第三章习题

习题 1 在一般网格剖分下，对微分方程

$$Lu = -\frac{d}{dx}(p\frac{du}{dx}) + qu = f, \ a < x < b \tag{1}$$

建立中心差分格式.

解: **第一步，网格剖分**

做一般网格剖分:

$$a = x_0 < x_1 < \cdots < x_N = b$$

其中，分点

$$x_i = x_{i-1} + h_i, \ i = 1, \cdots, N$$

第 i 个剖分单元的剖分步长

$$h_i = x_i - x_{i-1}, \ i = 1, \cdots, N$$

第二步: 节点处微分方程的离散化

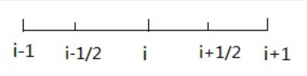
节点 x_i 处相应的微分方程为

$$\left[-\frac{d}{dx}(p\frac{du}{dx})+qu\right]_{x_i}=f(x_i) \tag{2}$$

有限差分离散的关键: 给出

$$\left[-\frac{d}{dx}(p\frac{du}{dx})\right]_{x_i}$$

的离散(近似)公式.



由一阶差商公式有

$$\left[-\frac{d}{dx}(p\frac{du}{dx})\right]_{x_i}\approx-\frac{2}{h_i+h_{i+1}}\left[p_{i+1/2}\left[\frac{du}{dx}\right]_{i+1/2}-p_{i-1/2}\left[\frac{du}{dx}\right]_{i-1/2}\right]$$

而

$$\left[\frac{du}{dx}\right]_{x_{i+1/2}} \approx \frac{u_{i+1}-u_i}{h_{i+1}}, \quad \left[\frac{du}{dx}\right]_{x_{i-1/2}} \approx \frac{u_i-u_{i-1}}{h_i}$$

故

$$\left[-\frac{d}{dx}\left(p\frac{du}{dx}\right)\right]_{x_i} \approx -\frac{2}{h_i+h_{i+1}}\left[p_{i+1/2}\frac{u_{i+1}-u_i}{h_{i+1}} - p_{i-1/2}\frac{u_i-u_{i-1}}{h_i}\right] \tag{3}$$

把 (3) 代入 (2) 可得到如下差分方程

$$-\frac{2}{h_i+h_{i+1}}\left[p_{i+1/2}\frac{u_{i+1}-u_i}{h_{i+1}} - p_{i-1/2}\frac{u_i-u_{i-1}}{h_i}\right] + q_iu_i = f_i \tag{4}$$

上式即为 (1) 的中心差分格式. □

习题 2 求差分方程

$$
\begin{aligned}
L_h u_i \quad \equiv \quad & -\frac{2}{h_i + h_{i+1}}\left[p_{i+1/2}\frac{u_{i+1}-u_i}{h_{i+1}} - p_{i-1/2}\frac{u_i - u_{i-1}}{h_i}\right] \\
& + r_i \frac{u_{i+1}-u_{i-1}}{h_i + h_{i+1}} + q_i u_i = f_i, \ i = 1, \cdots, N-1
\end{aligned} \tag{5}
$$

的截断误差(pp.94).

解: 把 (5) 中的数值解换成相应真解值后, 左端减右端, 有

$$
\begin{aligned}
R_i(u) \quad = \quad & L_h u(x_i) - f(x_i) \\
= \quad & -\frac{2}{h_i + h_{i+1}}\left[p(x_{i+1/2})\frac{u(x_{i+1})-u(x_i)}{h_{i+1}} - p(x_{i-1/2})\frac{u(x_i)-u(x_{i-1})}{h_i}\right] \\
& + r(x_i)\frac{u(x_{i+1})-u(x_{i-1})}{h_i + h_{i+1}} + q(x_i)u(x_i) - f(x_i)
\end{aligned} \tag{6}
$$

利用 Taylor 展开可知:

$$
\frac{u(x_{i+1})-u(x_{i-1})}{h_i + h_{i+1}} = [\frac{du}{dx}]_i + \frac{h_{i+1}-h_i}{2}[\frac{d^2u}{dx^2}]_i + O(h^2) \tag{7}
$$

$$
\begin{aligned}
p(x_{l-\frac{1}{2}})\frac{u(x_l)-u(x_{l-1})}{h_l} &= [p\frac{du}{dx}]_{l-\frac{1}{2}} + \frac{h_l^2}{24}[\frac{d^3u}{dx^3}]_{l-\frac{1}{2}} + O(h_l^3) \\
&= [p\frac{du}{dx}]_{l-\frac{1}{2}} + \frac{h_l^2}{24}[\frac{d^3u}{dx^3}]_l + O(h_l^3), \quad l = i-1, i
\end{aligned} \tag{8}
$$

