Interstellar Medium 2020
Problem set 4

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Problem 1: [O III] optical line ratio

In this problem we are going to analyse the [O III] λ4364.4 Å/λ5008.2 Å line ratio, observed from an H II region. The energy level diagram is given in Draine, Fig. 18.1 (right-hand panel). As you can see, this is treated as a 5-level system here (levels labeled from 0 to 4), but fortunately we will be able to make some useful simplifications.

(a) Discuss the sensitivity of this line ratio to temperature and density.

(b) Now make the following (good) approximations: (i) the only important spontaneous transitions are the transitions indicated by arrows in Draine Fig. 18.1 (right-hand panel); these are 4→3, 4→1, 3→2 and 3→1; all other spontaneous transitions can be ignored; (ii) the large majority of the ions is in the ground-state (level 0); (iii) we are in the low-density limit. With these approximations, write down expressions for the rate of change of the populations of levels 4 and 3 (so for dn_4/dt and dn_3/dt).

(c) Using statistical equilibrium, show that the population ratio n_4/n_3 is independent of density.

(d) Using the fact the lines are optically thin, show that the line ratio is given by

$$\frac{I(4364\text{ Å})}{I(5008\text{ Å})} = \frac{\nu_{43}}{\nu_{32}}\frac{A_{43}}{A_{32}}\frac{(A_{32}+A_{31})\,\Omega_{04}e^{-h\nu_{43}/kT}}{(A_{43}+A_{41})\,\Omega_{03}+A_{43}\Omega_{04}e^{-h\nu_{43}/kT}},$$

where on the right-hand side of the =-sign the subscripts denote the energy levels indicated in Draine, Fig. 18.1, and the symbols have their usual meaning. Another thing that you need to know is that the upward collision strength (Ω_{lu} where $E_l < E_u$) follows from

$$k_{lu} = 8.629\cdot 10^{-8} T_4^{-1/2}\frac{\Omega_{lu}}{g_l}e^{-E_{ul}/kT}.$$

This implies that $\Omega_{ul} = \Omega_{lu}$ (both Ω's are dimensionless numbers close to 1, and approximately independent of T). In the lectures I had only mentioned the downward collision strength, but here you will need the upward collision strength as given above.

Solution:

(a) Since the upper level temperatures are significantly different (relative to the typical H II region temperature of ~ 8000 K), the line ratio is sensitive to temperature. In the line ratio, the density dependence drops out in the low-density limit (when the density is much lower than the critical density of both lines) or in the high-density limit (when the density is much higher than the critical density of both lines).

(b) It is important to understand what these approximations imply. The implications of approximation (i) are clear. The second approximation implies that collisional excitation is only important from level 0, since all other levels have very low populations. The third approximation implies that the density is much lower than the critical density (for both lines), so collisional deexcitation can be ignored. Taking all of this into account, the rate of change of the level populations can be written as

$$\begin{aligned}\frac{dn_4}{dt} &= -A_{43}n_4 - A_{41}n_4 + k_{04}n_0n_e \\ \frac{dn_3}{dt} &= -A_{32}n_3 - A_{31}n_3 + k_{03}n_0n_e + A_{43}n_4.\end{aligned}$$

(c) Statistical equilibrium implies that the derivatives calculated above can be set to zero, which gives

$$\begin{aligned}\frac{n_4}{n_0} &= \frac{k_{04}n_e}{A_{43}+A_{41}} \\ \frac{n_3}{n_0} &= \frac{n_e}{A_{32}+A_{31}}\left(k_{03} + \frac{A_{43}k_{04}}{A_{43}+A_{41}}\right).\end{aligned}$$

Taking the ratio of these gives the ratio n_4/n_3 and n_e drops out, so the ratio is independent of density.

