Interstellar Medium 2020
Problem Set 3

Paul van der Werf

Oktober 22, 2020

1 Problem 1: A density-tracing line ratio

We have seen that certain line ratios are temperature-independent and can be used to measure density in a certain density range. But in the limits of high or low densities these line ratios become constant (see for instance Fig. 18.4 in Draine). Here we are going to calculate expressions for the values that the line ratios obtain in the high-density limit and the low-density limit. To be specific, we are going to consider a ratio like that of the [O II] lines at 3729.8 Å and 3727.1 Å. The energy level diagram is given in Draine, Fig. 18.3. We are going to simplify this by only considering levels 0, 1 and 2 (and ignore the higher levels). So we are considering a 3-level system. In addition we make the simplification that there are no transitions (radiative or collisional) between levels 1 and 2, and that the energies of levels 1 and 2 are so close that we can ignore the energy difference (these are good approximations). Recall that the collisional deexcitation rate k_{10} can be written as

$$k_{10} = \frac{8.629 \cdot 10^{-8}}{\sqrt{T_4}} \frac{\Omega_{10}}{g_1},$$

T_4 is the temperature in units of 10^4 K and Ω_{10} is a dimensionless number (called the collision strength), which does not depend on T. Derive expressions (in terms of only collision strengths, Einstein A values and level degeneracies) for the line ratio in the low-density limit and in the high-density limit. Assume that both lines are optically thin. You will also need to use the relation between the collisional excitation rate and deexcitation rates, which is given by

$$k_{01} = \frac{g_1}{g_0} k_{10} e^{-E_{10}/kT}.$$

Solution

Since the lines are optically thin, we can ignore absorption (from any level). Also, since we are in the optical regime, we can ignore stimulated emission. So we only have to consider spontaneous emission and collisional (de)excitation. Furthermore, since there are no transitions between levels 1 and 2, we have $A_{21} = k_{21} = k_{12} = 0$. We can now set up the equations of statistical equilibrium:

$$\frac{dn_2}{dt} = k_{02} n_e n_0 - k_{20} n_2 n_e - A_{20} n_2$$
$$\frac{dn_1}{dt} = k_{01} n_e n_0 - k_{10} n_1 n_e - A_{10} n_1.$$

In statistical equilibrium these derivatives should be 0, so we obtain

$$\begin{aligned}
\frac{n_2}{n_0} &= \frac{k_{02} n_e}{A_{20} + k_{20} n_e} \\
\frac{n_1}{n_0} &= \frac{k_{01} n_e}{A_{10} + k_{10} n_e}.
\end{aligned}$$

Now recall that for an optically thin emission line (not affected by stimulated emission) we can write for the total intensity (integrated over the line):

$$I_{20} = \frac{h\nu_{20}}{4\pi} A_{20} N_2,$$

and similarly for I_{10}. Using the fact that the upper levels are very close together (so $\nu_{20} \approx \nu_{10}$), we can write the line ratio as

$$\frac{I_{20}}{I_{10}} = \frac{A_{20}}{A_{10}} \frac{k_{02}}{k_{01}} \frac{A_{10} + k_{10} n_e}{A_{20} + k_{20} n_e}.$$

Now first take the low-density limit. Here we find

$$\frac{I_{20}}{I_{10}} = \frac{k_{02}}{k_{01}}$$

(why is this exactly the result we expected?), which can be converted to

$$\frac{I_{20}}{I_{10}} = \frac{g_2}{g_1} \frac{k_{20}}{k_{10}} = \frac{\Omega_{20}}{\Omega_{10}}.$$

So the ratio is in this case a constant, depending only on the collision strengths.

In the high density limit, we obtain

$$\frac{I_{20}}{I_{10}} = \frac{A_{20}}{A_{10}} \frac{k_{02}}{k_{20}} \frac{k_{10}}{k_{01}} = \frac{A_{20}}{A_{10}} \frac{g_2}{g_1},$$

again a constant, now depending only on atomic parameters.

2 Problem 2: Collisional mixing of ortho- and para-H_2

In class we discussed the properties of the H_2 molecule. When the proton spins in H_2 are parallel, we have para-H_2 which can have rotational quantum numbers $J = 0, 2, 4, ...$ If the spins are antiparallel, we have ortho-H_2 for which only odd values of J are possible. Radiative transitions between ortho- and para-H_2 are strongly forbidden; however, transitions between ortho- and para-H_2 may occur by collisions (at a low rate). Here we are going to estimate if this process is sufficiently effective that significant mixing between the ortho- and para-species occurs.

Let $H_2(J)$ denote H_2 with rotational quantum number J. The rate coefficient for the collisional process

$$\mathrm{H_2}(1) + \mathrm{H_2}(0) \to \mathrm{H_2}(0) + \mathrm{H_2}(0) \tag{1}$$

is estimated to be $k_{10} = 1.56 \cdot 10^{-28}\ \mathrm{cm^3\ s^{-1}}$. The energy difference between the $J = 1$ and $J = 0$ states is $\Delta E/k = 170.5$ K.

