstates are not identical to their propagation mass eigenstates. The relationship between these two eigenstates can be described by the following equation:

$$|v_\alpha\rangle = \sum_i U_{\alpha i}|v_i\rangle \tag{1.1.1}$$

$$|v_i\rangle = \sum_{\alpha} U^*_{\alpha i}|v_\alpha\rangle \tag{1.1.2}$$

Where the unitary matrix is called the *Pontecorvo–Maki–Nakagawa–Sakata matrix* (also called the *PMNS matrix*). [14]It is the analogue of the CKM matrix describing the analogous mixing of quarks. Because of it is not identity matrix so the flavour eigenstate does not equal to the mass eigenstates, that's the reason for neutrino oscillation comes from. The neutrino mixing matrix which has four degrees of freedom can be parameterized by three mixing angles and a CP breaking phase. The PMNS matrix describes the probability of a certain flavour $\alpha$ entering the mass eigenstate $i(1,2,3)$. These probabilities are proportional to $|\ U_{\alpha i}|^2$.

$$\begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{bmatrix} = \begin{bmatrix} \cos\theta_{12}\cos\theta_{13} & \sin\theta_{12}\cos\theta_{13} & \sin\theta_{13}e^{-i\delta} \\ -\sin\theta_{12}\cos\theta_{23} - \cos\theta_{12}\sin\theta_{23}\sin\theta_{13}e^{i\delta} & \cos\theta_{12}\cos\theta_{23} - \sin\theta_{12}\sin\theta_{23}\sin\theta_{13}e^{i\delta} & \sin\theta_{23}\cos\theta_{13} \\ \sin\theta_{12}\sin\theta_{23} - \cos\theta_{12}\cos\theta_{23}\sin\theta_{13}e^{i\delta} & -\cos\theta_{12}\sin\theta_{23} - \sin\theta_{12}\cos\theta_{23}\sin\theta_{13}e^{i\delta} & \cos\theta_{23}\cos\theta_{13} \end{bmatrix}$$

$Fig$ (1.1.1)

With the latest data from NuFIT.org, the newest range and the best fit for the mixing angels and CP breaking phase angle the $3\boldsymbol{\sigma}$ ranges (99.7% confidence)[4]:

$$\begin{aligned} \theta_{12} &= 33.44^{\circ}{}^{+0.77^{\circ}}_{-0.74^{\circ}} \\ \theta_{23} &= 49.2^{\circ}{}^{+1.0^{\circ}}_{-1.3^{\circ}} \\ \theta_{13} &= 8.57^{\circ}{}^{+0.13^{\circ}}_{-0.12^{\circ}} \\ \delta_{\mathrm{CP}} &= 194^{\circ}{}^{+52^{\circ}}_{-25^{\circ}} \end{aligned}$$

$Fig$ (1.1.2)

In this form the relationship between flavour eigenstates and mass eigenstates can be expressed as:

$$\begin{pmatrix} v_e \\ v_\mu \\ v_\tau \end{pmatrix} = U_{PMNS}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \tag{1.1.3}$$

# Theoretical framework of Neutrino Oscillation

According to the experiments and the discoveries of the phenomenon in recent years, they have demonstrated some properties that beyond the standard model obviously. It's hard to achieve a full understanding of every aspect of the neutrino oscillation. In this paper the theoretical background of the neutrino oscillation property is without quantum-field-theory. The result for anti-neutrinos can be described by $\delta_{cp} \to -\delta_{cp},\ V \to -V$ , also for comparison in this paper we considered and did some plots for the oscillation with antineutrinos and without it. The Lagrangian for neutrinos is:[3]

$$\begin{aligned} L = i\overline{v_{jL}}\partial^\mu\gamma_\mu v_{jL} + i\overline{v_{jR}}\partial^\mu\gamma_\mu v_{jR} - \overline{v_{jL}}m_{jk}v_{kR} - \overline{v_{jR}}\left(m^\dagger\right)_{jk}v_{kL} \\ + \left[\frac{g}{\sqrt{2}}\overline{e_{jL}}\gamma^\mu W^-_\mu U^*_{jk}v_{kl} + \frac{g}{4\cos\theta_W}\overline{v_{jL}}\gamma^\mu Z^0_\mu v_{jL} + h.c.\right] \end{aligned} \tag{2.1.1}$$

Where the $e_{jL}$ is left-handed charged lepton field components with $j = 1\sim3$, $W$ and $Z$ are gauge Boson in weak interaction, g is the coupling constant for weak interaction. $v_{jL}$ and $v_{jR}$ are left-handed and right-handed in their spinors, the neutrino field is in the basis that the assumed Dirac neutrino mass diagonal matrix $m_{jk}$. $U^*_{jk}$ here is the unitary matrix as PMNS matrix introduced in the previous paragraph, and the $v_{kl}$ is the weak interaction eigenstate, flavour eigenstate.

# Neutrino Oscillations in Vacuum

## Theoretical background and assumptions

The initial state of the neutrino is in a certain flavour eigenstate, however during propagation it transforms into mass eigenstates and reverts to the flavour eigenstate at the time of the final measurement. [5]For the initial state can be expressed as:

$$|v_\alpha(0)\rangle = U_{\alpha i}|v_i\rangle \tag{2.2.1}$$

Here in the following common denote, the flavour eigenstates use (e, μ, τ) by Greek indices and mass eigenstates use (1, 2, 3) by Latin indices as the same form in formula (1.1.3).[5] After the time evolution and the final flavour eigenstate is: (where the t is time and L is the distance, in natural unit t is approximate to L)

$$|v(t)\rangle = U_{\alpha i}e^{-i(E_i t - p_i L)}|v_i\rangle \tag{2.2.2}$$

Therefore, the neutrino oscillation probability can be expressed as:

$$P(v_\alpha \to v_\beta) = \left|\langle v_\beta|v(t)\rangle\right|^2 \tag{2.2.3}$$

Starting with the time-independent Schrodinger equation:

$$i\frac{d}{dt}\psi = H\psi \tag{2.2.4}$$

Where the Hamiltonian in vacuum can be defined as:

$$H = U\begin{pmatrix} E+\frac{m_1^2}{2E} & 0 & 0 \\ 0 & E+\frac{m_2^2}{2E} & 0 \\ 0 & 0 & E+\frac{m_3^2}{2E} \end{pmatrix}U^\dagger \tag{2.2.5}$$

In this common notation, the wave function is the sum of every flavour eigenstate's contribution:

$$\psi = \psi_e|v_e\rangle + \psi_\mu|v_\mu\rangle + \psi_\tau|v_\tau\rangle$$

According to the equation in Background part, the neutrino time-evolution process is given by a diagonal matrix, which is called S-matrix:

$$S = e^{-iHt} = U\begin{pmatrix} e^{-i\left(E+\frac{m_1^2}{2E}\right)t} & 0 & 0 \\ 0 & e^{-i\left(E+\frac{m_2^2}{2E}\right)t} & 0 \\ 0 & 0 & e^{-i\left(E+\frac{m_3^2}{2E}\right)t} \end{pmatrix}U^\dagger \tag{2.2.6}$$

By using this S-matrix, the probability is the squared module of the S-matrix elements with final flavour eigenstate β and initial eigenstate α:

$$P(v_\alpha \to v_\beta) = \left|S_{\beta\alpha}\right|^2 \tag{2.2.7}$$

## The two-flavour case

For electron-neutrino oscillate to muon-neutrino two flavour case with orthogonal basis, the rotate matrix is a sub matrix of PMNS matrix and the form is like:

$$\begin{pmatrix} v_e \\ v_\mu \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \cos\theta_{12} & -\sin\theta_{12} \\ \sin\theta_{12} & \cos\theta_{12} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2.2.8}$$

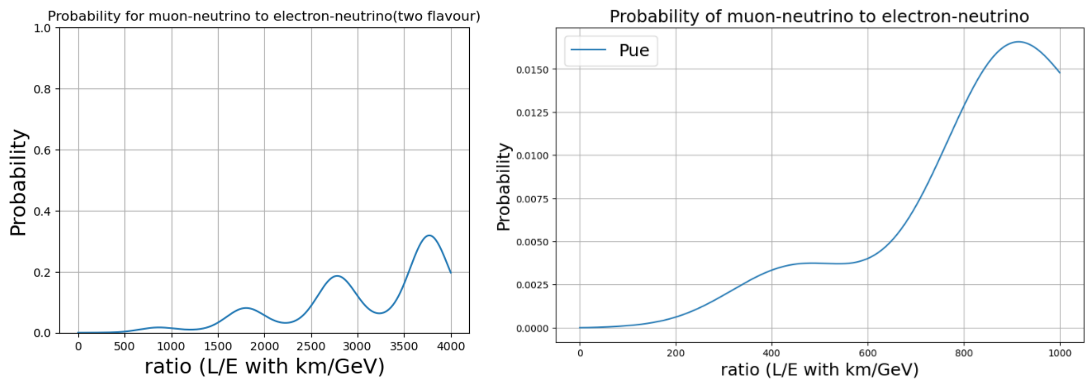
Take $\begin{pmatrix} v_e \\ v_\mu \end{pmatrix}$ as the wave function ψ into the time-independent Schrödinger equation as the expression in (2.2.7):

$$P(v_e \to v_\mu) = \left| -\sin\theta_{12}\cos\theta_{12}\, e^{\frac{-im_1^2 L}{2E}} + \cos\theta_{12}\sin\theta_{12}\, e^{\frac{-im_2^2 L}{2E}} \right|^2 \tag{2.2.9}$$

In order to simplify the expression, by using $\cos\theta_{12}\sin\theta_{12} = \frac{1}{2}\sin 2\theta_{12}$:

