Interstellar Medium 2020
Problem set 1

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1 Problem 1: Observing H I emission

An extended local atomic hydrogen cloud is located between us and the cosmic microwave background with temperature $T_{\mathrm{CMB}} = 2.7255$ K. We observe the cloud with a radiotelescope in the H I 21 cm line. Suppose that the H I in the cloud has spin temperature $T_{\mathrm{s}} = 50$ K, and that the optical depth in the centre of the line is $\tau_0 = 0.1$. What will be the Rayleigh-Jeans brightness temperature (expressed in K) and intensity (expressed in Jansky per steradian) at the centre of the line?

Solution

In brightness temperature, the equation of transfer for this problem is (at the centre of the line):

$$T_{\mathrm{B}} = T_{\mathrm{CMB}}\, e^{-\tau_0} + T_{\mathrm{s}} \left(1 - e^{-\tau_0}\right), \tag{1}$$

which yields $T_{\mathrm{B}} = 7.224$ K.

In intensity, the equation of transfer is (at the centre of the line)

$$I_\nu = B_\nu(T_{\mathrm{CMB}})\, e^{-\tau_0} + B_\nu(T_{\mathrm{s}}) \left(1 - e^{-\tau_0}\right). \tag{2}$$

Since we are in the Rayleigh-Jeans regime, this can be written to good accuracy as

$$I_\nu \approx \frac{2k}{\lambda^2} \left[T_{\mathrm{CMB}}\, e^{-0.1} + T_{\mathrm{s}} \left(1 - e^{-0.1}\right)\right], \tag{3}$$

which gives the result in erg cm^{-2} s^{-1} Hz^{-1} sr^{-1} if all values (k, λ and temperatures) are in cgs units. So we find

$$\begin{aligned} I_\nu &= \frac{2k}{(21.11\ \mathrm{cm})^2} \cdot 7.224\ \mathrm{K} && (4)\\ &= 4.476 \cdot 10^{-18}\ \mathrm{erg\ cm^{-2}\ s^{-1}\ Hz^{-1}\ sr^{-1}} && (5)\\ &= 4.476 \cdot 10^{5}\ \mathrm{Jy\ sr^{-1}}. && (6) \end{aligned}$$

2 Problem 2: Calculating H I mass

In class, the following expression was derived for calculating H I mass from optically thin H I 21 cm line emission:

$$M(\text{H I}) = \frac{2.343 \cdot 10^5}{M_\odot} \left(\frac{D}{\text{Mpc}}\right)^2 \left(\frac{\int F_\nu dv}{\text{Jy km s}^{-1}}\right).$$

Derive this relation, starting from the expression for calculating column density for optically thin H I emission:

$$\frac{N(\text{H I})}{\text{cm}^{-2}} = 1.813 \cdot 10^{18} \frac{\int T_\text{b}\, dv}{\text{K km s}^{-1}}.$$

Solution

To calculate the H I mass from a H I column density, we need to integrate over the area A of the source, and multiply by the mass of an H I atom (m_HI). So

$$M(\text{H I}) = m_\text{HI} \int N(\text{H I})\, dA.$$

Now dA is related to the solid angle $d\Omega$ subtended by the source, by the expression $d\Omega = dA/D^2$, where D is the distance to the source. So we can write

$$M(\text{H I}) = m_\text{HI} D^2 \int N(\text{H I})\, d\Omega$$

which yields

$$M(\text{H I}) = 1.813 \cdot 10^{18} m_\text{HI} D^2 \int\int T_\text{b}\, dv\, d\Omega,$$

with v in km s^{-1} and the rest in cgs units. Since we are in the Rayleigh-Jeans regime, we can go from brightness temperature units to specific intensity units by the expression $T_\text{b} = (\lambda^2/2k)I_\nu$. We can then write

$$M(\text{H I}) = 1.813 \cdot 10^{18} m_\text{HI} D^2 \frac{\lambda^2}{2k} \int\int I_\nu\, d\Omega\, dv,$$

