第五章习题

习题 1 导出下列常系数线性抛物型方程初边值问题

$$
\begin{cases}
\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + f(x,t), & 0 < x < l,\ 0 < t \leqslant T \\
u(x,0) = \phi(x), & 0 < x < l \\
u(0,t) = u(l,t) = 0, & 0 \leqslant t \leqslant T
\end{cases} \tag{1}
$$

的向后差分格式

$$
\frac{u_j^{k+1} - u_j^k}{\tau} = a\frac{u_{j+1}^{k+1} - 2u_j^{k+1} + u_{j-1}^{k+1}}{h^2} + f_j^{k+1} \tag{2}
$$

六点对称格式

$$
\frac{u_j^{k+1} - u_j^k}{\tau} = \frac{a}{2}\Big[\frac{u_{j+1}^{k+1} - 2u_j^{k+1} + u_{j-1}^{k+1}}{h^2} + \frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{h^2}\Big] + \frac{1}{2}[f_j^{k+1} + f_j^k] \tag{3}
$$

Richardson 格式

$$
\frac{u_j^{k+1} - u_j^{k-1}}{2\tau} = a\frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{h^2} + f_j^k \tag{4}
$$

的截断误差.

解: 首先, 计算向后差分格式的截断误差

把 (2) 中数值解换为真解, 并左端减右端, 有

$$
\begin{aligned}
R_j^k(u) \quad = \quad & \frac{u(x_j,t_{k+1})-u(x_j,t_k)}{\tau} - a\frac{u(x_{j+1},t_{k+1})-2u(x_j,t_{k+1})+u(x_{j-1},t_{k+1})}{h^2} \\
& -f(x_j,t_{k+1}) \qquad (5)
\end{aligned}
$$

利用 Taylor 展式, 有

$$
\frac{u(x_j,t_{k+1})-u(x_j,t_k)}{\tau} = \frac{\partial u}{\partial t}(x_j,t_k)+O(\tau) \qquad (6)
$$

$$
\frac{u(x_{j+1},t_{k+1})-2u(x_j,t_{k+1})+u(x_{j-1},t_{k+1})}{h^2} = \frac{\partial^2 u}{\partial x^2}(x_j,t_{k+1})+O(h^2) \qquad (7)
$$

$$
f(x_j,t_{k+1}) = f(x_j,t_k)+O(\tau) \qquad (8)
$$

把 (6), (7) 和 (8) 代入 (5), 并利用微分方程 (1), 有

$$R_j^k(u) = \left(\frac{\partial u}{\partial t} - a\frac{\partial^2 u}{\partial x^2} - f\right)\Bigg|_{(x_j,t_k)} + O(\tau + h^2) = O(\tau + h^2) \quad (9)$$

其次，计算六点对称格式的截断误差

把 (3) 中数值解换为真解，并左端减右端，有

$$R_j^k(u) = \frac{u(x_j,t_{k+1})-u(x_j,t_k)}{\tau} - \frac{a}{2}\Big(\frac{u(x_{j+1},t_{k+1})-2u(x_j,t_{k+1})+u(x_{j-1},t_{k+1})}{h^2}$$
$$\frac{u(x_{j+1},t_k)-2u(x_j,t_k)+u(x_{j-1},t_k)}{h^2}\Big) + \frac{1}{2}(f(x_j,t_{k+1})+f(x_j,t_k)) \quad (10)$$

