Macroscopic Supercurrent in the superconductor

July 23, 2023

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Assume the wave function in supercurrent is:

$$\psi(r) = |\psi| e^{i\phi(r)} \tag{1}$$

So the density of probability current can be derived as:

$$\rho(r) = \psi^*(r)\psi(r) \tag{2}$$

Let the probability flow as the flow of particles, then the current density will be (where v is the velocity of the particle):

$$j_s = \rho \cdot v = q \cdot v \cdot \psi^*(r)\psi(r) \tag{3}$$

If there's a external magnetic field existed, the vector potential is $\vec{A} = \nabla \times \vec{B}$ Then the Hamiltonian for particle in the magnetic field is:

$$H = \frac{(\vec{p} - q\vec{A})^2}{2m} \tag{4}$$

And the velocity according to the equation above can be represented as:

$$v = \frac{dr}{dt} = \frac{\partial H}{\partial p} = \frac{1}{m}(\vec{p} - q\vec{A}) \tag{5}$$

And momentum p is $p = \frac{\hbar}{i}\frac{\partial}{\partial r} = \frac{\hbar}{i}\nabla$, let it substitute into the expression for velocity:

$$v = \frac{1}{m}(\frac{\hbar}{i}\nabla - q\vec{A}) \tag{6}$$

Using the expression for v and ρ to the expression for the current density j_s:

$$j_s = \psi^*(r)q \cdot \frac{1}{m}(\frac{\hbar}{i}\nabla - q\vec{A})\psi(r) \tag{7}$$

$$= \psi^*(r) \cdot [\frac{\hbar}{im}\nabla - \frac{q^2\vec{A}}{m}]\psi(r) \tag{8}$$

$$= \frac{\hbar q}{im} \cdot \nabla\psi^2(r) - \frac{q^2\vec{A}}{m}\psi^*(r)\psi(r) \tag{9}$$

In order to get the expression by using the expression for the probability current density $\rho(r)$, where $\phi(r)$ is the phase function of variable r:

$$\psi(r) = \sqrt{\rho(r)} \cdot e^{i\phi(r)} \tag{10}$$

So the gradient for the wave function can be derived by the chain rule:

$$\nabla\psi(r) = \nabla|\psi| e^{i\phi(r)} + i\nabla\phi(r)|\psi| e^{i\phi(r)} \tag{11}$$

Which means $|\psi|^2 \cdot |\phi(r)|^2$ is equals to $\psi^*\nabla\psi - \psi\nabla\psi^*$. Then the current density will be:

$$j_s(r) = -\frac{q\hbar}{2m} \cdot |\psi|^2 \cdot |\phi(r)|^2 - \frac{q^2\vec{A}}{m}\psi^*\psi \tag{12}$$

$$= -\frac{q\hbar}{2m}[\psi^*\nabla\psi - \psi\nabla\psi^*] - \frac{q^2\vec{A}}{m}\psi^*\psi \tag{13}$$

And the last term can be reduced:

$$\psi^*\psi = |\psi|^2 \longrightarrow \frac{q^2\vec{A}}{m}\psi^*\psi = \frac{q^2\vec{A}}{m}|\psi|^2 \tag{14}$$

Because in the supercurrent the carrier is cooper-pair, the mass will be $2m_e$ and it takes the charge for $-2e$. Substitute the value into the expression:

$$j_s = -\frac{e}{m}|\psi|^2(\hbar\nabla\phi + 2e\vec{A}) \tag{15}$$

Those are the derivation process for the supercurrent density of cooper-pair in the superconductor.