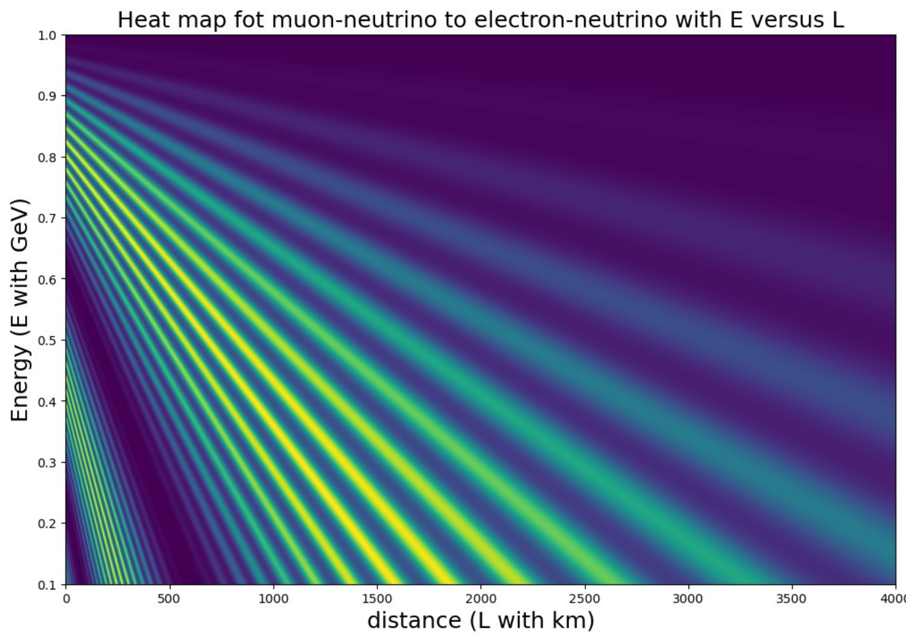
$$P(v_\alpha \to v_\beta) = \left| \sin 2\theta_{12} \sin\frac{\Delta m^2 L}{4E} \right|^2 \tag{2.2.10}$$

Here shows the max oscillation will happened when the distance $L = \frac{2\pi E}{\Delta m^2}$, the two figures below demonstrate the oscillation of muon-neutrino to electron-neutrino in vacuum with different ratio scale. Noticed that the second figure are in $\frac{1}{4}$ range of the first figure.




$Fig$ (2.2.1)

The heat map for energy versus distance in $P(v_\mu \to v_e)$ channel can be plotted; the light color means the higher probability:



$Fig\ (2.2.2)$

## The three-flavour case

The evolution operator in the mass eigenstate basis given by:

$$S = \begin{pmatrix} e^{i\varphi_1} & 0 & 0 \\ 0 & e^{i\varphi_2} & 0 \\ 0 & 0 & e^{i\varphi_3} \end{pmatrix} \tag{2.2.11}$$

With $\varphi_i = -\frac{m_i^2 L}{2|p|}$, so the probability to observe oscillation is:

$$P(v_\alpha \to v_\beta) = \sum_{ij} \left[U_{\beta i} U_{\alpha i}^* U_{\beta j}^* U_{\alpha j}\right] e^{i\varphi_{ij}} \tag{2.2.12}$$

In fact, the neutrino can be regard as the relativistic particle with $E \approx |p|$, so $\varphi_{ij} = \frac{[(m_j^2 - m_i^2)L]}{2E}$

$$P(v_\alpha \to v_\beta) = \sum_i |U_{\beta i}|^2 |U_{\alpha i}|^2 + 2Re\left[\sum_{i>j} \left[U_{\beta i} U_{\alpha i}^* U_{\beta j}^* U_{\alpha j}\right] e^{i\varphi_{ij}}\right] \tag{2.2.13}$$

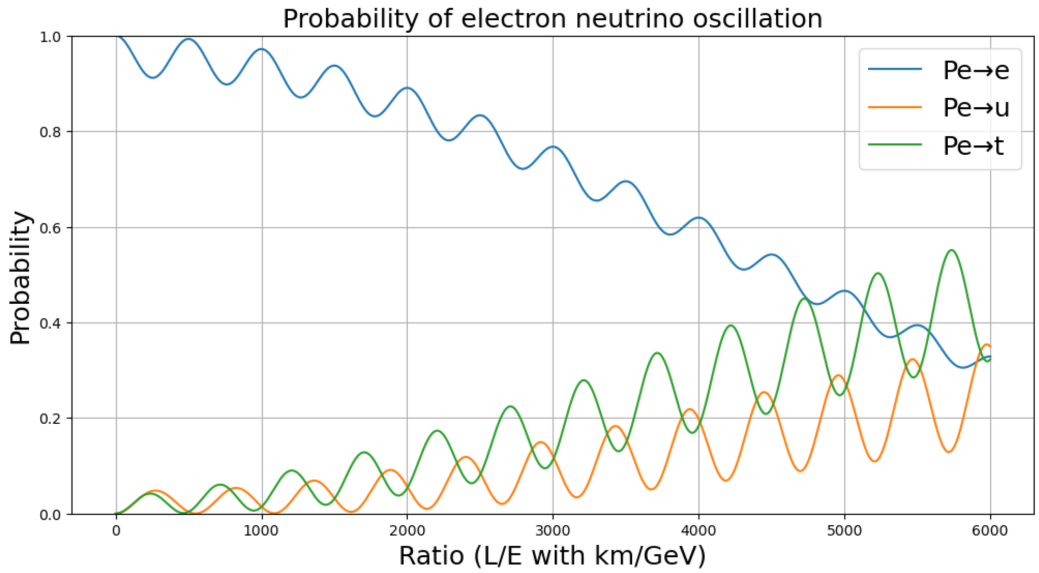
Another efficient expression for the oscillation probability is:

$$P(v_\alpha \to v_\beta) = \delta_{\alpha\beta} - 4Re\left[\sum_{i>j} \left[U_{\beta i} U_{\alpha i}^* U_{\beta j}^* U_{\alpha j}\right] e^{i\varphi_{ij}}\right] \sin^2 \frac{\varphi_{ij}}{2}$$
$$-2\left[\sum_{i>j} Im\left[U_{\beta i} U_{\alpha i}^* U_{\beta j}^* U_{\alpha j}\right] e^{i\varphi_{ij}}\right] \sin \varphi_{ij} \tag{2.2.14}$$

Where the $\delta_{\alpha\beta}$ becomes 1 when the flavour eigenstate $\alpha = \beta$, $\varphi_{ij}$ is the multiple of squared mass difference between $m_i$ ,$m_j$ and the propagate distance then over 2 times the energy. By using the expression above, a python simulation program had been created, which was designed to show the three-flavour oscillation. The figure below includes the electron neutrino survival probability and oscillation probability, via the formula for survival probability:

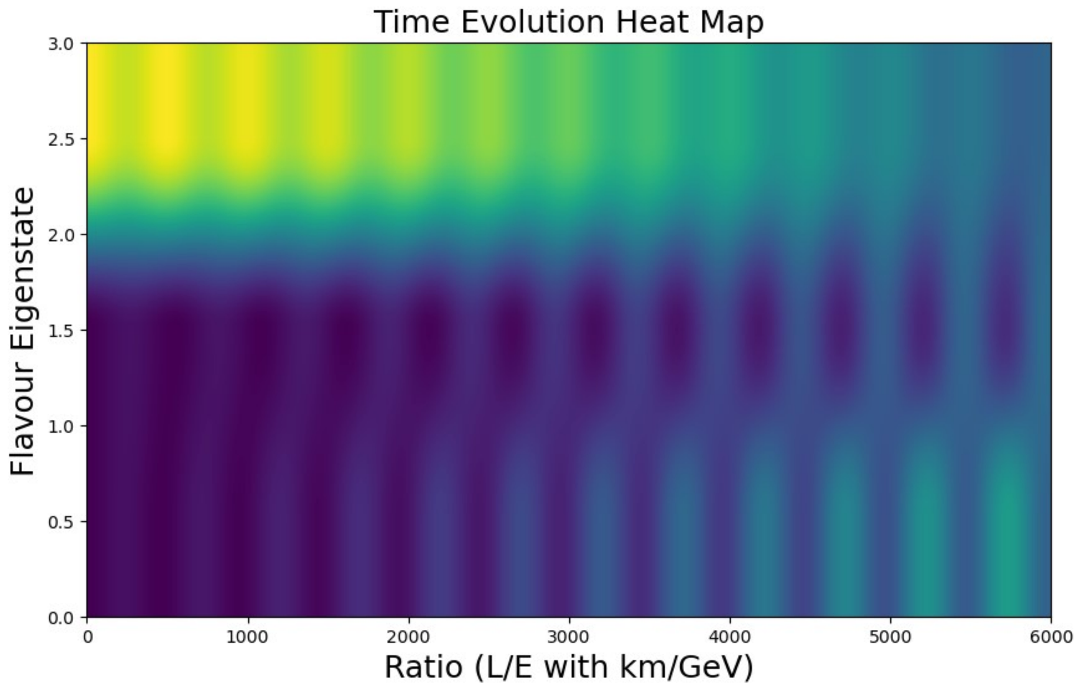
$$P(v_e \to v_e) = 1 - 4Re\left[\sum_{i>j} \left[U_{ei} U_{ei}^* U_{ej}^* U_{ej}\right] e^{i\varphi_{ij}}\right] \sin^2 \frac{\varphi_{ij}}{2}$$
$$-2\left[\sum_{i>j} Im\left[U_{ei} U_{ei}^* U_{ej}^* U_{ej}\right] e^{i\varphi_{ij}}\right] \sin \varphi_{ij} \tag{2.2.15}$$

$P(v_e \to v_\mu)$ and $P(v_e \to v_\tau)$ are calculated by the same method, the figure below shows the probability in three flavour case.