利用 Taylor 展式，有

$$\frac{u(x_j,t_{k+1})-u(x_j,t_k)}{\tau} = \frac{\partial u}{\partial t}(x_j,t_k) + \frac{\tau}{2}\frac{\partial^2 u}{\partial t^2}(x_j,t_k) + O(\tau^2) \qquad (11)$$

$$
\begin{aligned}
&\frac{1}{2}\Big(\frac{u(x_{j+1},t_{k+1})-2u(x_j,t_{k+1})+u(x_{j-1},t_{k+1})}{h^2}\\
&\quad+\frac{u(x_{j+1},t_k)-2u(x_j,t_k)+u(x_{j-1},t_k)}{h^2}\Big)\\
&=\frac{1}{2}\Big(\frac{\partial^2 u}{\partial x^2}(x_j,t_{k+1})+\frac{\partial^2 u}{\partial x^2}(x_j,t_k)\Big)+O(h^2)\\
&=\frac{\partial^2 u}{\partial x^2}(x_j,t_k)+\frac{\tau}{2}\frac{\partial^3 u}{\partial t\partial x^2}(x_j,t_k)+O(\tau^2)+O(h^2)
\end{aligned}
\tag{12}
$$

$$
\frac{1}{2}\left(f(x_j,t_{k+1})+f(x_j,t_k)\right)=f(x_j,t_k)+\frac{\tau}{2}\frac{\partial f}{\partial t}(x_j,t_k)+O(\tau^2)
\tag{13}
$$

把 (11), (12) 和 (13) 代入 (10), 并利用微分方程 (1), 有

$$
\begin{aligned}
R_j^k(u) &= \left(\frac{\partial u}{\partial t} - a\frac{\partial^2 u}{\partial x^2} - f\right)\Big|_{(x_j,t_k)} \\
&\quad + \frac{\tau}{2}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} - a\frac{\partial^2 u}{\partial x^2} - f\right)\Big|_{(x_j,t_k)} + O(\tau^2 + h^2) \\
&= O(\tau^2 + h^2)
\end{aligned}
$$

最后，计算 Richardson 格式的截断误差

把 (4) 中数值解换为真解，并左端减右端，有

$$R_j^k(u) = \frac{u(x_j,t_{k+1})-u(x_j,t_{k-1})}{2\tau} - a\frac{u(x_{j+1},t_k)-2u(x_j,t_k)+u(x_{j-1},t_k)}{h^2} - f(x_j,t_k) \tag{14}$$

利用 Taylor 展式，有

$$\frac{u(x_j,t_{k+1})-u(x_j,t_{k-1})}{2\tau} = \frac{\partial u}{\partial t}(x_j,t_k) + O(\tau^2) \tag{15}$$

$$\frac{u(x_{j+1},t_k)-2u(x_j,t_k)+u(x_{j-1},t_k)}{h^2} = \frac{\partial^2 u}{\partial x^2}(x_j,t_k) + O(h^2) \tag{16}$$

把 (15), (16) 代入 (14), 并利用微分方程 (1), 有

$$
\begin{aligned}
R_j^k(u) &= \left(\frac{\partial u}{\partial t} - a\frac{\partial^2 u}{\partial x^2} - f\right)\bigg|_{(x_j,t_k)} + O(\tau^2 + h^2) \\
&= O(\tau^2 + h^2)
\end{aligned}
$$

习题 2 将向前差分格式和向后差分格式作加权平均，得到如下格式:

$$\frac{u_j^{k+1}-u_j^k}{\tau}=\frac{a}{h^2}\left[\theta\left(u_{j+1}^{k+1}-2u_j^{k+1}+u_{j-1}^{k+1}\right)+(1-\theta)\left(u_{j+1}^k-2u_j^k+u_{j-1}^k\right)\right] \tag{17}$$

其中 $0\leqslant\theta\leqslant 1$. 试计算截断误差，并证明当 $\theta=\frac{1}{2}-\frac{1}{12r}$ 时，截断误差的阶最高($O(\tau^2+h^4)$).

解: 首先，求格式 (24) 的截断误差.