still with v in km s^{-1} and the rest in cgs units. Now since the flux density F_ν is the specific intensity integrated over solid angle, i.e., $F_\nu = \int I_\nu\, d\Omega$, we obtain

$$M(\text{H I}) = 1.813 \cdot 10^{18} m_\text{HI} D^2 \frac{\lambda^2}{2k} \int F_\nu\, dv,$$

still with v in km s^{-1} and the rest in cgs units. Now, making use of the facts that 1 pc $= 3.084 \cdot 10^{18}$ cm, that $\lambda = 21.106$ cm, that 1 Jy $= 10^{-23}$ erg s^{-1} cm^{-2} Hz^{-1} and that $m_\text{HI} = 8.514 \cdot 10^{-58}$ M$_\odot$, we find

$$M(\text{H I}) = (1.813 \cdot 10^{18})(8.514 \cdot 10^{-58})(3.084 \cdot 10^{24})^2 \left(\frac{D}{\text{Mpc}}\right)^2 \frac{21.106^2}{2 \cdot 1.38 \cdot 10^{-16}} \cdot 10^{23} \left(\frac{\int F_\nu\, dv}{\text{Jy km s}^{-1}}\right),$$

with $M(\text{H I})$ in M$_\odot$. This works out to

$$M(\text{H I}) = 2.343 \cdot 10^5 \text{ M}_\odot \left(\frac{D}{\text{Mpc}}\right)^2 \left(\frac{\int F_\nu dv}{\text{Jy km s}^{-1}}\right).$$