$Fig$ (2.2.3)

What's more, for better demonstration a time-evolution heat map has been created, the vertical axis means the flavour eigenstates, that the vertical values mapping to the three channels $(3,2,1) \rightarrow (P(v_e \rightarrow v_e), P(v_e \rightarrow v_\mu), P(v_e \rightarrow v_\tau))$, the light/dark stripes show the oscillation of probabilities:




$Fig$ (2.2.4)

# Neutrino Oscillations in Matter

## Theoretical background and assumptions

This section is to show the MSW effect and the interaction between neutrino and the dark matter. Note that the electron-neutrino can have charged current interaction with the electron. They will exchange a W boson, the $e^-$ emits a charged $W^-$ boson and then it converts into an electron neutrino, during the propagation $W^-$ boson is then absorbed by an electron neutrino which converts to an electron:

$$e^- \rightarrow v_e + W^- \tag{2.3.1}$$

$$v_e + W^- \rightarrow e^- \tag{2.3.2}$$

So, the oscillation is relevant to the density of electron in the medium, but others two flavours do not join the CC-interaction, what did them joined is the neutral current interaction. Take this account into the new Hamiltonian which is notated as $H_{eff} = H_0 + V$, the time-independent Schrödinger equation in matter is:[5]

$$i\frac{\partial}{\partial t}\psi = H_{eff}\psi = \big(H_0 + V(t)\big)\psi \tag{2.3.3}$$

Where $G_F$ is the Fermi constant, and the matter potential matrix V is defined by:

$$V(t) = \pm\sqrt{2}G_F n_e(t) \tag{2.3.4}$$

The sign $+/-$ is referring to the neutrino and anti-neutrino. For non-relativistic unpolarized medium, the matter potential is the sum of charged current (CC) interaction contribution and neutral current (NC) interaction contribution, the relationship is:

$$V_\alpha \ = \ A_{CC}\delta_{\alpha e} + A_{NC} \tag{2.3.5}$$

The NC interaction with $Z^0$ Boson does not couple to the PMNS matrix from the Lagrangian given in (2.1.1), that $\frac{g}{4\cos\theta_W}\overline{v_{jL}}\gamma^\mu Z_\mu^0 v_{jL}$. So, it will be neglected because it has no changing for flavour eigenstate, which means it has none effect to the probability.

But the CC interaction with $W^-$ Boson term $\frac{g}{\sqrt{2}}\overline{e_{jL}}\gamma^\mu W_\mu^- U_{jk}^* v_{kl}$ in (2.1.1) is connected to the PMNS matrix, hence $V$ can be represented as a diagonal matrix, so the time-evolution equation is:

$$\begin{aligned} i\frac{\partial}{\partial t}\psi &= \left[U\begin{pmatrix} E+\frac{m_1^2}{2E} & 0 & 0 \\ 0 & E+\frac{m_2^2}{2E} & 0 \\ 0 & 0 & E+\frac{m_3^2}{2E} \end{pmatrix}U^\dagger + V\right]\psi \\ &= \left[U\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\Delta m_{21}^2}{2E} & 0 \\ 0 & 0 & \frac{\Delta m_{31}^2}{2E} \end{pmatrix}U^\dagger + \begin{pmatrix} \sqrt{2}G_F n_e & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]\psi \end{aligned} \tag{2.3.6}$$

# The two-flavour case in matter

In two flavour case, the matter potential will be $diag(V, 0)$, so the effective Hamilitionian is:[7][12]

$$H_{eff} = \frac{\Delta m^2}{2E}\begin{pmatrix} 1-\cos 2\theta + \frac{2EV}{\Delta m^2} & \sin 2\theta \\ \sin 2\theta & 1+\cos 2\theta \end{pmatrix} \tag{2.3.7}$$

The eigenvalues for $H_{eff}$ are:

$$E_{1m} = \frac{V}{2} + \frac{\Delta m^2}{4E}\left(1 - \sqrt{\sin^2 2\theta + \left(\cos 2\theta - \frac{2EV}{\Delta m^2}\right)}\right) \quad (2.3.8)$$

$$E_{2m} = \frac{V}{2} + \frac{\Delta m^2}{4E}\left(1 + \sqrt{\sin^2 2\theta + \left(\cos 2\theta - \frac{2EV}{\Delta m^2}\right)}\right) \quad (2.3.9)$$

Here defines the new mixing angle in the matter that $\sin 2\theta_m$ as:

$$\sin 2\theta_m = \frac{\sin 2\theta}{\sqrt{\sin^2 2\theta + \left(\cos 2\theta - \frac{2EV}{\Delta m^2_{\beta\alpha}}\right)^2}} \quad (2.3.10)$$

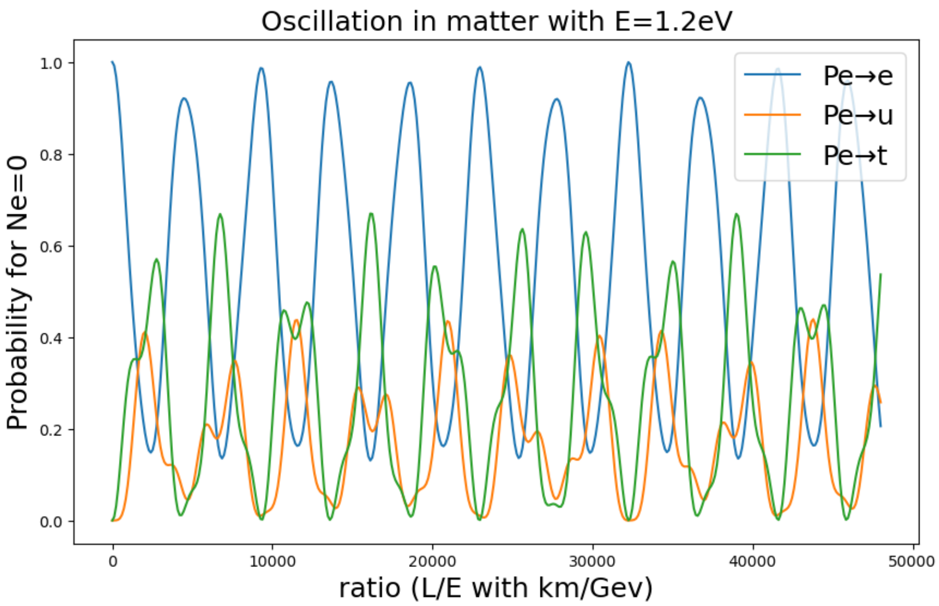
By using the inverse function, the formula for $\theta_m$ can be represented as:

$$\theta_m = \frac{1}{2}\sin^{-1}\sin 2\theta_m = \frac{1}{2}\sin^{-1}\frac{\sin 2\theta}{\sqrt{\sin^2 2\theta + \left(\cos 2\theta - \frac{2EV}{\Delta m^2_{\beta\alpha}}\right)^2}} \quad (2.3.11)$$

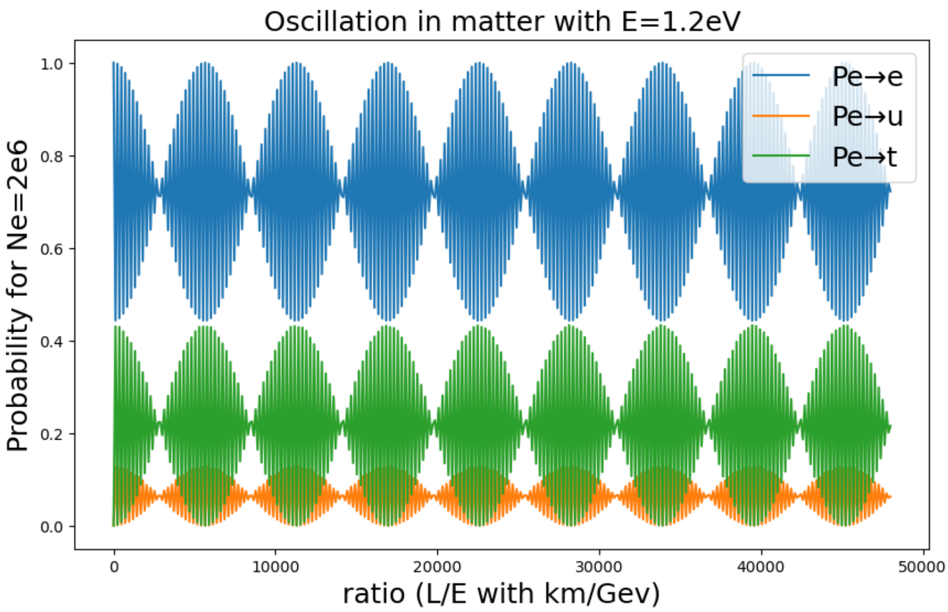
The probability for neutrino oscillation in $v_\alpha \rightarrow v_\beta$ channel is:

$$P(v_\alpha \rightarrow v_\beta) = \sin^2 2\theta_m \sin^2\left(\frac{\Delta m^2_{\beta\alpha} L}{4E}\sqrt{\sin^2 2\theta + \left(\cos 2\theta - \frac{2EV}{\Delta m^2_{\beta\alpha}}\right)^2}\right) \quad (2.3.12)$$

The figures below show the oscillations when $n_e = 0\ mm^{-3}$ and $n_e = 2e6\ mm^{-3}$, as we can see the max oscillation amplitude will reduce both in $v_e \rightarrow v_\mu$ and $v_e \rightarrow v_\tau$ channel when the $n_e$ increased at $E = 1.2eV$:




$Fig$ (2.3.1)




$Fig$ (2.3.2)

## MSW resonance

Another phenomenon is the MSW Resonance, it will appear when:[6][12]

$$A_{cc} = \Delta m^2 \cos 2\theta \tag{2.3.13}$$

The resonance electron number density is defined by:

$$N_e^R = \frac{A_{cc}}{2\sqrt{2}EG_F} = \frac{\Delta m^2 \cos 2\theta}{2\sqrt{2}EG_F} \tag{2.3.14}$$

With the effective squared mass difference in matter:

$$\Delta {m^2}_m = \Delta m^2 \sin 2\theta \tag{2.3.15}$$

Then let the matter potential and the electron density with the condition we defined, with $V = \sqrt{2}G_F n_e$ that:

$$\left(\cos 2\theta - \frac{2EV}{\Delta m_{\beta\alpha}^2}\right)^2 = \left(\cos 2\theta - \frac{2E\sqrt{2}\dfrac{\Delta m_{\beta\alpha}^2 \cos 2\theta}{2\sqrt{2}EG_F}G_F}{\Delta m_{\beta\alpha}^2}\right)^2 = (\cos 2\theta - \cos 2\theta)^2 = 0 \tag{2.3.16}$$