把 (24) 中的数值解换为真解，并左端减右端，有

$$\begin{aligned}R_j^k(u) &= \frac{u(x_j,t_{k+1})-u(x_j,t_k)}{\tau}\\ &-\frac{a}{h^2}[\theta(u(x_{j+1},t_{k+1})-2u(x_j,t_{k+1})+u(x_{j-1},t_{k+1}))\\ &+(1-\theta)(u(x_{j+1},t_k)-2u(x_j,t_k)+u(x_{j-1},t_k))]\end{aligned} \tag{18}$$

利用 Taylor 展式, 有

$$\frac{u(x_j,t_{k+1})-u(x_j,t_k)}{\tau}=\frac{\partial u}{\partial t}(x_j,t_k)+\frac{\tau}{2}\frac{\partial^2 u}{\partial t^2}(x_j,t_k)+O(\tau^2) \tag{19}$$

$$\begin{aligned}
&\frac{u(x_{j+1},t_{k+1})-2u(x_j,t_{k+1})+u(x_{j-1},t_{k+1})}{h^2}\\
&=\frac{\partial^2 u}{\partial x^2}(x_j,t_{k+1})+\frac{h^4}{12}\frac{\partial^4 u}{\partial x^4}(x_j,t_{k+1})+O(h^4)\\
&=\frac{\partial^2 u}{\partial x^2}(x_j,t_{k+1/2})+\frac{\tau}{2}\frac{\partial^3 u}{\partial t\partial x^2}(x_j,t_{k+1/2})+\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(x_j,t_{k+1})+O(\tau^2+h^4)
\end{aligned} \tag{20}$$

$$\begin{aligned}
&\frac{u(x_{j+1},t_k)-2u(x_j,t_k)+u(x_{j-1},t_k)}{h^2}\\
&=\frac{\partial^2 u}{\partial x^2}(x_j,t_k)+\frac{h^4}{12}\frac{\partial^4 u}{\partial x^4}(x_j,t_k)+O(h^4)\\
&=\frac{\partial^2 u}{\partial x^2}(x_j,t_{k+1/2})-\frac{\tau}{2}\frac{\partial^3 u}{\partial t\partial x^2}(x_j,t_{k+1/2})+\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(x_j,t_k)+O(\tau^2+h^4)
\end{aligned} \tag{21}$$

利用 (20) 和 (21), 有

$$\begin{aligned}
&\frac{1}{h^2}\left[\theta\left(u(x_{j+1},t_{k+1})-2u(x_j,t_{k+1})+u(x_{j-1},t_{k+1})\right)\right.\\
&\qquad\left.+(1-\theta)\left(u(x_{j+1},t_k)-2u(x_j,t_k)+u(x_{j-1},t_k)\right)\right]\\
=&\frac{\partial^2 u}{\partial x^2}(x_j,t_{k+1/2})+(2\theta-1)\frac{\tau}{2}\frac{\partial^3 u}{\partial t\partial x^2}(x_j,t_{k+1/2})\\
&\quad+\theta\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(x_j,t_{k+1})+(1-\theta)\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(x_j,t_k)+O(\tau^2+h^4)\\
=&\frac{\partial^2 u}{\partial x^2}(x_j,t_k)+\frac{\tau}{2}\frac{\partial^3 u}{\partial t\partial x^2}(x_j,t_k)+(2\theta-1)\frac{\tau}{2}\frac{\partial^3 u}{\partial t\partial x^2}(x_j,t_k)\\
&\quad+\theta\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(x_j,t_{k+1})+(1-\theta)\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(x_j,t_k)+O(\tau^2+h^4)\\
=&\frac{\partial^2 u}{\partial x^2}(x_j,t_k)+\theta\tau\frac{\partial^3 u}{\partial t\partial x^2}(x_j,t_k)\\
&\quad+\theta\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(x_j,t_{k+1})+(1-\theta)\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(x_j,t_k)+O(\tau^2+h^4)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial^2 u}{\partial x^2}(x_j, t_k) + \theta\tau \frac{\partial^3 u}{\partial t \partial x^2}(x_j, t_k) \\
&\quad + \theta \frac{h^2}{12} \frac{\partial^4 u}{\partial x^4}(x_j, t_k) + (1-\theta) \frac{h^2}{12} \frac{\partial^4 u}{\partial x^4}(x_j, t_k) + O(\tau^2 + h^4) \\
&= \frac{\partial^2 u}{\partial x^2}(x_j, t_k) + \theta\tau \frac{\partial^3 u}{\partial t \partial x^2}(x_j, t_k) \\
&\quad + \frac{h^2}{12} \frac{\partial^4 u}{\partial x^4}(x_j, t_k) + O(\tau^2 + h^4)
\end{aligned} \tag{22}
$$