3 Problem 3: Radio continuum optical depth

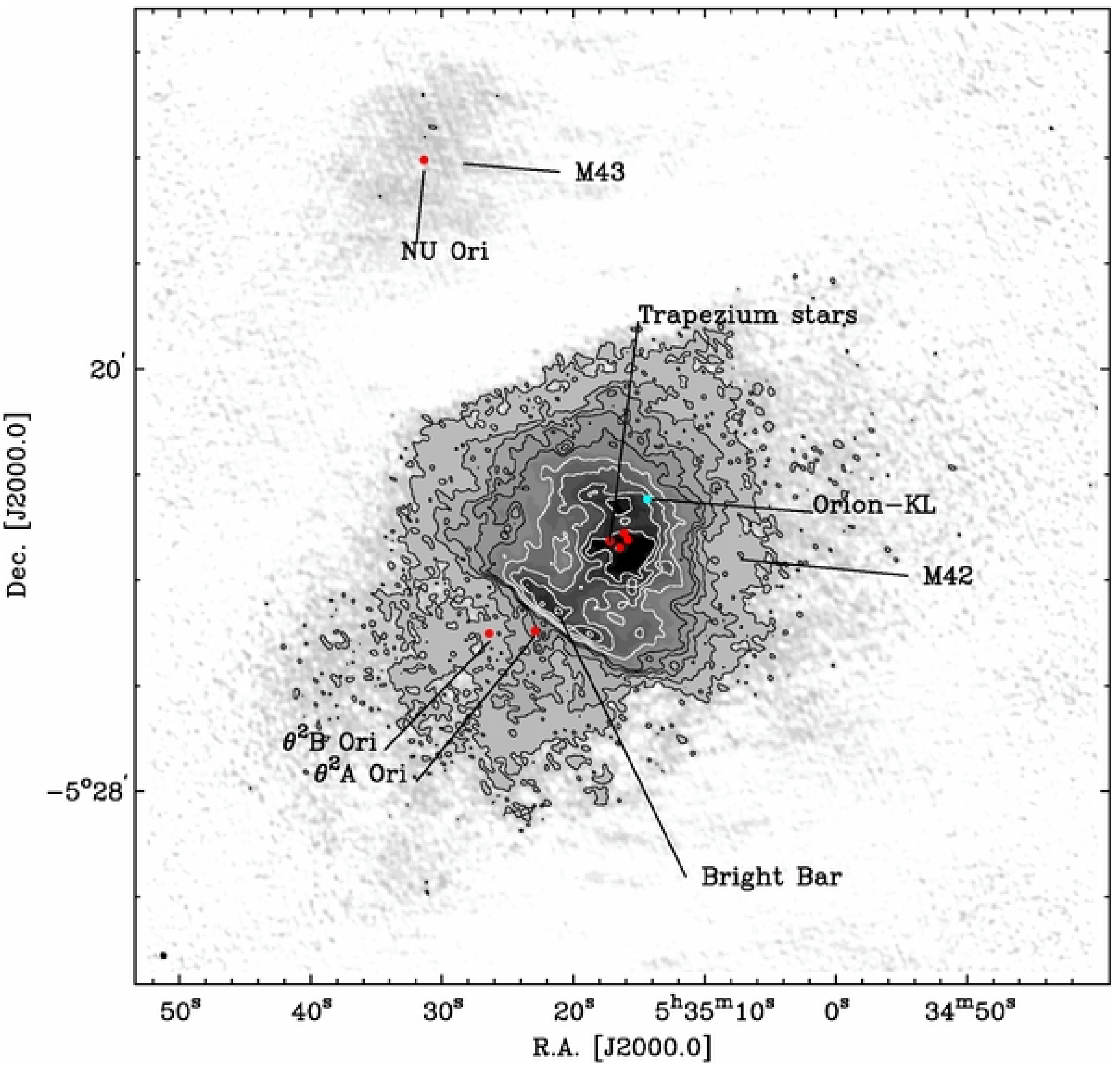

Figure 1: Continuum emission of the Orion Nebula at 1420.4 MHz. Contours indicate specific intensity levels of 20, 40, 60, 80, and 100 mJy beam^{-1} (black contours) and 150, 200, 250, 300, and 350 mJy beam^{-1} (white contours). The small ellipse in the lower left-hand corner indicates the FWHM size and the orientation of the synthesized beam ($7''.2 \times 5''.7$ at a position angle of $29^\circ.7$).

Figure 1 shows a 1420.4 MHz radio continuum image of the Orion Nebula, observed with the JVLA (from Van der Werf et al., 2013, ApJ, 762, 101). The temperature of the Orion Nebula is about 8400 K (this is also the excitation temperature). Use the information in the caption and derive the 1420.4 MHz continuum optical depth of the Orion Nebula at the brightest position. Note that the synthesized beam indicated in the caption is a 2-dimensional Gaussian. You need to take this into account.

Solution

The tricky part here is calculating the solid angle of the beam, Ω_b. It would be wrong to simply calculate it as $\Omega_b = 7\farcs2 \times 5\farcs7 = 41$ arcsec^2, since that would mean assuming the beam is a rectangular block, while in reality it is a 2-D Gaussian (with peak unity). The integral of such a Gaussian is $\Omega_b = 2\pi\sigma_x\sigma_y$, where σ_x and σ_y are the dispersions along the principal axes. Since FWHM $= \sqrt{8\ln 2}\,\sigma$, the beam solid angle is then $\Omega_b = (\pi/4\ln 2)\,\mathrm{FWHM}_x\,\mathrm{FWHM}_y = 1.133\,\mathrm{FWHM}_x\,\mathrm{FWHM}_y$. Inserting numbers, we get $\Omega_b = 46.5$ arcsec$^2 = 1.09 \cdot 10^{-9}$ sr. The peak specific intensity (read off from the highest contour) is then 350 mJy beam^{-1} $= 3.2 \cdot 10^{-15}$ erg s^{-1} cm^{-2} Hz^{-1} sr^{-1}. Now convert into brightness temperature by multiplying with $\lambda^2/2k$, so we find a peak brightness temperature $T_\mathrm{b} = 5170$ K. The optical depth now follows from

$$T_\mathrm{b} = T_\mathrm{e}\left(1 - e^{-\tau}\right),$$

with $T_\mathrm{b} = 5170$ K and $T_\mathrm{e} = 8400$ K, which yields $\tau = 0.96$.