If $\theta_m = \frac{\pi}{4}$, the resonance density then will disappear:

$$P(v_\alpha \to v_\beta) = \sin^2 2\theta_m \sin^2\left(\frac{\Delta m_{\beta\alpha}^2 L}{4E}\sqrt{\sin^2 2\theta + \left(\cos 2\theta - \frac{2EV}{\Delta m_{\beta\alpha}^2}\right)^2}\right)$$

$$= \sin^2\left(\frac{\Delta m_{\beta\alpha}^2 L}{4E}\sqrt{\sin^2 2\theta_{\alpha\beta} + \left(\cos 2\theta_{\alpha\beta} - \frac{2EV}{\Delta m_{\beta\alpha}^2}\right)^2}\right) = \sin^2\left(\frac{\Delta m_{\beta\alpha}^2 L}{4E}\sin 2\theta_{\alpha\beta}\right) \tag{2.3.17}$$

# Oscillation in Dark Matter

## Theoretical background and assumptions

Without too much Quantum-Field-Theory knowledge, the oscillation in dark matter can be regarded as the time-independent perturbations. [13]With the assumption that the energy-momentum operator is conserved:

$$P^{\mu} = P_0^{\mu} + \delta P^{\mu} \tag{2.3.18}$$

and $N_0$ is:

$$N_0(t,x) = e^{-iH_0 t} N(0,x) e^{iH_0 t} \tag{2.3.19}$$

So, with the subscript indicating this field satisfy the free field equation:

$$P^{\mu}[N(t,x)] = e^{iH_0 t} P^{\mu}[N_0(t,x)] e^{-iH_0 t} \tag{2.3.20}$$

For a given neutrino direction with a small correction $\delta_m$:

$$P = P_0 = H - e^{iH_0 t}[\delta_m H + \delta H_{int}(t)] e^{-iH_0 t} \tag{2.3.21}$$

Where $\delta H_{int}(t) = -\int dx\, \delta L\big[N_0, N_0^{\dagger}\big]$, here $L$ is the Lagrangian for neutrino. Follow the steps in this paper which were used in previous non-dark matter case, the effective Hamiltonian beyond the standard model with the dark potential is:

$$H_{eff} = H_{SM} + \delta V_{eff} \tag{2.3.22}$$

The time-evolution equation for Weyl neutrino propagates in the dark matter:

$$i\frac{\partial}{\partial t}|v(x)\rangle = \delta H|v(x)\rangle = \big(H_{SM} + \delta V_{eff}\big)|v(x)\rangle \tag{2.3.23}$$

Then the propagation factor $e^{iPL}$ between source and detector has the same physical meaning with the effective Hamiltonian:

$$H_{eff} = \langle v'|\delta H|v\rangle = \begin{pmatrix} U\left(\frac{mm^*}{2E}\right)U^{\dagger} + a(\hat{p}) & b(\hat{p}) \\ b^{\dagger}(\hat{p}) & U^*\left(\frac{mm^*}{2E}\right)U^T - a^*(\hat{p}) \end{pmatrix} \tag{2.3.24}$$

Where $a(\hat{p}) = -V_0 + \hat{p}\cdot V$, and $b(\hat{p}) = -4\sqrt{2}(E + iB)\cdot\hat{\epsilon}(\hat{p})$, with the property $a(\hat{p}) = a^{\dagger}(\hat{p})$, $b(\hat{p}) = -b^T(\hat{p})$. $(E + iB)$ described the dark electric and magnetic field. In this notation with $\hat{p} = \frac{p}{|p|}$, $|p| = E$, and $U$ is the PMNS matrix.

So $H_{eff}$ is a $6\times 6$ matrix, include four $3\times 3$ sub-matrix elements, the numerical definition for $a(\hat{p})$ and $b(\hat{p})$ are:

$$a(\hat{p}) = \begin{pmatrix} a_{ee} & a_{e\mu} & a_{e\tau} \\ a_{e\mu}^* & a_{\mu\mu} & a_{\mu\tau} \\ a_{e\tau}^* & a_{\mu\tau}^* & a_{\tau\tau} \end{pmatrix}, b(\hat{p}) = \begin{pmatrix} 0 & b_{e\mu} & b_{e\tau} \\ -b_{e\mu} & 0 & b_{\mu\tau} \\ -b_{e\tau} & -b_{\mu\tau} & 0 \end{pmatrix} \tag{2.3.25}$$

This time the time-evolution equation needs to obtain the antineutrinos, so the solution should include the anti-particles. Schrödinger equation is not suitable in this case, what we need is Dirac equation.

## Cases with different values for parameter matrix

These is the simple case shows the simple probability simulations for electron survival and the probability of oscillate to $\hat{\nu}_e$, when $a(\hat{p}) = 0, b(\hat{p})! = 0$, the effective Humiliation from (2.3.24) can be rewritten as:[13][9]

$$H_{eff} = \begin{pmatrix} 0_{3\times3} & b(\hat{p}) \\ b^{\dagger}(\hat{p}) & 0_{3\times3} \end{pmatrix} \tag{2.3.26}$$

So, the time-evolution operator can be derived as:

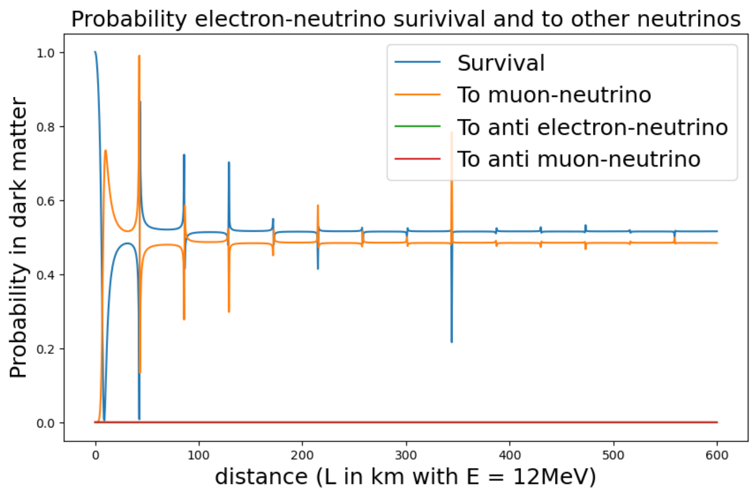
$$S(0,L) = e^{-iH_{eff}L} = 1_{6\times6} - iL_{osc}H_{eff}\sin\left(\frac{L}{L_{osc}}\right) - 2{L_{osc}}^2{H_{eff}}^2\sin\left(\frac{L}{2L_{osc}}\right) \tag{2.3.27}$$

Where the expression for $L_{osc}$ is:

$$L_{osc} = 4\sqrt{tr\left(H_{eff}\cdot H_{eff}\right)} \tag{2.3.28}$$

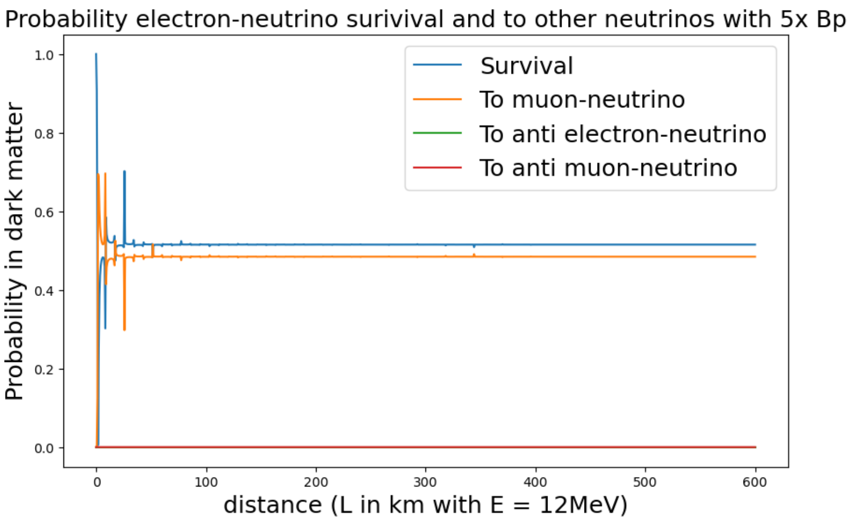
For a common plot to show a simple case (this case will have less probability for oscillate to anti-particles, a better case will in the both $a(\hat{p}), b(\hat{p})! = 0$ case), just define $E = 12MeV$ here, and setup with $b_{\alpha\beta}$ equals to its squared mass difference over 2 times of its energy, which the case may not exist in reality. The numerical expression for the input parameter matrix is:

$$b(\hat{p}) = \begin{pmatrix} 0 & \frac{\Delta m_{21}^2}{2E} & \frac{\Delta m_{31}^2}{2E} \\ -\frac{\Delta m_{21}^2}{2E} & 0 & \frac{\Delta m_{32}^2}{2E} \\ -\frac{\Delta m_{31}^2}{2E} & -\frac{\Delta m_{32}^2}{2E} & 0 \end{pmatrix} \tag{2.3.29}$$




$Fig$ (2.3.3)

In order to see the effect clearly, this time change the $b(\hat{p})$ with 5 times than before, $|b(\hat{p})| \rightarrow 5|b(\hat{p})|$, what we can see is that the oscillation amplitudes will decrease and cancel faster:




$Fig$ (2.3.4)