(d) If the lines are optically thin, their intensities can be calculated as

$$I_{43} = \frac{h\nu_{43}}{4\pi}A_{43}N_4$$

and similarly for I_{32}. The line ratio now involves taking the ratio of the column densities N_4 and N_3. This is the same as the ratio of the densities n_4 and n_3. Substitution then gives

$$\frac{I(4364\ \text{Å})}{I(5008\ \text{Å})} = \frac{\nu_{43}}{\nu_{32}}\frac{A_{43}}{A_{32}}\frac{k_{04}\left(A_{32}+A_{31}\right)}{k_{03}\left(A_{43}+A_{41}\right)+A_{43}k_{04}}.$$

Using then the relation between the collisional excitation coefficients and the upward collision strengths, this can be written as

$$\frac{I(4364\ \text{Å})}{I(5008\ \text{Å})} = \frac{\nu_{43}}{\nu_{32}}\frac{A_{43}}{A_{32}}\frac{\Omega_{04}\left(A_{32}+A_{31}\right)e^{-E_{40}/kT}}{\Omega_{03}\left(A_{43}+A_{41}\right)e^{-E_{30}/kT}+\Omega_{04}A_{43}e^{-E_{40}/kT}}.$$

Since $E_{40} = E_{43} + E_{30}$, this leads to

$$\begin{aligned}\frac{I(4364\ \text{Å})}{I(5008\ \text{Å})} &= \frac{\nu_{43}}{\nu_{32}}\frac{A_{43}}{A_{32}}\frac{\Omega_{04}\left(A_{32}+A_{31}\right)e^{-E_{43}/kT}}{\Omega_{03}\left(A_{43}+A_{41}\right)+\Omega_{04}A_{43}e^{-E_{43}/kT}} \\ &= \frac{\nu_{43}}{\nu_{32}}\frac{A_{43}}{A_{32}}\frac{\left(A_{32}+A_{31}\right)\Omega_{04}e^{-h\nu_{43}/kT}}{\left(A_{43}+A_{41}\right)\Omega_{03}+A_{43}\Omega_{04}e^{-h\nu_{43}/kT}}.\end{aligned}$$

Problem 2: Cooling with the [C II] 158 μm line

One of the most important coolants of the ISM is the [C II] 158 μm line. This is a fine-structure line in the ground state ($^2P_{3/2} -^2 P_{1/2}$), with an Einstein A coefficient of $A_{10} = 2.4 \cdot 10^{-6}$ cm^{-3} and an upper level with $(E_1 - E_0)/k = 91.2$ K.

(a) Write down expressions for the number density of C$^+$ ions in the upper (n_1) and lower state (n_0), in terms of the total number density of ions (n_{C^+}). Assume that the effect of absorption and stimulated emission can be ignored.

(b) We now assume that collisional excitation of the upper state is primarily due to electrons, and the collisional deexcitation coefficient for this process is $k_{10} = 3.9{\cdot}10^{-7}(T/100\ \mathrm{K})^{-1/2}$ cm^3 s^{-1}. Calculate the critical density of the transition for this process.

(c) Assume that $n_e \ll n_{\mathrm{crit}}$ and write down an expression for the cooling rate Λ_{C^+} (in units erg cm^{-3} s^{-1}), in terms of the densities of electrons and C$^+$ ions.

(d) Do the same thing for the case $n_e \gg n_{\mathrm{crit}}$.

(e) Now assume that there is heating rate from the photoelectric effect that is given by

$$\Gamma_{\mathrm{pe}} = 5 \cdot 10^{-26} n_{\mathrm{H}}\ \mathrm{erg\ s^{-1}\ cm^{-3}}, \tag{1}$$

where n_{H} is the density of hydrogen nuclei. Assume that the carbon abundance is $3 \cdot 10^{-4}$ and that all carbon is in the form of C$^+$. Furthermore, assume that $n_e = 50$ cm^{-3}. What is the resulting temperature in this medium?

