Interstellar Medium 2020
Problem set 5

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Problem 1: Behaviour of population ratios in the optically thick case

Consider a 2-level system (levels 0 and 1), where the Einstein coefficient for spontaneous emission from the upper level is A_{10}. Show that, if the line is optically thick, the population ratio n_1/n_0 becomes independent of A_{10}. Ignore any external radiation field and stimulated emission. (*Hint:* use the escape probability approach).

Solution

The population ratio has been calculated a number of times before and this is

$$\frac{n_1}{n_0} = \frac{k_{01}n_{\rm c}}{k_{10}n_{\rm c} + A_{10}}, \tag{1}$$

where $n_{\rm c}$ is the density of the collision partner and k_{01} and k_{10} are the collisional excitation and deexcitation rates. For the optically thick case, we replace A_{10} by βA_{10} and β is given by $\beta \approx 1/(1+0.5\tau_0) \approx 2/\tau_0$ where τ_0 is the peak optical depth of the line. An expression for τ_0 can be found in Draine Eq. (19.14) which (note that in this equation $b = \sqrt{2}\sigma_v$) can be written as

$$\tau_0 = N_0 \left(1 - \frac{n_1}{n_0}\frac{g_0}{g_1}\right) \frac{\lambda^3}{8\pi\sqrt{2\pi}\sigma_v}\frac{g_1}{g_0}A_{10}. \tag{2}$$

Substituting into the equation for n_1/n_0, we find that A_{10} drops out. This is only the case for an optically thick line, not for an optically thin line.

Problem 2: Molecular hydrogen mass from CO(1—0)

Recall that $X_{\rm CO} \equiv N({\rm H_2})/\int T_{\rm b}\, dv$ gives the relation between $N({\rm H_2})$ and the brightness temperature $T_{\rm b}$ (integrated over radial velocity v) of the CO J=1→0 line in an extended sources.

(a) Suppose that we observe CO J=1 → 0 line emission from a galaxy at distance D, with a total flux in the line $W_{\rm CO} \equiv \int F_\nu dv$, where F_ν is the flux density, v is the radial velocity, and the integral extends over the full range of radial velocities in the galaxy. Derive an expression giving the mass $M({\rm H_2})$ of $\rm H_2$ in terms of $W_{\rm CO}$, $X_{\rm CO}$, and D (and some constants).

(b) The galaxy NGC 7331, at a distance $D = 14.7$ Mpc, has $W_{\rm CO} = 4090$ Jy km s^{-1}. Calculate $M({\rm H_2})$. Assume that $X_{\rm CO} = 4 \cdot 10^{20}$ cm^{-2} (K km s^{-1})$^{-1}$.

Solution:

(a) Using in the following the notation $\Sigma({\rm H_2})$ for the mass surface density of $\rm H_2$ and $m_{\rm H}$ for the mass of an H atom:

$$
\begin{aligned}
W_{\rm CO} &= \int F_\nu \, dv \\
&= \int\int I_\nu \, d\Omega \, dv \\
&= \int\int \frac{2k\nu^2}{c^2} T_{\rm b} \, d\Omega \, dv \\
&= \int \frac{2k}{\lambda^2} \, d\Omega \cdot \int T_{\rm b} \, dv \\
&= \int \frac{2k}{\lambda^2} \, d\Omega \cdot \frac{N({\rm H_2})}{X_{\rm CO}} \\
&= \frac{2k}{\lambda^2} \frac{1}{X_{\rm CO}} \int N({\rm H_2}) \, \frac{dA}{D^2} \\
&= \frac{2k}{\lambda^2} \frac{1}{X_{\rm CO}} \frac{1}{2m_{\rm H} D^2} \int \Sigma({\rm H_2}) \, dA \\
&= \frac{k}{m_{\rm H}\lambda^2} \frac{1}{X_{\rm CO}} \frac{M({\rm H_2})}{D^2} \\
M({\rm H_2}) &= \frac{m_{\rm H}\lambda^2}{k} X_{\rm CO} D^2 W_{\rm CO}.
\end{aligned}
$$

(b)

$$
\begin{aligned}
M({\rm H_2}) = \; & \frac{1.67 \cdot 10^{-24}\ {\rm g} \cdot (0.26\ {\rm cm})^2}{1.38 \cdot 10^{-16}\ {\rm erg\ K^{-1}}} \frac{4 \cdot 10^{20}\ {\rm cm^{-2}}}{\rm K\ km\ s^{-1}} (14.7\ {\rm Mpc})^2 \times \\
& 4090 \cdot 10^{-23} {\rm erg\, cm^{-2}\ s^{-1}\ Hz^{-1}} \\
& = 2.75 \cdot 10^{43}\ {\rm g} \\
& = 1.38 \cdot 10^{10}\ {\rm M_\odot}.
\end{aligned}
$$