(a) Calculate the rate coefficient k_{01} for the process

$$\mathrm{H_2}(0) + \mathrm{H_2}(0) \to \mathrm{H_2}(1) + \mathrm{H_2}(0), \tag{2}$$

keeping the kinetic temperature T a a free parameter. *Careful:* there are two factors to consider for the statistical weight: the angular momentum quantum number of the entire molecule and the total spin of the 2 nuclei.

(b) In a molecular cloud with $T = 50$ K, what is the equilibrium ratio $n(J = 1)/n(J = 0)$ if only collisions with H_2 are acting?

(c) If the ortho-para ratio at $t = 0$ differs from the equilibrium value, the deviation from equilibrium will decay exponentially with an e-folding time τ, where

$$\tau = \frac{1}{n(\mathrm{H_2})(k_{01} + k_{10})}.$$

Calculate τ for $n(H_2) = 10^6\ \mathrm{cm^{-3}}$ and $T = 50$ K, assuming that the only processes causing ortho-para conversion are collision-induced transitions between the $J = 1$ and $J = 0$ states, with $H_2(0)$ as the collision partner. What do you conclude?

Solution

(a) The statistical weights are $g(J) = (2J + 1)g_\mathrm{n}(J_\mathrm{n})$. The first factor is for the rotational quantum number of the molecule. The second factor is for the total spin of the 2 nuclei, and this total nuclear spin J_n is 1 for parallel spins (para-H_2, J even) and 0 for antiparallel spins (ortho-H_2, J odd). So $g(0) = 1$ and $g(1) = 3 \times 3 = 9$. Therefore, using the relation between collisional excitation and deexcitation rates

$$k_{01} = 1.404 \cdot 10^{-27} e^{-170.5\,\mathrm{K}/T}. \tag{3}$$

(b) Since there are no radiative transitions, only collisions, the equilibrium situation is given by the Boltzmann equation and therefore

$$n(J = 1)/n(J = 0) = 9e^{-170.5\,\mathrm{K}/T} = 0.297. \tag{4}$$

(c) We obtain

$$\tau = \frac{1}{n(\mathrm{H_2})(k_{01} + k_{10})} = 1.6 \cdot 10^{14}\ \mathrm{yr}. \tag{5}$$

This is much longer than the age of the Universe, so collisional conversion between ortho- and para-H_2 is in practice totally irrelevant.

3 Problem 3: Temperature and density in an H II region

The observed spectrum of an H II region has

$$\frac{I([\text{O III}]4364.4\ \text{Å})}{I([\text{O III}]5008.2\ \text{Å})} = 0.003 \tag{6}$$

$$\frac{I([\text{O II}]3729.9\ \text{Å})}{I([\text{O II}]3727.1\ \text{Å})} = 1.2 \tag{7}$$

(a) Ignoring the effects of interstellar dust, estimate the electron temperature T_e and the electron density n_e of the H II region. (*Hint:* use the diagrams in Draine Chapter 18, taking into account that in these diagrams $T_4 \equiv T_\mathrm{e}/10^4$ K).

(b) Now suppose it is discovered that there is foreground dust, absorbing more strongly at shorter wavelengths, such that $A(4364.4\ \text{Å}) - A(5008.2\ \text{Å}) = 0.31$, where $A(\lambda)$ is the extinction at wavelength λ, in magnitudes. Calculate again T_e and n_e.

Solution

(a) Reading off from Draine Fig. 18.2: if $n_e \lesssim 10^4\ \text{cm}^{-3}$ (i.e., in the low-density limit), then $I([\text{O III}]4364.4\ \text{Å})/I([\text{O III}]5008.2\ \text{Å}) = 0.003$ implies $T_\mathrm{e} \approx 8000$ K. At higher densities, different temperatures will result, so we will have to verify that we are actually in the low-density limit.

Reading off from Draine Fig. 18.4: $I([\text{O II}]3729.9\ \text{Å})/I([\text{O II}]3727.1\ \text{Å}) = 1.2$ implies $n_e T_4^{-1/2} \approx 240\ \text{cm}^{-3}$ (note that $T_4 = T_\mathrm{e}/10^4$). This confirms that we are in the low-density limit (since $T_4 \sim 1$ for H II regions). So we find $T_\mathrm{e} \approx 8000$ K and $n_e \approx 215\ \text{cm}^{-3}$.

(b) If $A(4364.4\ \text{Å}) - A(5008.2\ \text{Å}) = 0.31$ magnitudes, then the line ratio corrected for extinction is $I([\text{O III}]4364.4\ \text{Å})/I([\text{O III}]5008.2\ \text{Å}) = 0.003 \times 10^{0.4\times 0.31} = 0.004$ (recall that 2.5^m of extinction is an attenuation by a factor 10). Then, from Draine Fig. 18.2, we infer $T_\mathrm{e} \approx 8600$ K.

Because the two [O II] lines are very close in wavelength, the line ratio need not be corrected for differential extinction. Thus $n_e \approx 240 \times (0.86)^{1/2} \approx 220\ \text{cm}^{-3}$.