把 (19) 和 (22) 代入 (18), 并利用该差分格式所对应的微分方程, 有

$$
\begin{aligned}
R_j^k &= \left(\frac{\partial u}{\partial t} - a\frac{\partial^2 u}{\partial x^2}\right)\Bigg|_{(x_j,t_k)} + \frac{\tau}{2}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t} - a\frac{\partial^2 u}{\partial x^2}\right)\Bigg|_{(x_j,t_k)} \\
&\quad -a\tau(\theta - \frac{1}{2})\frac{\partial^3 u}{\partial t\partial x^2}(x_j,t_k) - a\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(x_j,t_k) + O(\tau^2+h^4) \\
&= -a\tau(\theta - \frac{1}{2})\frac{\partial^3 u}{\partial t\partial x^2}(x_j,t_k) - a\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(x_j,t_k) + O(\tau^2+h^4) \\
&= \left[-a^2\tau(\theta - \frac{1}{2})\frac{\partial^4 u}{\partial x^4}(x_j,t_k) - a\frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(x_j,t_k)\right] + O(\tau^2+h^4) \\
&= a\left[-a\tau(\theta - \frac{1}{2}) - \frac{h^2}{12}\right]\frac{\partial^4 u}{\partial x^4}(x_j,t_k) + O(\tau^2+h^4) \qquad (23)
\end{aligned}
$$

当 $a\tau(\theta - \frac{1}{2}) = -\frac{h^2}{12}$ 时，即 $\theta = \frac{1}{2} - \frac{1}{12r}$, (23) 的截断误差的阶最高, 且为 $O(\tau^2 + h^4)$.

习题 3 用判别稳定性的直接估计法(矩阵法), 求证差分格式

$$\frac{u_j^{k+1}-u_j^k}{\tau}=\frac{a}{h^2}\left[\theta\left(u_{j+1}^{k+1}-2u_j^{k+1}+u_{j-1}^{k+1}\right)+(1-\theta)\left(u_{j+1}^k-2u_j^k+u_{j-1}^k\right)\right] \tag{24}$$

当 $\frac{1}{2}\leqslant\theta\leqslant 1$ 恒稳定, 当 $0\leqslant\theta<\frac{1}{2}$ 时, 稳定的充要条件是

$$r\leqslant\frac{1}{2(1-2\theta)}.$$

证明: Step 1: 写出差分格式的矩阵表示.

(24)$\Leftrightarrow$

$$u_j^{k+1}-u_j^k=r\left[\theta\left(u_{j+1}^{k+1}-2u_j^{k+1}+u_{j-1}^{k+1}\right)+(1-\theta)\left(u_{j+1}^k-2u_j^k+u_{j-1}^k\right)\right]$$

$\Leftrightarrow$

$$(1+2\theta r)u_j^{k+1}-\theta r\left(u_{j+1}^{k+1}+u_{j-1}^{k+1}\right)=(1-2(1-\theta)r)\,u_j^k+(1-\theta)r\left(u_{j+1}^k+u_{j-1}^k\right)$$

$\Leftrightarrow$

$$[(1+2\theta r)\mathrm{I}-\theta r\mathrm{S}]\, U^{k+1} = [(1-2(1-\theta)r)\,\mathrm{I}+(1-\theta)r\mathrm{S}]\, U^{k}$$

其中 I 为 $N-1$ 阶的单位矩阵，$U^k = (u_1^k, u_2^k, \cdots, u_{N-1}^k)^T$

$$S = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 0 & 1 \\ & & & 1 & 0 \end{pmatrix}_{(N-1)\times(N-1)}$$

$\Leftrightarrow$

$$U^{k+1} = CU^k$$

其中

$$C = [(1+2\theta r)\mathrm{I}-\theta r\mathrm{S}]^{-1}\, [(1-2(1-\theta)r)\,\mathrm{I}+(1-\theta)r\mathrm{S}]$$

Step 2: 判别差分格式的稳定性.