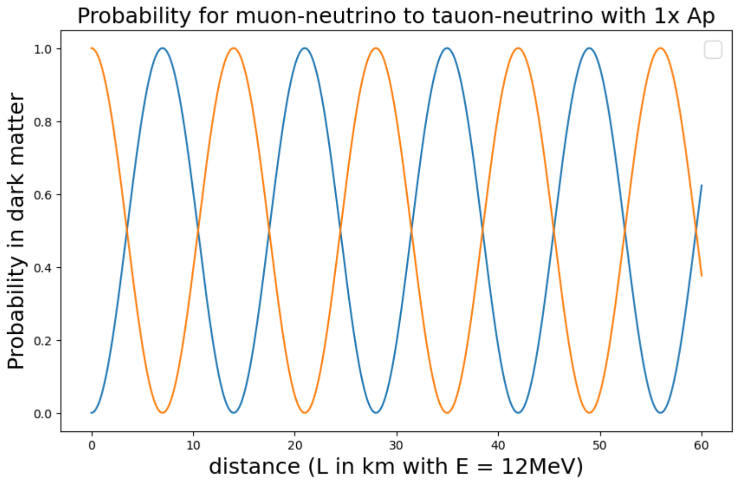
An alternative case is $a(\hat{p})! = 0, b(\hat{p}) = 0$: we can neglect the $\theta_{13}$ and $\Delta m^2_{SOL}$ term because of there's no entries for $e\alpha$. Considering about the $P(v_\mu \to v_\tau)$ channel, then the effective Humiliation in this case from (2.3.24) can be rewritten as:

$$H_{eff} = \begin{pmatrix} U_{2\times2}\left(\frac{mm^*}{2E}\right){U_{2\times2}}^{\dagger} + a(\hat{p}) & 0_{2\times2} \\ 0_{2\times2} & U_{2\times2}\left(\frac{mm^*}{2E}\right){U_{2\times2}}^{\dagger} - a(\hat{p})^* \end{pmatrix} =$$
$$\begin{pmatrix} U_{2\times2}\left(\frac{mm^*}{2E}\right){U_{2\times2}}^{\dagger} + \begin{pmatrix} a_{\mu\mu} & a_{\mu\tau} \\ a^*_{\mu\tau} & a_{\tau\tau} \end{pmatrix} & 0_{2\times2} \\ 0_{2\times2} & U_{2\times2}\left(\frac{mm^*}{2E}\right){U_{2\times2}}^{\dagger} - \begin{pmatrix} a_{\mu\mu} & a_{\mu\tau} \\ a^*_{\mu\tau} & a_{\tau\tau} \end{pmatrix}^* \end{pmatrix} \quad (2.3.30)$$

Then the time-evolution operator can be written as:

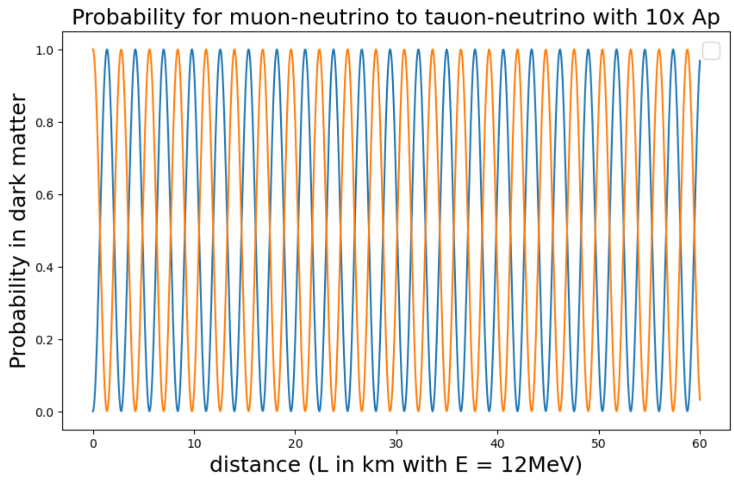
$$S(0,L) = e^{-iH_{eff}L} = e^{-i\begin{pmatrix} U_{2\times2}\left(\frac{mm^*}{2E}\right){U_{2\times2}}^{\dagger} + \begin{pmatrix} a_{\mu\mu} & a_{\mu\tau} \\ a^*_{\mu\tau} & a_{\tau\tau} \end{pmatrix} & 0_{2\times2} \\ 0_{2\times2} & U_{2\times2}\left(\frac{mm^*}{2E}\right){U_{2\times2}}^{\dagger} - \begin{pmatrix} a_{\mu\mu} & a_{\mu\tau} \\ a^*_{\mu\tau} & a_{\tau\tau} \end{pmatrix}^* \end{pmatrix}L} =$$
$$1_{4\times4}\cos\left(\frac{L}{L_{osc}}\right) - iL_{osc}\, H_{eff}\sin\left(\frac{L}{L_{osc}}\right) \quad (2.3.31)$$

Where $L_{osc}$ has the same expression as $a(\hat{p}) = 0$ case, then the plot can be shown with $a_{\alpha\beta}$ equals to each the squared mass difference, where the initial energy $E = 12MeV$, and the input parameter matrix $a(\hat{p})_{2\times2} = \begin{pmatrix} 0 & \frac{\Delta m^2_{32}}{2E} \\ (\frac{\Delta m^2_{32}}{2E})^* & 0 \end{pmatrix}$:




$Fig\ (2.3.5)$

If we twice the module of input $|a(\hat{p})| \to 2|a(\hat{p})|$, then the oscillation distance in figure will decreased and oscillate more frequently:




$Fig\ (2.3.6)$

For a more flexible model and to show the oscillation of particles to anti-particles case, by adding our customized $b(\hat{p})$ inputs:

$$H_{eff} = \begin{pmatrix} U\left(\frac{mm^*}{2E}\right)U^\dagger + a(\hat{p}) & b(\hat{p}) \\ b^\dagger(\hat{p}) & U^*\left(\frac{mm^*}{2E}\right)U^T - a^*(\hat{p}) \end{pmatrix} =$$

$$\begin{pmatrix} U\left(\frac{mm^*}{2E}\right)U^\dagger + \begin{pmatrix} a_{ee} & a_{e\mu} & a_{e\tau} \\ a_{e\mu}^* & a_{\mu\mu} & a_{\mu\tau} \\ a_{e\tau}^* & a_{\mu\tau}^* & a_{\tau\tau} \end{pmatrix} & \begin{pmatrix} 0 & b_{e\mu} & b_{e\tau} \\ -b_{e\mu} & 0 & b_{\mu\tau} \\ -b_{e\tau} & -b_{\mu\tau} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & b_{e\mu} & b_{e\tau} \\ -b_{e\mu} & 0 & b_{\mu\tau} \\ -b_{e\tau} & -b_{\mu\tau} & 0 \end{pmatrix}^\dagger & U^*\left(\frac{mm^*}{2E}\right)U^T - \begin{pmatrix} a_{ee} & a_{e\mu} & a_{e\tau} \\ a_{e\mu}^* & a_{\mu\mu} & a_{\mu\tau} \\ a_{e\tau}^* & a_{\mu\tau}^* & a_{\tau\tau} \end{pmatrix}^* \end{pmatrix} \quad (2.3.32)$$

If the effective Hamiltonian is dominated by $a(\hat{p})$ and $b(\hat{p})$, in this assumption it becomes to:

$$H_{eff} \approx \begin{pmatrix} a(\hat{p}) & b(\hat{p}) \\ b^{\dagger}(\hat{p}) & a^{*}(\hat{p}) \end{pmatrix} \approx \left( \begin{matrix} \begin{pmatrix} a_{ee} & a_{e\mu} & a_{e\tau} \\ a_{e\mu}^{*} & a_{\mu\mu} & a_{\mu\tau} \\ a_{e\tau}^{*} & a_{\mu\tau}^{*} & a_{\tau\tau} \end{pmatrix} & \begin{pmatrix} 0 & b_{e\mu} & b_{e\tau} \\ -b_{e\mu} & 0 & b_{\mu\tau} \\ -b_{e\tau} & -b_{\mu\tau} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & b_{e\mu} & b_{e\tau} \\ -b_{e\mu} & 0 & b_{\mu\tau} \\ -b_{e\tau} & -b_{\mu\tau} & 0 \end{pmatrix}^{\dagger} & -\begin{pmatrix} a_{ee} & a_{e\mu} & a_{e\tau} \\ a_{e\mu}^{*} & a_{\mu\mu} & a_{\mu\tau} \\ a_{e\tau}^{*} & a_{\mu\tau}^{*} & a_{\tau\tau} \end{pmatrix}^{*} \end{matrix} \right) \tag{2.3.33}$$

Then the time-evolution operator for this case is:

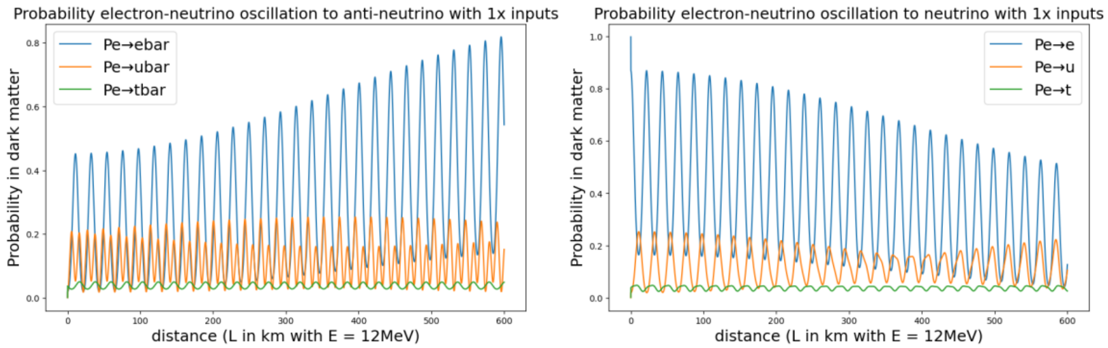
$$U(0,L) = e^{-iH_{eff}L} = e^{-i\begin{pmatrix} a(\hat{p}) & b(\hat{p}) \\ b^{\dagger}(\hat{p}) & -a^{*}(\hat{p}) \end{pmatrix}L} \tag{2.3.34}$$

The oscillation distance is still in the previous expression but with the new effective Hamiltonian:

$$L_{osc} = 4\sqrt{tr(H_{eff}\cdot H_{eff})} = 4\sqrt{tr\left(\begin{pmatrix} a(\hat{p}) & b(\hat{p}) \\ b^{\dagger}(\hat{p}) & -a^{*}(\hat{p}) \end{pmatrix}\cdot\begin{pmatrix} a(\hat{p}) & b(\hat{p}) \\ b^{\dagger}(\hat{p}) & -a^{*}(\hat{p}) \end{pmatrix}\right)} \tag{2.3.35}$$

