组合数学 1.1 (2017.9.5)

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1.加法原理

2.乘法原理

3.排列与组合

一些符号

$$P_{n,m} = n(n-1)\cdots(n-m+1)$$

$$\binom{n}{k} = \frac{n!}{m!(n-m)!}$$

$$\binom{n}{m} = 0(m > n)$$

原有概念的推广

$$\binom{-1}{n} = \frac{(-1)(-2)\cdots(-n)}{n!} = (-1)^n$$

$$\binom{\frac{1}{2}}{n} = \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-n+1)}{n!} = \frac{(-1)^{n-1}(2n-2)!}{2^{2n-1}n!(n-1)!} = \frac{(-1)^{n-1}\binom{2n-1}{n-1}}{2^{2n-1}n}$$

$$(x+y)^n = \sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^i$$

$$(1+x)^n = \sum_{i=0}^{n}\binom{n}{i}x^i = \sum_{i\geq 0}\binom{n}{i}x^i$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots = \sum_{i\geq 0}(-1)^i x^i = \sum_{i\geq 0}\binom{-1}{i}x^i$$

推广

$$(1+x)^a=\sum_{i\geq 0}\binom{a}{i}x^i \quad (|x|<1)$$

(a任意)

应用1

$$\binom{n}{0}+\binom{n}{1}+\dots+\binom{n}{n}=(1+1)^n=2^n$$

$$\binom{n}{0}-\binom{n}{1}+\dots+\binom{n}{n}=(1-1)^n=0$$

$$\binom{n}{0}+\binom{n}{2}+\dots+\binom{n}{2*\lfloor n\rfloor}=2^{n-1}$$

$$\binom{n}{1}+\binom{n}{3}+\dots+\binom{n}{2*\lfloor n\rfloor}=2^{n-1}$$

应用2

$$\sum_(i\geq 0)\binom{n}{3i}$$

考虑1的三次单位根 1 $\quad \omega 1 \quad \omega 2$

其中

$$\omega 1=e^{\pi*2i/3} \quad \omega 2=e^{\pi*4i/3}$$

$$\omega 1^3=1$$

于是可以展开$(1+\omega)^n,(1+\omega^2)^n,(1+1)^n$

由$(1+\omega+\omega^2=0)$得

$$\sum_{i\geq 0}\binom{n}{3i}=\frac{(1+\omega)^n+(1+\omega^2)^n+(1+1)^n}{3}$$

> 练习: 化简上式

4.Vandermonde恒等式

1. 求和 $\sum_{i\geq 0} i\binom{n}{i} = \binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} =?$

对$(1+x)^n = \sum_{i\geq 0}\binom{n}{i}x^i$求导得到

$$n(1+x)^{n-1} = \sum_{i\geq 0} i\binom{n}{i}x^{i-1} = \binom{n}{1} + 2\binom{n}{2}x + \cdots + n\binom{n}{n}x^{n-1}$$

令$x=1$得到

$$\sum_{i\geq 0} i\binom{n}{i} = \binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = n\cdot 2^{n-1}$$

2. $(1+x)^n(1+x)^m = (1+x)^{m+n}$

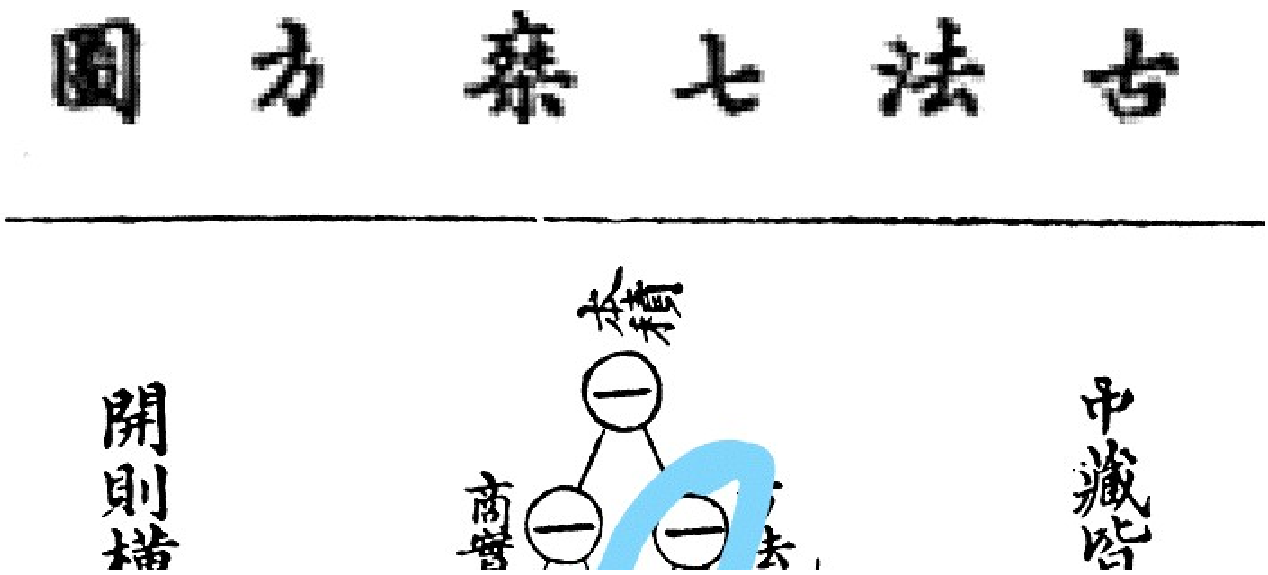
$(\sum_{i\geq 0}\binom{n}{i}x^i)(\sum_{j\geq 0}\binom{m}{j}x^j) = (\sum_{k\geq 0}\binom{n+m}{k}x^k)$