S 的特征值为

$$\lambda_j^S = 2\cos j\pi h,\ \ j = 1, 2, \cdots, N-1;\ \ h = \frac{1}{N}$$

又注意到该差分格式所对应的矩阵 C 是对称矩阵 S 的实系数有理函数，因此其特征值为

$$
\begin{aligned}
\lambda_j^C &= \frac{(1-2(1-\theta)r)+(1-\theta)r2\cos j\pi h}{(1+2\theta r)-\theta r2\cos j\pi h} \\
&= \frac{1-2(1-\theta)r(1-\cos j\pi h)}{1+2\theta r(1-\cos j\pi h)} \\
&= \frac{1-4(1-\theta)r\sin^2\frac{j\pi h}{2}}{1+4\theta r\sin^2\frac{j\pi h}{2}}, \ j=1,2,\cdots,N-1; \ h=\frac{1}{N}
\end{aligned}
$$

因此，由 pp.171 推论 5.2.1 知：差分格式 (24) 稳定的充要条件是 (对一切 $0<\tau\le\tau_0$)

$$
\begin{aligned}
& \left|\lambda_j^C\right| \leqslant 1+M\tau, \quad j=1,\cdots,N-1 \\
\Leftrightarrow \quad & \left|\frac{1-4(1-\theta)r\sin^2\frac{j\pi h}{2}}{1+4\theta r\sin^2\frac{j\pi h}{2}}\right| \leqslant 1+M\tau \\
\Leftrightarrow \quad & -1-M\tau \leqslant \frac{1-4(1-\theta)r\sin^2\frac{j\pi h}{2}}{1+4\theta r\sin^2\frac{j\pi h}{2}} \leqslant 1+M\tau
\end{aligned}
$$

$$\Leftrightarrow \qquad -1 \leqslant \frac{1-4(1-\theta)r\sin^2\frac{j\pi h}{2}}{1+4\theta r\sin^2\frac{j\pi h}{2}} \leqslant 1$$

上式右侧的不等式显然对任意的 $0 \leqslant \theta \leqslant 1$ 及 $r > 0$ 均成立. 故只需要考查左侧的不等式

$$\begin{aligned}
& -1-4\theta r\sin^2\frac{j\pi h}{2} \leqslant 1-4(1-\theta)r\sin^2\frac{j\pi h}{2} \\
\Leftrightarrow \qquad & -4(1-\theta)r\sin^2\frac{j\pi h}{2} + 4\theta r\sin^2\frac{j\pi h}{2} \geqslant -1-1 \\
\Leftrightarrow \qquad & -4(1-2\theta)r\sin^2\frac{j\pi h}{2} \geqslant -2 \\
\Leftrightarrow \qquad & 2(2\theta-1)r\sin^2\frac{j\pi h}{2} \geqslant -1 \qquad (25)
\end{aligned}$$

若 $2\theta - 1 \geqslant 0$, 即 $\frac{1}{2} \leqslant \theta < 1$, 上式恒成立, 所以差分格式 (24) 恒稳定.

若 $2\theta - 1 < 0$, 即 $0 \leqslant \theta < \frac{1}{2}$, (25) 等价于

$$2(1-2\theta) r \sin^2 \frac{j\pi h}{2} \leqslant 1, \quad j = 1, 2, \cdots, N-1; \; h = \frac{1}{N}$$