将 (7), (8) 代入 (6) 化简可得 (注意 $h_i + h_{i+1} > h_l, l = i-1, i$)

$$
\begin{aligned}
R_i(u) &= -\frac{2}{h_i+h_{i+1}}\left([p\frac{du}{dx}]_{i+\frac{1}{2}} + \frac{h_{i+1}^2}{24}[\frac{d^3u}{dx^3}]_i - [p\frac{du}{dx}]_{i-\frac{1}{2}} - \frac{h_i^2}{24}[\frac{d^3u}{dx^3}]_i\right) \\
&\quad + \left([r\frac{du}{dx}]_i + \frac{h_{i+1}-h_i}{2}[r\frac{d^2u}{dx^2}]_i\right) + q(x_i)u(x_i) - f(x_i) + O(h^2) \\
&= -\frac{2}{h_i+h_{i+1}}\left([p\frac{du}{dx}]_{i+\frac{1}{2}} - [p\frac{du}{dx}]_{i-\frac{1}{2}} + \frac{h_{i+1}^2-h_i^2}{24}[\frac{d^3u}{dx^3}]_i\right) \\
&\quad + \left([r\frac{du}{dx}]_i + \frac{h_{i+1}-h_i}{2}[r\frac{d^2u}{dx^2}]_i\right) + q(x_i)u(x_i) - f(x_i) \\
&\quad + O(h^2)
\end{aligned} \tag{9}
$$

又因为

$$[p\frac{du}{dx}]_{i+\frac{1}{2}} = [p\frac{du}{dx}]_i + \frac{h_{i+1}}{2}\frac{d}{dx}[p\frac{du}{dx}]_i + \frac{h_{i+1}^2}{8}\frac{d^2}{dx^2}[p\frac{du}{dx}]_i + O(h_i^3) \tag{10}$$

$$[p\frac{du}{dx}]_{i-\frac{1}{2}} = [p\frac{du}{dx}]_i - \frac{h_i}{2}\frac{d}{dx}[p\frac{du}{dx}]_i + \frac{h_i^2}{8}\frac{d^2}{dx^2}[p\frac{du}{dx}]_i + O(h_{i+1}^3) \tag{11}$$

(10) 和 (11) 相减，得

$$[p\frac{du}{dx}]_{i+\frac{1}{2}} - [p\frac{du}{dx}]_{i-\frac{1}{2}} = \frac{h_{i+1}+h_i}{2}\frac{d}{dx}[p\frac{du}{dx}]_i + \frac{h_{i+1}^2-h_i^2}{8}\frac{d^2}{dx^2}[p\frac{du}{dx}]_i + O((h_i+h_{i+1})^3) \tag{12}$$

将 (12) 代入 (9) 可得

$$
\begin{aligned}
R_i(u) &= -\frac{2}{h_i+h_{i+1}}\left(\frac{h_{i+1}+h_i}{2}\frac{d}{dx}[p\frac{du}{dx}]_i+\frac{h_{i+1}^2-h_i^2}{8}\frac{d^2}{dx^2}[p\frac{du}{dx}]_i\right.\\
&\quad\left.+\frac{h_{i+1}^2-h_i^2}{24}[\frac{d^3u}{dx^3}]_i\right)+\left([r\frac{du}{dx}]_i+\frac{h_{i+1}-h_i}{2}[r\frac{d^2u}{dx^2}]_i\right)\\
&\quad+q(x_i)u(x_i)-f(x_i)+O(h^2)\\
&= -\frac{d}{dx}[p\frac{du}{dx}]_i-\frac{h_{i+1}-h_i}{4}[\frac{d^2}{dx^2}(p\frac{du}{dx})]_i-\frac{h_{i+1}-h_i}{12}[p\frac{d^3u}{dx^3}]_i\\
&\quad+\left([r\frac{du}{dx}]_i+\frac{h_{i+1}-h_i}{2}[r\frac{d^2u}{dx^2}]_i\right)+q(x_i)u(x_i)-f(x_i)+O(h^2)\\
&= \left(-\frac{d}{dx}[p\frac{du}{dx}]_i+[r\frac{du}{dx}]_i+q(x_i)u(x_i)-f(x_i)\right)-\frac{h_{i+1}-h_i}{4}[\frac{d^2}{dx^2}(p\frac{du}{dx})]_i\\
&\quad-\frac{h_{i+1}-h_i}{12}[p\frac{d^3u}{dx^3}]_i+\frac{h_{i+1}-h_i}{2}[r\frac{d^2u}{dx^2}]_i+O(h^2)\\
&= -(h_{i+1}-h_i)\left(\frac{1}{4}[\frac{d^2}{dx^2}(p\frac{du}{dx})]_i+\frac{1}{12}[p\frac{d^3u}{dx^3}]_i-\frac{1}{2}[r\frac{d^2u}{dx^2}]_i\right)+O(h^2)
\end{aligned}
$$