4 Problem 4: Masers

In class we discussed maser emission. Can a maser also be seen in absorption? Prove your answer.

Solution

In class we looked at emission and absorption by subtracting the continuum level (T_c) from the spectrum. Call the resulting spectrum T_sub, then we can write

$$T_\mathrm{sub} = T_\mathrm{b} - T_\mathrm{c} = (T_\mathrm{ex} - T_\mathrm{c})\left(1 - e^{-\tau_\nu}\right).$$

Since for a maser $T_\mathrm{ex} < 0$, the first factor is negative. But a maser also has $\tau_\nu < 0$, so the second factor is also negative. So $T_\mathrm{sub} > 0$, hence a maser can only be observed in emission.

5 Problem 5: Conditions for maser emission

Maser emission can occur when the process of populating a certain energy level is easier than that of other, lower levels. Here we will consider an idealized molecule to illustrate how this works. Our molecule will have three levels, denoted 0, 1 and 2, ordered according to increasing energy, $E_0 < E_1 < E_2$. The degeneracies of these levels will be g_0, g_1 and g_2. Now suppose that there is a radiation field at energy $h\nu_{20} = E_2 - E_0$, consisting of radiation from an external source plus any emission in the 2→0 transition.

Let ζ_{02} be the absorption probability per unit time for a molecule in level 0, with a transition to level 2. Let A_{20}, A_{21} and A_{10} be the Einstein A coefficients for decays 2→0, 2→1 and 1→0. Ignore collisions.

(a) Ignoring possible absorption or stimulated emission in the 2→1 and 1→0 transitions, obtain an expression for the ratio n_1/n_0, where n_i is the number density of molecules in level i.

(b) How large must ζ_{02} be for this molecule to act as a maser in the 1→0 transition?

(c) Is it possible to have maser emission in the 2→1 transition? If so, what condition(s) must be satisfied?

Solution

(a) Radiation with $h\nu = E_2 - E_0$ will generate both absorption (proportional to the Einstein coefficient B_{02}) and stimulated emission (proportional to the Einstein coefficient B_{20}). Therefore, using the relation between B_{02} and B_{20}, if ζ_{02} is the probability per unit time for a molecule in level 0 to absorb a photon, the probability per unit time for a molecule in level 2 to undergo stimulated emission is $(g_0/g_2)\zeta_{02}$. From statistical equilibrium, the population of level 2 must therefore satisfy

$$n_0\zeta_{02} = n_2[A_{21} + A_{20} + (g_0/g_2)\zeta_{02}] \tag{7}$$

$$\frac{n_2}{n_0} = \frac{\zeta_{02}}{A_{21} + A_{20} + (g_0/g_2)\zeta_{02}}, \tag{8}$$

while for level 1 we have

$$n_2 A_{21} = n_1 A_{10} \tag{9}$$

$$\frac{n_2}{n_1} = \frac{A_{10}}{A_{21}}, \tag{10}$$

so that

$$\frac{n_1}{n_0} = \frac{\zeta_{02}}{A_{21} + A_{20} + (g_0/g_2)\zeta_{02}} \frac{A_{21}}{A_{10}}. \tag{11}$$

(b) To have a maser in this transition, we must have $T_{\rm ex} < 0$, which happens if $n_1/n_0 > g_1/g_0$. Therefore the requirement on ζ_{02} will be

$$\zeta_{02} > \frac{g_1}{g_0} \frac{A_{10}(A_{21} + A_{20})}{A_{21} - (g_1/g_2)A_{10}}. \tag{12}$$

What this means is that the rate of populating level 2 must be fast enough to cause an inversion in the levels below it.

(c) To have a maser in the $2 \to 1$ transition, we must have $n_2/n_1 > g_2/g_1$, which will occur provided

$$\frac{A_{10}}{A_{21}} > \frac{g_2}{g_1}. \tag{13}$$

Note that this does not depend on ζ_{02}, it is just a property of the levels.