$\Leftrightarrow$

$$r \leqslant \frac{1}{2(1-2\theta)}$$

即 $0 \leqslant \theta < \frac{1}{2}$, 格式 (26) 稳定的充要条件是

$$r \leqslant \frac{1}{2(1-2\theta)}$$

$\square$

习题 4 用 Fourier 方法证明差分格式

$$\frac{u_j^{k+1}-u_j^k}{\tau}=\frac{a}{h^2}\left[\theta\left(u_{j+1}^{k+1}-2u_j^{k+1}+u_{j-1}^{k+1}\right)+(1-\theta)\left(u_{j+1}^k-2u_j^k+u_{j-1}^k\right)\right] \tag{26}$$

稳定的充要条件 $(r=a\frac{\tau}{h^2})$

$$r\leqslant\frac{1}{2(1-2\theta)},\quad 0\leqslant\theta<\frac{1}{2}$$

证明:

(26)$\Leftrightarrow$

$$\begin{aligned}&(1+2\theta r)u_j^{k+1}-\theta r\left(u_{j+1}^{k+1}+u_{j-1}^{k+1}\right)\\&=\left(1-2(1-\theta)r\right)u_j^k+(1-\theta)r\left(u_{j+1}^k+u_{j-1}^k\right)\end{aligned} \tag{27}$$

取通项

$$u_{j+m}^{k+q}=v_{k+q}e^{i\alpha(x_j+mh)},\ \ \alpha=2p\pi \tag{28}$$

将 (28) 代入 (27), 有

$$\begin{aligned} & v^{k+1} e^{i\alpha x_j}\left[(1+2\theta r)-\theta r\left(e^{i\alpha h}+e^{-i\alpha h}\right)\right] \\ & = v^{k} e^{i\alpha x_j}\left[(1-2(1-\theta) r)-(1-\theta) r\left(e^{i\alpha h}+e^{-i\alpha h}\right)\right] \end{aligned}$$

消去 $e^{i\alpha x_j}$, $\Leftrightarrow$

$$\begin{aligned} & v^{k+1}\left[(1+2\theta r)-\theta r\left(e^{i\alpha h}+e^{-i\alpha h}\right)\right] \\ & = v^{k}\left[(1-2(1-\theta) r)-(1-\theta) r\left(e^{i\alpha h}+e^{-i\alpha h}\right)\right] \end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned} v^{k+1} = v^{k} & \left[(1+2\theta r)-\theta r\left(e^{i\alpha h}+e^{-i\alpha h}\right)\right]^{-1} \\ & \left[(1-2(1-\theta) r)-(1-\theta) r\left(e^{i\alpha h}+e^{-i\alpha h}\right)\right] \end{aligned}$$

由此可知，增长因子为

$$
\begin{aligned}
G(ph,\tau) &= \left[(1+2\theta r)-\theta r\left(e^{i\alpha h}+e^{-i\alpha h}\right)\right]^{-1} \\
&\quad\times\left[(1-2(1-\theta)r)-(1-\theta)r\left(e^{i\alpha h}+e^{-i\alpha h}\right)\right] \\
&= \frac{(1-2(1-\theta)r)-2(1-\theta)r\cos\alpha h}{(1+2\theta r)-2\theta r\cos\alpha h} \\
&= \frac{1-4(1-\theta)r\sin^2\frac{\alpha h}{2}}{1+4\theta r\sin^2\frac{\alpha h}{2}}
\end{aligned}
$$

由 Von Neumann 条件可知, 格式 (26) 稳定的充分条件是

$$\left|\frac{1-4(1-\theta)r\sin^2\frac{\alpha h}{2}}{1+4\theta r\sin^2\frac{\alpha h}{2}}\right| \leqslant 1+M\tau$$

剩下的证明过程与习题 3 完全一样, 即 $0 \leqslant \theta < \frac{1}{2}$, 格式 (26) 稳定的充要条件是

$$r \leqslant \frac{1}{2(1-2\theta)}$$

□

习题 5 证明差分格式

$$\frac{u_j^{k+1}-u_j^k}{\tau}=a\frac{u_{j+1}^k-2u_j^k+u_{j-1}^k}{h^2}+b\frac{u_{j+1}^k-u_{j-1}^k}{2h}+cu_j^k \tag{29}$$

稳定的充要条件是网比 $r=a\frac{\tau}{h^2}\leqslant\frac{1}{2}$, 其中 $a>0, b, c$ 有界.