Taking them into the time-evolution equation and setting up the values for each parameter with $\frac{\Delta m_{\beta\alpha}^2}{2E}$. Then the oscillation is shown in the figures below:

$$a(\hat{p}) = \begin{pmatrix} 0 & \frac{\Delta m_{21}^2}{2E} & \frac{\Delta m_{31}^2}{2E} \\ \frac{\Delta m_{21}^2}{2E} & 0 & \frac{\Delta m_{32}^2}{2E} \\ \frac{\Delta m_{31}^2}{2E} & \frac{\Delta m_{32}^2}{2E} & 0 \end{pmatrix}, b(\hat{p}) = \begin{pmatrix} 0 & \frac{\Delta m_{21}^2}{2E} & \frac{\Delta m_{31}^2}{2E} \\ -\frac{\Delta m_{21}^2}{2E} & 0 & \frac{\Delta m_{32}^2}{2E} \\ -\frac{\Delta m_{31}^2}{2E} & -\frac{\Delta m_{32}^2}{2E} & 0 \end{pmatrix} \tag{2.3.36}$$



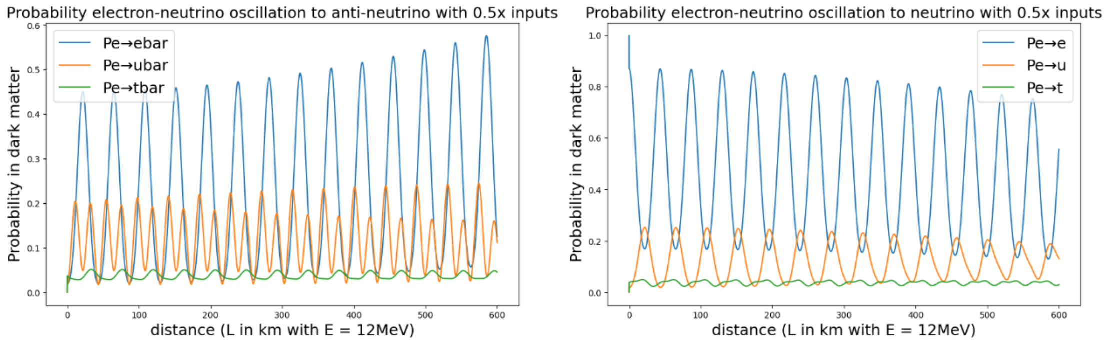

$Fig$ (2.3.7)

The left figure is the oscillation for electron neutrino oscillate to anti-neutrinos with $a(\hat{p})! = 0$, and $b(\hat{p})! = 0$, the right one is the oscillation for electron neutrino oscillate to neutrinos. Then if we setup the module of $a(\hat{p}), b(\hat{p})$ with both multiply the coefficient 0.5 and 3, the figures below will proof the conclusion we gave before:

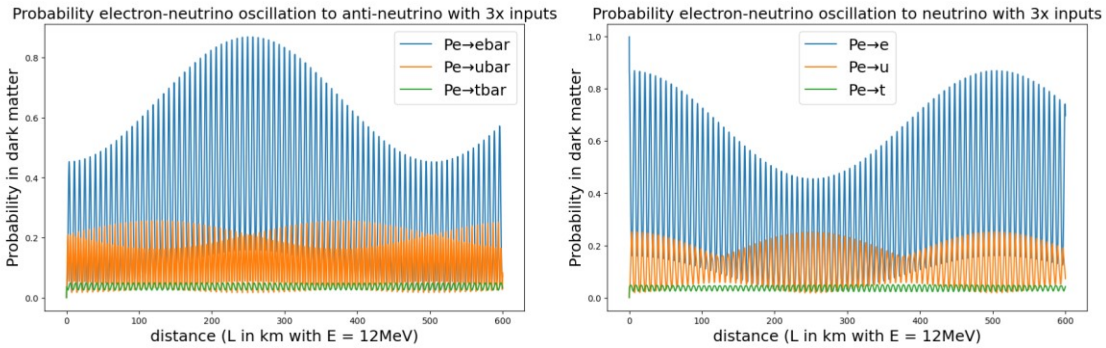
$$\mathrm{F}_{osc} \propto |a(\hat{p})| \tag{2.3.37}$$

$$L_{osc} \propto \left|\frac{1}{b(\hat{p})}\right| \tag{2.3.38}$$

Where $L_{osc}$ is the approximately oscillation distance between two nodes of the wave packet, $\mathrm{F}_{osc}$ is the oscillation estimative frequency.




$Fig$ (2.3.8)




$Fig$ (2.3.9)

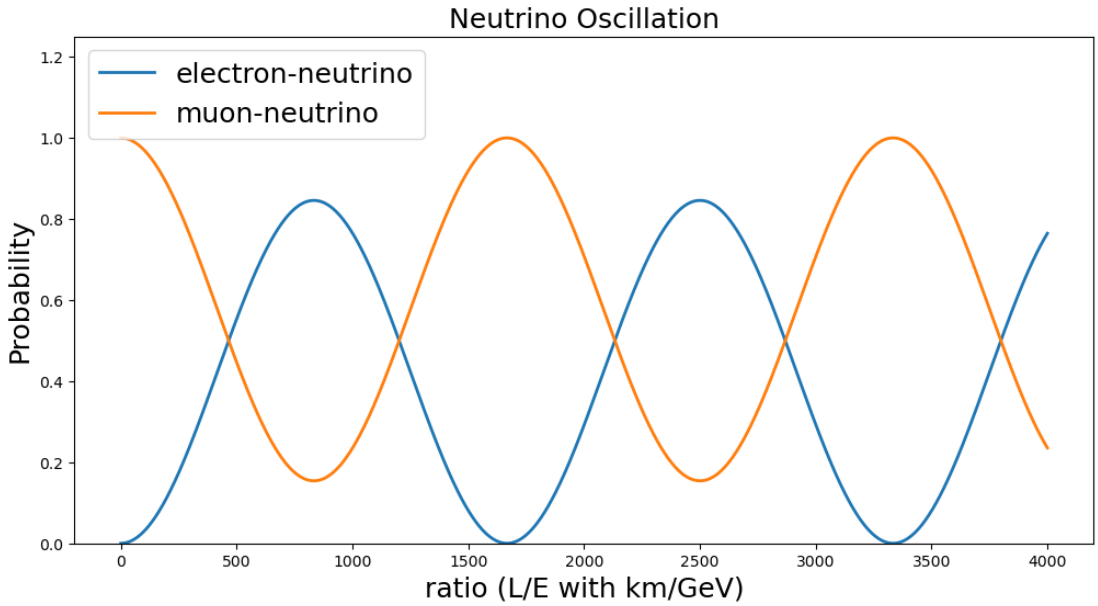
For more details that how will the $a(\hat{p}), b(\hat{p})$ effect the oscillation and if the $a(\hat{p}), b(\hat{p})$ are not in static as the constant, these do not include in this paper and them may be discussed in the future research.

# Calculation and Simulation Method

# Simulation on Python Program

During the Taste of Research program period, some simulation plots and calculation functions had been created. Also, there are a lot of assumptions and approximations in order to simplify the form and less performance cost. Some approximations were used in the calculation. In order to get a better plot understanding, traveling distance and energy become ratio of them instead, which is the x-axis in the following plot. In two flavour case, another flavour eigenstate is just neglected. In three flavours case, the CP-violating phase is setting up with $\pi$. Note that in oscillation analysis, the controlled variable usually is the ratio of distance over energy $\frac{L}{E}$ not distance $L$. And the time term is set up equals to the distance because the propagate speed is as same as the light speed, which is 1 in natural unit.

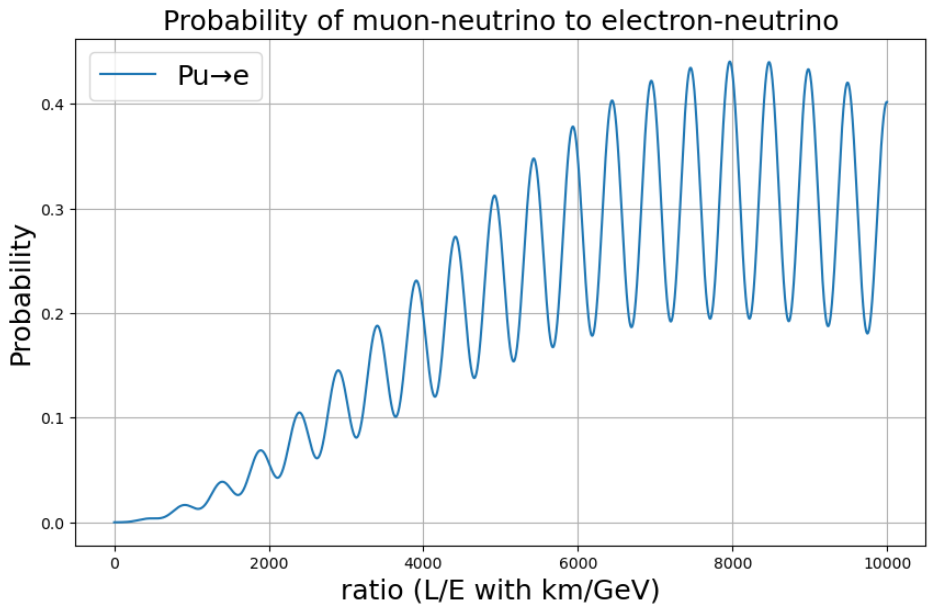
First go to the neutrino oscillation in two flavours simple case, notice that flavour1 is electron-neutrino, and flavour2 is muon-neutrino, oscillation for $v_\mu \to v_e$ channel is shown by the figure below:




$Fig$ (3.1.1)

But in three flavour case, an efficient notation is used if the anti-neutrinos have been added, that $\widetilde{P_{\alpha\beta}} = P_{\alpha\beta}(\theta_{23} \to \theta_{23} + \frac{\pi}{2})$. In this case the figure shows the oscillation probability which can be represented as:

$$P(v_\mu \to v_e) = P_{e\mu} - P_{\mu\tau} + \widetilde{P_{\mu\tau}} \tag{3.1.1}$$




$Fig$ (3.1.2)