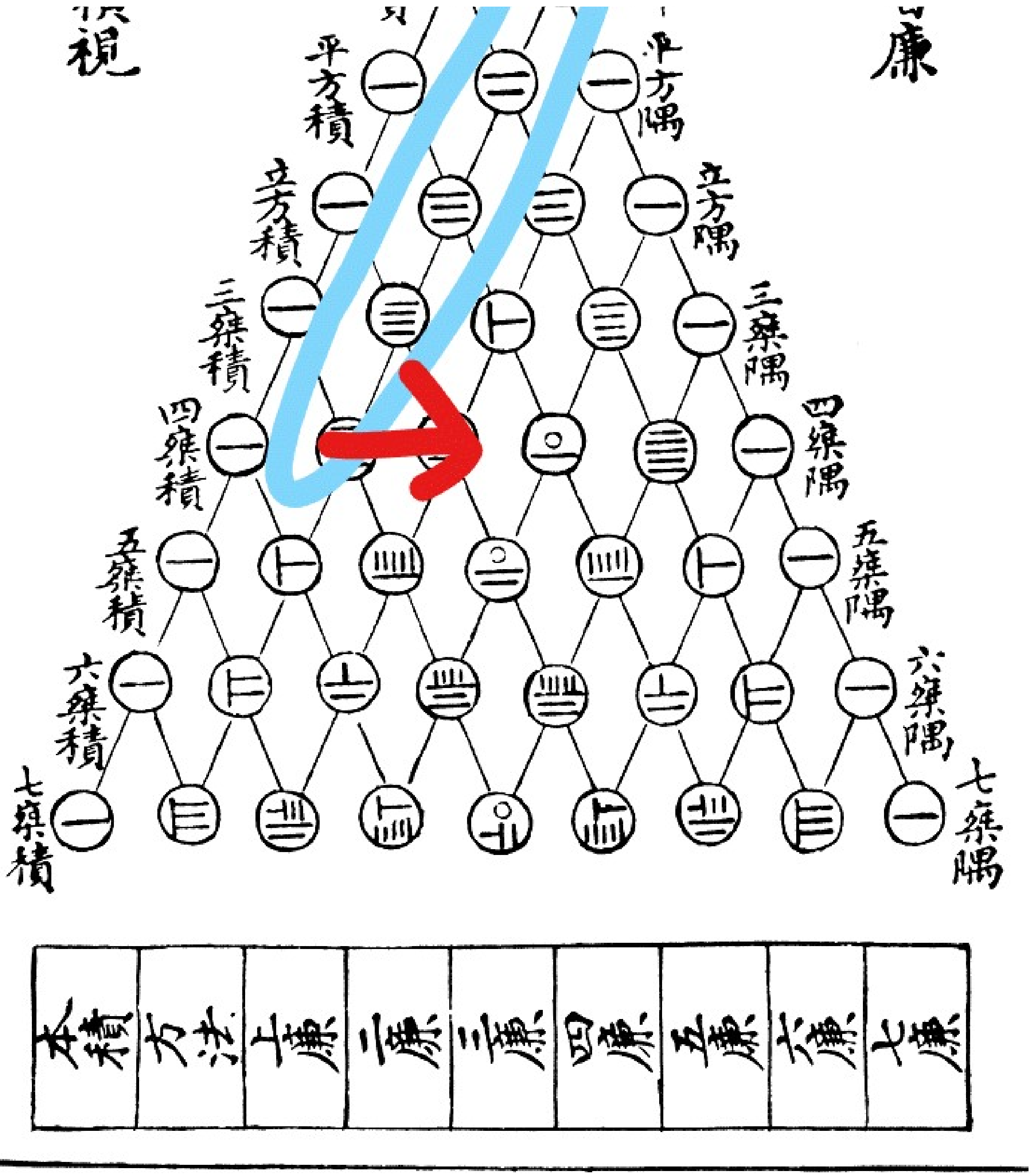
$$\underbrace{\Rightarrow \sum_{i=0}^{k}\binom{n}{i}\binom{m}{k-i} = \binom{n+m}{k}}_{Vandermonde\quad formula}$$

令$m=n=k$可得

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}^2 = 2^{2n} - 2\sum_{i=0}^{n-1}\binom{2n}{i}$$

在杨辉三角中，有以下关系





写成公式形式：

$$\binom{k}{k}+\binom{k+1}{k}+\binom{k+2}{k}+\ldots+\binom{n}{k}=\binom{n+1}{k+1}$$

5.组合数的多项式性质

考察

$$1^2+2^2+3^2+\ldots+n^2=\frac{n*(n(+1)*(2n-1)}{6}$$

其可用组合数的性质证明如下：

$$\sum \binom{k}{2} = \sum \frac{k(k+1)}{2} = \sum (k^2/2 - k/2)$$

(1)

$$\sum_{k=1}^{n} k^2 = 2\sum_{k=1}^{n} \binom{k}{2} + \sum_{k=1}^{n} k$$

(2)

> 练习: $1^3 + 2^3 + 3^3 + ... + n^3 = ?$

Tips:

$$x^2 = x(x-1) + x = 2\binom{x}{2} + \binom{x}{1}$$

(1)

$$x^3 = 6\binom{x}{3} + 2\binom{x}{2} + \binom{x}{1}$$

(2)

$$x^k == ?\binom{x}{n} + ... + ?\binom{x}{1}$$

(3)
可以发现

$$\binom{x}{n}, ..., \binom{x}{1}$$

(4)
是一组线性无关的基.

> 思考：见作业1