证明:

(29)$\Leftrightarrow$

$$u_j^{k+1}-u_j^k=r\left(u_{j+1}^k-2u_j^k+u_{j-1}^k\right)+\frac{b\tau}{2h}\left(u_{j+1}^k-u_{j-1}^k\right)+c\tau u_j^k$$

$\Leftrightarrow$

$$u_j^{k+1}=\left(r+\frac{b\tau}{2h}\right)u_{j+1}^k+(1-2r+c\tau)\,u_j^k+\left(r-\frac{b\tau}{2h}\right)u_{j-1}^k \tag{30}$$

取通项

$$u_{j+m}^{k+q}=v_{k+q}e^{i\alpha(x_j+mh)},\ \ \alpha=2p\pi \tag{31}$$

将 (31) 代入 (30), 有

$$v^{k+1}e^{i\alpha x_j} = v^k e^{i\alpha x_j}\left[\left(r+\frac{b\tau}{2h}\right)e^{i\alpha h} + (1-2r+c\tau) + \left(r-\frac{b\tau}{2h}\right)e^{-i\alpha h}\right]$$

消去 $e^{i\alpha x_j}$, $\Leftrightarrow$

$$v^{k+1} = v^k\left[\left(r+\frac{b\tau}{2h}\right)e^{i\alpha h} + (1-2r+c\tau) + \left(r-\frac{b\tau}{2h}\right)e^{-i\alpha h}\right]$$

由此可得, 增长因子为

$$\begin{aligned} G(ph,\tau) &= \left(r+\frac{b\tau}{2h}\right)e^{i\alpha h} + (1-2r+c\tau) + \left(r-\frac{b\tau}{2h}\right)e^{-i\alpha h} \\ &= r\left[e^{i\alpha h}+e^{-i\alpha h}\right] + (1-2r+c\tau) \\ &\quad +\frac{b\tau}{2h}\left[e^{i\alpha h}-e^{-i\alpha h}\right] \\ &= 2r\cos\alpha h + (1-2r+c\tau) + \frac{b\tau}{2h}\sin\alpha h \\ &= 1-4r\sin^2\frac{\alpha h}{2} + \frac{b\tau}{2h}\sin\alpha h + c\tau \end{aligned}$$

由 Von Neumann 条件可知 (可以证明 Von Neumann 条件可改写为 $|G(ph,\tau)| \le 1 + O(\tau^{1/2})$), 格式 (29) 稳定的充分条件是

$$\left|1 - 4r\sin^2\frac{\alpha h}{2} + \frac{b\tau}{2h}\sin\alpha h + c\tau\right| \leqslant 1 + M\tau^{1/2}$$

$\Leftrightarrow$

$$-1 - M\tau^{1/2} \leqslant 1 - 4r\sin^2\frac{\alpha h}{2} + (c + \frac{b}{2h}\sin\alpha h)\tau \leqslant 1 + M\tau^{1/2} \qquad (32)$$

(32) 右侧不等式显然对任意的 $r > 0$ 恒成立. 而 (32) 左侧不等式等价于

$$-1 + 4r\sin^2\frac{\alpha h}{2} - (c + \frac{b}{2h}\sin\alpha h)\tau \leqslant 1 + M\tau^{1/2}$$

$\Leftrightarrow$ (注意 $\frac{b}{2h}(\sin\alpha h)\tau = O(\frac{\tau}{h}) = O(h) = O(\tau^{1/2})$ (因为 $r = \frac{\tau}{h^2}$ 是常数))

$$-1 + 4r\sin^2\frac{\alpha h}{2} \leqslant 1 \;\Leftrightarrow\; 4r\sin^2\frac{\alpha h}{2} \leqslant 2$$

$\Leftrightarrow$

$$4r \leqslant 2 \quad \Leftrightarrow \quad r \leqslant \frac{1}{2}$$

习题 1^* 求

$$S = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}_{(N-1)\times(N-1)}$$

的特征值和特征向量.