Problem 3: Determining CO column densities using isotopic ratios

We measure the ^{12}CO and ^{13}CO line emission of a molecular cloud in the J=1→0 line. Both lines are measured to be Gaussian, with a velocity dispersion $\sigma_v = 2$ km s^{-1}. At the peak of the line, the brightness temperature is measured to be 12.4 K for ^{12}CO(1−0) and 4.9 K for ^{13}CO(1−0). Assume that the ^{12}CO line is optically thick and the abundance ratio [^{12}CO]/[^{13}CO] = 65. Approximate by considering only the lowest 2 levels. Calculate the column density of ^{12}CO, ignoring any external radiation field, and stimulated emission. You will need these parameters for the ^{13}CO $J = 1 \rightarrow 0$ line: $A_{10} = 6.78 \cdot 10^{-8}$ s^{-1}, $\nu_{10} = 110.2$ GHz and $\lambda_{10} = 2.72$ mm. Assume that the gas density is 10^4 cm^{-3}. Explain why this last bit of information (on the density) is crucial (*Hint:* for this last point, take a look at Draine Section 19.3.2; in the rest of the problem, do not use escape probabilities but the usual Equation of Transfer).

Solution:

To begin with the last point: in Draine Section 19.3.2 we find that the critical density for CO(1−0) is given by $n_{\rm crit} = 1100\,(T_{\rm kin}/100\ {\rm K})^{0.2}$ cm^{-3} in the optically thin case. In the optically thick case (valid for the ^{12}CO line) the critical density gets multiplied by the escape probability β, which makes the critical density for ^{12}CO even lower. We conclude that $n \gg n_{\rm crit}$ for both lines. In this regime we have $T_{\rm ex} = T_{\rm kin}$ for both lines. Note that if this was not the case, we would probably have different excitation temperatures for ^{12}CO and ^{13}CO, since the line trapping for the optically thick ^{12}CO line makes its critical density much lower than that of ^{13}CO.

Now write down the equations of transfer at the line peaks:

$$T_{\rm b}(^{12}{\rm CO}) = T_{\rm ex}(^{12}{\rm CO})\left(1 - e^{-\tau_0(^{12}{\rm CO})}\right) \tag{3}$$

$$T_{\rm b}(^{13}{\rm CO}) = T_{\rm ex}(^{13}{\rm CO})\left(1 - e^{-\tau_0(^{13}{\rm CO})}\right). \tag{4}$$

Together with the above remarks on $T_{\rm ex}$ and the information on the optical depths, this leads to

$$T_{\rm b}(^{12}{\rm CO}) = T_{\rm kin} \tag{5}$$

$$T_{\rm b}(^{13}{\rm CO}) = T_{\rm kin}\left(1 - e^{-\tau_0(^{13}{\rm CO})}\right). \tag{6}$$

Inserting numbers, we find $T_{\rm kin} = 12.4$ K and $\tau_0(^{13}{\rm CO}) = 0.5$. We can use this to find $N(^{13}{\rm CO})$. Start with the general expression in Draine Eq.(19.12), which we write here as follows:

$$\tau_\nu = \frac{g_1}{g_0}\frac{A_{10}}{8\pi}\lambda_{10}^2\phi_\nu N_0\left(1 - \frac{n_1}{n_0}\frac{g_0}{g_1}\right) \tag{7}$$

$$= \frac{g_1}{g_0}\frac{A_{10}}{8\pi}\lambda_{10}^2\phi_\nu N_0\left(1 - e^{-h\nu_{10}/kT_{\rm ex}}\right), \tag{8}$$

where the last step simply uses the definition of $T_{\rm ex}$. To proceed, we note that

$$\phi_\nu = \frac{1}{\sqrt{2\pi}\sigma_\nu}e^{-(\nu-\nu_0)^2/2\sigma_\nu^2}. \tag{9}$$

Note that this is a function of frequency, but we need to go to velocity, using $v/c = (\nu_0 - \nu)/\nu_0$. Therefore $\sigma_v = \sigma_\nu c/\nu_{10} = \sigma_\nu \lambda_{10}$ (note the difference between σ_v and σ_ν). Combining, the peak optical depth becomes

$$\tau_0 = \frac{g_1}{g_0} \frac{A_{10}}{8\pi\sqrt{2\pi}\sigma_v} \lambda_{10}^3 N_0 \left(1 - e^{-h\nu_{10}/kT_{\mathrm{kin}}}\right). \tag{10}$$

Using $A_{10} = 6.78 \cdot 10^{-8}$ s^{-1}, $\nu_{10} = 110.2$ GHz and $\lambda_{10} = 2.72$ mm, we then find $N_0 = 4.7 \cdot 10^{15}$ cm^{-2}. However, this is not the total column density, which is $N_1 + N_0$. We find N_1 from

$$\frac{N_1}{N_0} = \frac{g_1}{g_0} e^{-h\nu_{10}/kT_{\mathrm{ex}}}. \tag{11}$$

The resulting total column density is $N(^{13}\mathrm{CO}) = 1.4 \cdot 10^{16}$ cm^{-2}. The ^{12}CO column density then becomes $N(^{12}\mathrm{CO}) = 9.0 \cdot 10^{17}$ cm^{-2}.