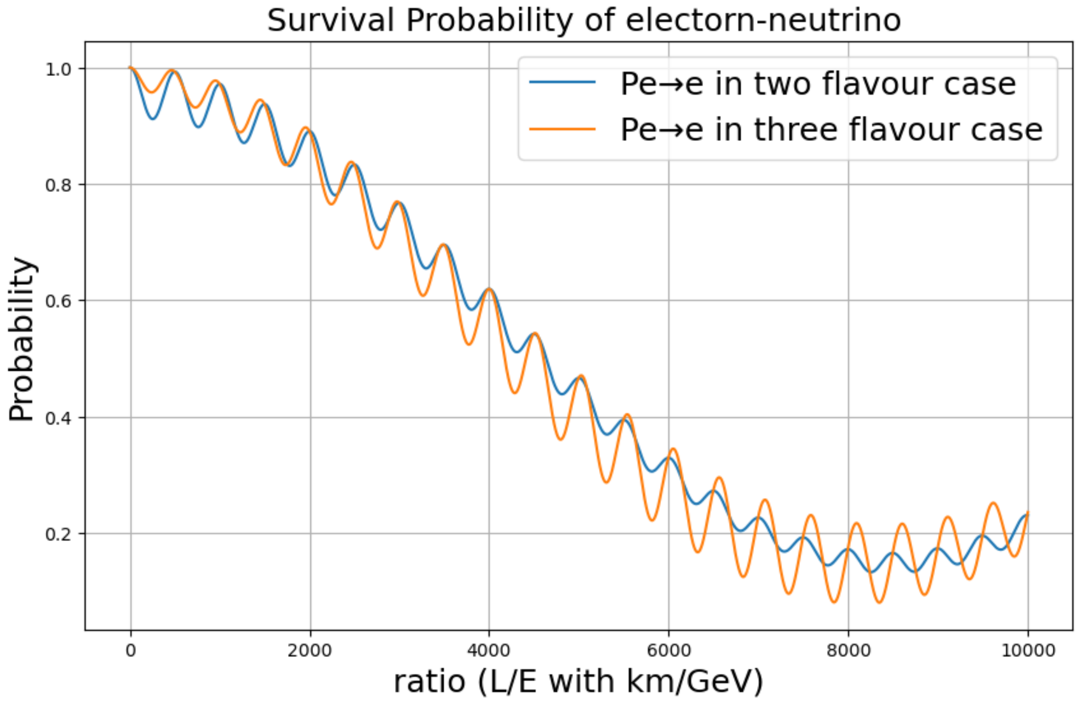
In order to verify the validity and error of two types of calculation, there is an additional comparison for the survival probability for electron-neutrino in two flavour case(comparing with the result of three flavour case), the formula for the survival probability in three flavour case is the formula like (2.2.15):

$$P(v_e \to v_\mu) = P_{\mu e} + P_{\mu\tau} - \widetilde{P_{\mu\tau}} \tag{3.1.2}$$

$$P(v_e \to v_e) = 1 - P_{e\mu} - \widetilde{P_{e\mu}} \tag{3.1.3}$$

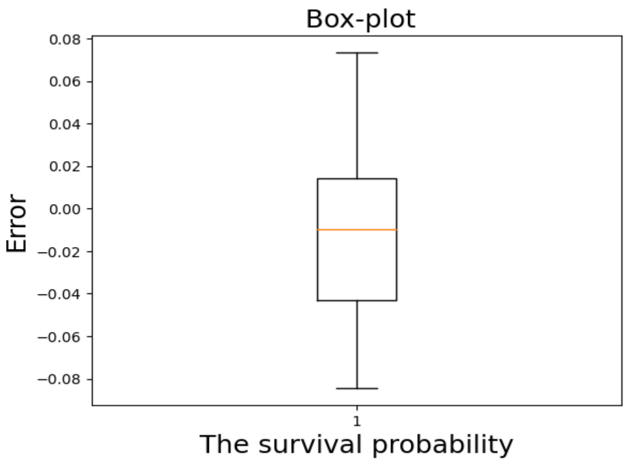
For the simple way in two flavour case, using the expression in (2.2.15), the electron survival probability is:

$$P(v_e \to v_e) = 1 - P_{e\mu} - P_{e\tau} = 1 - |S_{\tau e}|^2 - |S_{\tau e}|^2 \tag{3.1.4}$$



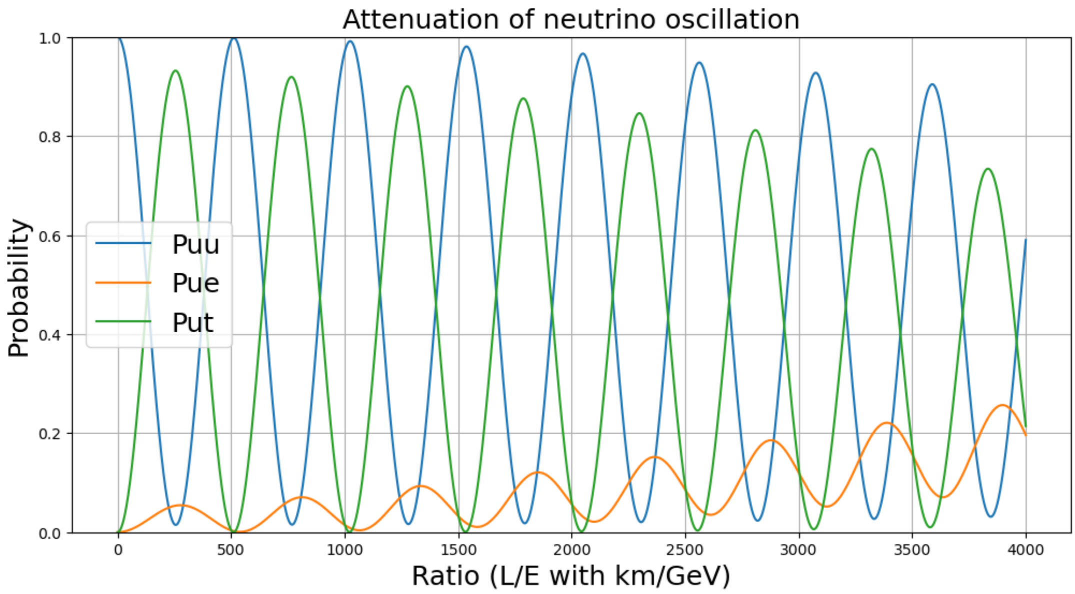

$Fig$ (3.1.3)

The relevant coefficient of these two curves is 0.9924, and the covariance is: $\begin{pmatrix} 0.0933 & 0.0903 \\ 0.0903 & 0.0887 \end{pmatrix}$, according to this analysis, the simulation results are very close with two flavour case and three flavour case, and has enough transformation consistency.




$Fig$ (3.1.4)

During the program, in order to get a better research performance, the input of function can be customized in 3-flavour case, the oscillation for tauon-neutrino to muon-neutrino flavour eigenstate is shown in the below picture, the oscillation amplitude will decrease with the increasing input ratio value. With the convergent of oscillation, finally the oscillation will gradually disappear if the ratio shifts to significant large.




$Fig$ (3.1.5)

# Calculation for Neutrino Masses

For neutrinos their very small masses are difficult to measure in existing experiments, however, the squared difference of their masses during oscillations are very sensitive, and their approximate mass distribution can be derived. Be based on the previous code, an additional feature had been added to investigate the effect of different mass differences on oscillations. In the previous assumption, the flavour eigenstate distribution in mass eigenstate can be derivatized, with $|v_\alpha\rangle = \sum_i U_{\alpha i}|v_i\rangle$ we defined before:

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = U_{PMNS}^{\dagger} \begin{pmatrix} v_e \\ v_\mu \\ v_\tau \end{pmatrix} \tag{3.2.1}$$

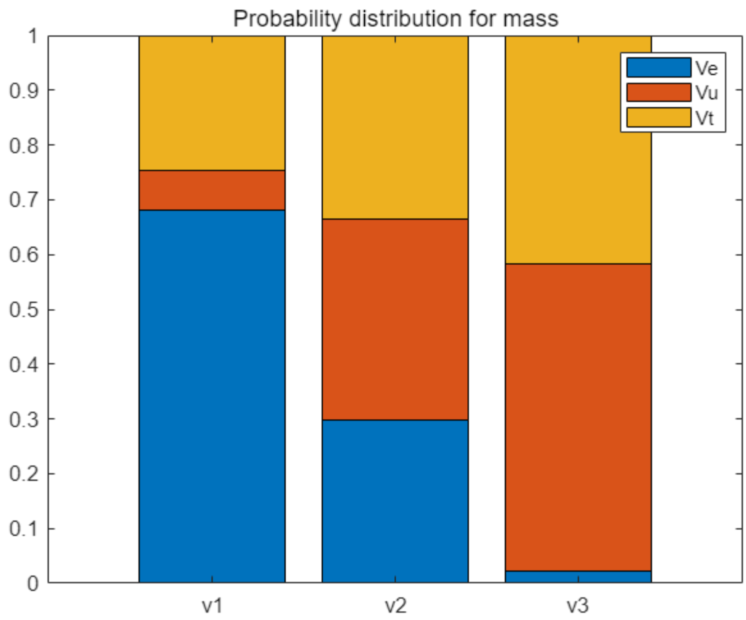
That's from the expression in (1.1.3) but transferred to different basis of eigenstate. Where PMNS matrix is unitary so the $U_{PMNS}^{\dagger} = U_{PMNS}^{-1}$. According to the calculation by using Python:

$$|v_1\rangle \approx 0.519|v_e\rangle + 0.169|v_\mu\rangle + 0.312|v_\tau\rangle \tag{3.2.2}$$

$$|v_2\rangle \approx 0.315|v_e\rangle + 0.350|v_\mu\rangle + 0.335|v_\tau\rangle \tag{3.2.3}$$