Solution:

(a) Considering the processes (de)populating the upper level, we can write the equilibrium condition

$$\frac{dn_1}{dt} = k_{01} n_{\mathrm{c}} n_0 - k_{10} n_{\mathrm{c}} n_1 - A_{10} n_1 = 0, \tag{2}$$

where n_{c} is the density of the collision partner, and k_{01} an k_{10} are the collisional excitation and deexcitation rate. With $n_0 + n_1 = n_{\mathrm{C}^+}$ this works out to

$$n_0 = n_{\mathrm{C}^+} \frac{k_{10} n_{\mathrm{c}} + A_{10}}{(k_{01} + k_{10}) n_{\mathrm{c}} + A_{10}} \tag{3}$$

$$n_1 = n_{\mathrm{C}^+} \frac{k_{01} n_{\mathrm{c}}}{(k_{01} + k_{10}) n_{\mathrm{c}} + A_{10}}. \tag{4}$$

(b) The critical density is given by

$$n_{\mathrm{crit}} = \frac{A_{10}}{k_{10}} \tag{5}$$

so we find

$$n_{\mathrm{crit}} = 6.2 \left(\frac{T}{100\ \mathrm{K}}\right)^{\frac{1}{2}}\ \mathrm{cm^{-3}}. \tag{6}$$

(c) The number of photons emitted per unit volume and per unit of time is $A_{10}n_1$. Every photon carries an energy $h\nu$ where ν is the frequency of the 158 μm line. The cooling rate is then

$$\Lambda_{\mathrm{C}^+} = h\nu\, A_{10} n_{\mathrm{C}^+} \frac{k_{01} n_e}{(k_{01} + k_{10}) n_e + A_{10}}. \tag{7}$$

However, we also know that $n_e \ll n_{\mathrm{crit}}$, so that the collisional deexcitation term can be neglected (compared to the spontaneous emission term). So we obtain

$$\Lambda_{\mathrm{C}^+} = h\nu\, A_{10} n_{\mathrm{C}^+} \frac{k_{01} n_e}{k_{01} n_e + A_{10}}. \tag{8}$$

(d) Now the spontaneous emission can be ignored and we obtain

$$\Lambda_{\mathrm{C}^+} = h\nu\, A_{10} n_{\mathrm{C}^+} \frac{k_{01}}{k_{01} + k_{10}}. \tag{9}$$

(e) With $n_e = 50$ cm^{-3} we seem to be in the regime where $n_e \gg n_{\mathrm{crit}}$ unless the resulting temperature is very high. This needs to be verified afterwards. Assuming then that $n_e \gg n_{\mathrm{crit}}$, we start with the result of the previous part. We furthermore write $n_{\mathrm{C}^+} = 3 \cdot 10^{-4} n_{\mathrm{H}}$. We also need the relation between k_{01} and k_{10} which is given by

$$k_{01} = k_{10} \frac{g_1}{g_0} e^{-91.2/T}. \tag{10}$$

Here we need the g_1 and g_0 which are 4 (for $J = 3/2$ in the upper level) and 2 (for $J = 1/2$ in the lower level. Collecting, we obtain

$$\Gamma_{\mathrm{pe}} = \Lambda_{\mathrm{C}^+} \tag{11}$$

$$5 \cdot 10^{-26} n_{\mathrm{H}} = h\nu\, A_{10}\, 3 \cdot 10^{-4} n_{\mathrm{H}} \frac{k_{01}}{k_{01} + k_{10}} \tag{12}$$

$$1.7 \cdot 10^{-22} = h\nu\, A_{10} \frac{2e^{-91.2/T}}{2e^{-91.2/T} + 1} \tag{13}$$

$$0.01 = \frac{2e^{-91.2/T}}{2e^{-91.2/T} + 1}. \tag{14}$$

This can easily be solved to give $T \approx 130$ K. At this temperature, $n_{\mathrm{crit}} \approx 8$ cm^{-3}, so our use of $n_e \gg n_{\mathrm{crit}}$ was justified.