解: 记 λ 为矩阵 S 的任意特征值, $u = (u_1, \cdots, u_{N-1})^T$ 为 λ 非平凡 (或非零) 特征向量. 由特征值和特征向量的定义知: λ 及 u 满足

$$Su = \lambda u$$

$\Leftrightarrow$ 差分方程

$$\begin{cases} l_h u := u_{j-1} + u_{j+1} = \lambda u_j, \ j = 1, \cdots, N-1 \\ u_0 = u_N = 0 \end{cases} \tag{33}$$

由盖尔圆盘定理可知, $|\lambda| \leq 2$, 故可令

$$\lambda = 2\cos\theta, \quad \theta \in [0, \pi] \tag{34}$$

则此时 (33) 可等价的表示为

$$l_h u := u_{j-1} + u_{j+1} = 2\cos\theta u_j, \; j = 1, \cdots, N-1 \tag{35}$$

$$u_0 = u_N = 0 \tag{36}$$

下面先给出二阶齐次线性差分方程 (35) 的通解序列 (向量) 表示, 其关键是求出 (35) 的两个线性无关的特解序列. 为此, 令

$$u_j = z^j, j = 0, 1, \cdots, N$$

其中 z 是待定(复)常数, 将它代人 (35), 有

$$z^2 - 2\cos\theta z + 1 = 0$$

上述一元二次方程有两个根

$$z_1 = e^{i\theta}, \quad z_2 = e^{-i\theta}$$

由此就得到了 (35) 的两个线性无关的解序列

$$u_j^{(1)} = e^{ij\theta}, \quad u_j^{(2)} = e^{-ij\theta}, j = 0, 1, \cdots, N$$

又由齐次线性差分方程解的理论知: 方程 (35) 的任意两个线性无关解序列的线性组合仍然是 (35) 的解, 因此

$$\frac{u_j^{(1)} + u_j^{(2)}}{2} = \cos j\theta, \quad \frac{u_j^{(1)} - u_j^{(2)}}{2i} = \sin j\theta, j = 0, 1, \cdots, N$$

也构成了(35) 的两个线性无关的解序列.

利用这两个线性无关的解序列 $\{\cos j\theta\}$ 和 $\{\sin j\theta\}$ 可知: 差分方程 (35) 的通解序列可表示为

$$a\cos j\theta + b\sin j\theta$$

其中 a 和 b 为任意实常数.

接着利用边值条件 (36) 确定上述通解中的系数 a 和 b.
首先由边值条件 $u_0 = 0$, 有

$$u_0 = a\cos 0 + b\sin 0$$

故 $a = 0$.
又注意 u 为 λ 的非平凡 (或非零) 特征向量, 所以常数 $b \neq 0$, 不妨取 $b = 1$ (因为特征向量乘以任意非零常数还是特征向量),
综上知: λ 的非平凡特征向量u 的分量可表示为

$$u_j = \sin j\theta, \quad j = 0, \cdots, N \tag{37}$$

下面确定待定系数 θ. 由边值条件 $u_N = 0$, 可得 θ 满足

$$\sin N\theta = 0 \tag{38}$$

由此并注意 $\theta \in (0, \pi)$ (非平凡解要求 $\theta \neq 0$或π) 可知: 有 $N-1$ 个 θ 值:

$$\theta_k = \frac{k\pi}{N}, \quad k = 1, \cdots, N-1 \tag{39}$$

满足 (38).

利用 (34), 以及 (39) 和 (37) 可得 $N-1$ 个特征值

$$\lambda_k = 2\cos\frac{k\pi}{N}, \quad k = 1, 2, \cdots, N-1$$

和 λ_k 相应的特征向量 (均对应 θ_k)

$$u = \left\{\sin\frac{jk\pi}{N}\right\}_{j=1}^{N-1}, \quad k = 1, 2, \cdots, N-1.$$

$\square$