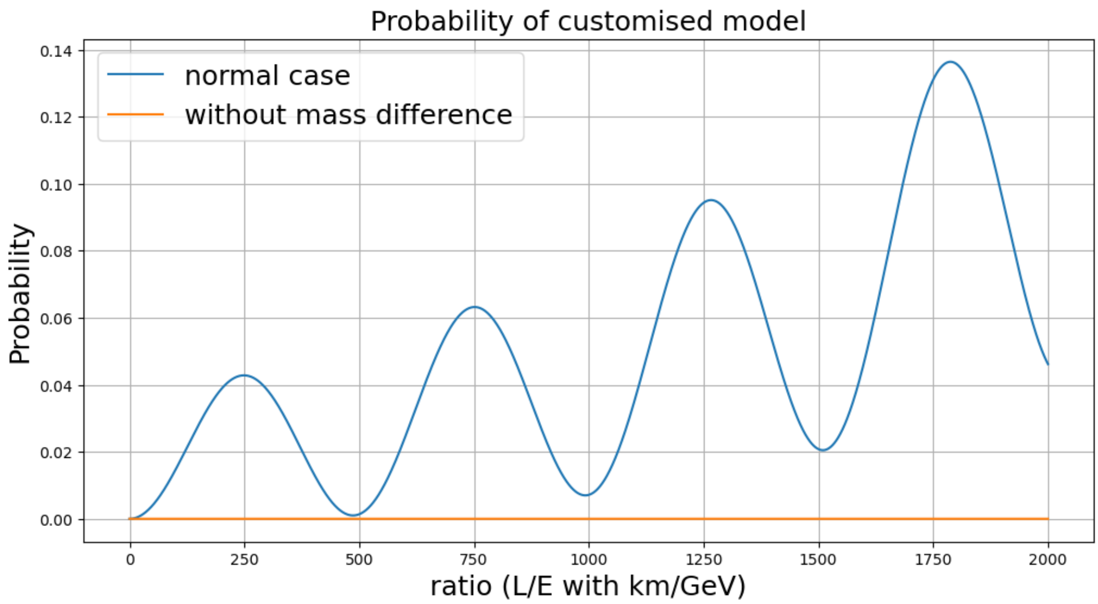
$$|v_3\rangle \approx 0.096|v_e\rangle + 0.475|v_\mu\rangle + 0.429|v_\tau\rangle \tag{3.2.4}$$

Until today, the exact distribution still cannot be measured or calculated exactly, all the results are from the experiments data analysis and fit. In this paper's assumption, the stack bar figure below shows the approximate probability distribution:




$Fig$ (3.2.1)

The squared differences of neutrino mass term act an important role in the oscillation, the diagonal Hamiltonian matrix is depended on this term. The oscillation will cancel if the squared mass difference was limited to zero. The figure below shows the compared situation if there are no mass differences for neutrino in different flavour eigenstates in $v_e \rightarrow v_\tau$ channel, it's clearly that the oscillation disappeared if the mass differences are zero and just without any chance oscillate to other flavour eigenstates.




$Fig$ (3.2.2)

# Simulation on Quantum Processor

For a single quantum-bit (also called qubit), it has two orthogonal eigenstates as usual, which are

$|0\rangle$ and $|1\rangle$. So, the actual state must be a linear superposition of these two states, with two complex coefficients:

$$|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{3.3.1}$$

In mathematics, two-flavor neutrino oscillations are involved in a two-dimension Hilbert space, which can be represented on a single controlled qubit, due to a mapping process they can be represented as:

$$|0\rangle = |v_\alpha\rangle = \begin{pmatrix}1\\0\end{pmatrix}, |1\rangle = |v_\beta\rangle = \begin{pmatrix}0\\1\end{pmatrix} \tag{3.3.2}$$

The rotation process of these two qubit states must be a unitary operator, like a two-dimension rotate matrix. The reduced PMNS matrix can properly satisfy its requirement, mapping the $2 \times 2$ reduced PMNS matrix to a general unitary transformation operation for a single qubit system is allowed, which called the U3-Gate with three parameters:

$$U3(\Theta, \Phi, \lambda) = \begin{pmatrix} \cos\frac{\Theta}{2} & -\sin\frac{\Theta}{2}e^{i\lambda} \\ \sin\frac{\Theta}{2}e^{i\Phi} & \cos\frac{\Theta}{2}e^{i(\lambda+\Phi)} \end{pmatrix} \tag{3.3.3}$$

According to let $|v_\alpha\rangle \to e^{-i\Phi}|v_\alpha\rangle$, the parameter $\Phi$ can be removed, because the field which under the Standard Model has invariant Lagrangian, so $\Phi$ can be set up with zero easily without loss of generality. [11]The Phase $\lambda$ does not enter the exponential for the neutrino oscillation because the Hamiltonian has the form $\frac{U(mm^*)U^\dagger}{2E}$.

The relationship between $2 \times 2$ reduced PMNS matrix and U3-Gate is:

$$U_{PMNS}^{2\times2} = U3(2\Theta, 0, 0) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{3.3.4}$$

By using the same process as classical calculation, the oscillation probabilities can be calculated by time-evolution equation for the initial flavour state vector with the proper time-evolution operator $U$ in measured flavour basis, the operator $U$ can be embedded in the S Gate:

$$U(t) = S(\Phi) = \begin{pmatrix} 1 & 0 \\ 0 & \cos\theta \end{pmatrix} \tag{3.3.5}$$

Where φ $= \frac{\Delta m^2 t}{2Eh} = \frac{\Delta m^2 L}{2E}$ in natural unit, then the calculation could be finished by using the same process as in classical method.

# Limitations and Future Research

This paper is focus on the modeling of the neutrino oscillation; some approximations may obey the reality experiment data. The formulas and technology languages are all without QFT background, it's not that strict and theoretical in mathematics derivations. What's more, not all the cases include the anti-neutrino consideration, which should evolve in Dirac equation. The outlook for future research is that the calculation and simulation have the potential that entirely set up in quantum computing environment during NISQ era, it's possible to apply the simulation by using quantum algorithm.

# Contributions and Conclusions

In this paper, we have given an introduction to the neutrino oscillation, point out that the weak interaction eigenstate is not equals to the propagate mass eigenstate exactly, the relationship between these two groups of basis in mathematics can be done by a unitary transformation. In fact, this transformation matrix has already been an outcome of serval experiments, which named PMNS matrix. Under the quantum mechanical the time-evolution method is introduced to analysis the oscillation here, then we analyzed the error and difference between two and three flavour case. Even though the oscillation in matter and the dark matter is not as same as in vacuum, but we have already finished the simulations by giving an effective Hamiltonian with some approximations. In the analysis of interaction with dark matter part we compared the difference of oscillation amplitude and node distance of neutrino wave packet with different parameter matrix $a(\hat{p}), b(\hat{p})$ numerical values. and evaluated the influence of $a(\hat{p}), b(\hat{p})$ in expression $(2.3.37)$ and $(2.3.38)$. Another thing is we calculated and plotted for the probability distribution for flavour eigenstate in mass eigenstate. What's more, we discussed how the quantum computing can improve the calculation process, it is acceptable to use quantum algorithm to simulate the oscillation and fit the classical result on a real quantum computer. This may be a new way to help the future research and get profit from the power of quantum computing.

# References


[1] Marta Losada, Yosef Nir, Gilad Perez, Inbar Savoray, Yogev Shpilman: “Parametric resonance in neutrino oscillations induced by ultra-light dark matter and implications for KamLAND and JUNO”, 2022; arXiv:2205.09769.

[2] Fukuda, Y.; et al. Evidence for oscillation of atmospheric neutrinos. Physical Review Letters. 1998, 81 (8): 1562–1567. arXiv:hep-ex/9807003;doi:10.1103/PhysRevLett.81.1562

[3] 30 years after the detection of SN1987A neutrinos. Super-Kamiokande. 2017-02-23 [2017-04-14].A. Broncano, M. B. Gavela, E. Jenkins: “The Effective Lagrangian for the Seesaw Model of Neutrino Mass and Leptogenesis”, 2002, Phys.Lett.B552:177-184,2003; Erratum-ibid.B636:330-331,2006; arXiv:hep-ph/0210271. DOI: 10.1016/j.physletb.2006.04.003

[4] Three-neutrino fit based on data available in October 2021. http://www.nu-fit.org/?q=node/238

[5] G. Bellini, L. Ludhova, G. Ranucci, F. L. Villante: “Neutrino oscillations”, 2013, Advances in High Energy Physics, vol. 2014, Article ID 191960, 28 pages, 2014; arXiv:1310.7858. DOI: 10.1155/2014/191960.

[6] Ara Ioannisian, Stefan Pokorski: “Three Neutrino Oscillations in Matter”, 2018; arXiv:1801.10488. DOI: 10.1016/j.physletb.2018.06.001.

[7] Xun-Jie Xu: “Why Is The Neutrino Oscillation Formula Expanded In $Δ m_{21}^{2}/Δm_{31}^{2}$ Still Accurate Near The Solar Resonance In Matter?”, 2015; arXiv:1502.02503. DOI: 10.1007/JHEP10(2015)090.

[8] Hisakazu Minakata, Stephen J Parke: “Simple and Compact Expressions for Neutrino Oscillation Probabilities in Matter”, 2015; arXiv:1505.01826. DOI: 10.1007/JHEP01(2016)180.

[9] Ki-Young Choi, Eung Jin Chun, Jongkuk Kim: “Neutrino Oscillations in Dark Matter”, 2019; arXiv:1909.10478.

[10] B. Kayser, On the quantum mechanics of neutrino oscillation, Phys. Rev. D24 (1981), 110.

[11] C. A. Argüelles, B. J. P. Jones: “Neutrino Oscillations in a Quantum Processor”, 2019, Phys. Rev. Research 1, 033176 (2019); arXiv:1904.10559. DOI: 10.1103/PhysRevResearch.1.033176.

[12] Lunardini, C.; Smirnov, A. Yu. (7 March 2001). "Neutrinos from SN 1987A, Earth matter effects, and the large mixing angle solution of the solar neutrino problem". Physical Review D. 63 (7): 073009. arXiv:hep-ph/0009356;doi:10.1103/PhysRevD.63.073009. S2CID 119066653

[13] Francesco Capozzi, Ian M. Shoemaker, Luca Vecchi: “Neutrino Oscillations in Dark Backgrounds”, 2018; arXiv:1804.05117. DOI: 10.1088/1475-7516/2018/07/004.

[14] Maki, Z.; Nakagawa, M.; Sakata, S. (1962). "Remarks on the unified model of elementary particles". Progress of Theoretical Physics. 28 (5): 870. Bibcode:1962PThPh..28..870M.