习题 3 用有限体积法导出如下两点边值问题

$$\begin{cases} Lu = -\frac{d}{d}(p\frac{du}{dx}) + r\frac{du}{dx} + qu = f, \quad a < x < b \\ u(a) = \alpha, \quad u(b) = \beta, \end{cases} \tag{13}$$

的差分格式, 其中,

$$p \in C^1[I], p \geqslant p_{\min} > 0, r, q, f \in C(I), q(x) \geqslant 0, I = [a, b]$$

解: 第一步, 网格剖分

做一般网格剖分:

$$a = x_0 < x_1 < \cdots < x_N = b$$

其中, 分点

$$x_i = x_{i-1} + h_i, i = 1, \cdots, N$$

第 i 个剖分单元的剖分步长

$$h_i = x_i - x_{i-1}, i = 1, \cdots, N$$

引入对偶剖分

$$a = x_0 < x_{1/2} < \cdots < x_{N-1/2} < x_N = b$$

其中，分点

$$x_{i-1/2} = x_i - \frac{1}{2}h_i, i = 1, \cdots, N$$

第二步，在微分方程节点 x_i 处的离散化

将微分方程

$$Lu = -\frac{d}{dx}(p\frac{du}{dx}) + r\frac{du}{dx} + qu = f,\ \ a < x < b \tag{14}$$

在节点 x_i 所属的对偶单元 $[x_{i-1/2}, x_{i+1/2}]$ 上积分得：积分方程

$$W(x_{i-1/2}) - W(x_{i+1/2}) + \int_{x_{i-1/2}}^{x_{i+1/2}} r\frac{du}{dx}dx + \int_{x_{i-1/2}}^{x_{i+1/2}} qudx = \int_{x_{i-1/2}}^{x_{i+1/2}} fdx \tag{15}$$

其中

$$W = p\frac{du}{dx}$$

下面利用广义中矩阵公式得到如下含未知函数的项

$$W(x_{i+1/2}), W(x_{i-1/2}), \int_{x_{i-1/2}}^{x_{i+1/2}} r\frac{du}{dx}dx, \int_{x_{i-1/2}}^{x_{i+1/2}} qudx$$

做进一步的离散.

注意到 $u' = \frac{w}{p}$, 并利用广义中矩形公式, 可得

$$u_l - u_{l-1} = \int_{x_{l-1}}^{x_l} \frac{w(x)}{p(x)}dx \approx w(x_{l-1/2})\int_{x_{l-1}}^{x_l} \frac{1}{p(x)}dx, \quad l = i, i+1$$

注意 上述 $w(x_{l-1/2})$ 可视为 $w(x)$ 在点 $x_{l-1/2}$ 处的值, 也可视为 $w(x)$ 在单元 $[x_{l-1}, x_l]$ 上的平均值(积分意义下的量).

于是有

$$w(x_{l-1/2}) \approx a_l\frac{(u_l - u_{l-1})}{h_l} \tag{16}$$

其中

$$a_l = \left[\frac{1}{h_l}\int_{x_{l-1}}^{x_l} \frac{1}{p(x)}dx\right]^{-1}$$

类似的利用广义中矩形公式, 有

$$\int_{x_{i-1/2}}^{x_{i+1/2}} r\frac{du}{dx}dx \approx \frac{h_i+h_{i+1}}{2}r_i\left[\frac{du}{dx}\right]_i \approx \frac{1}{2}r_i\left(u_{i+1}-u_{i-1}\right)$$

$$\int_{x_{i-1/2}}^{x_{i+1/2}} qudx \approx \frac{h_i+h_{i+1}}{2}d_iu_i$$

其中

$$d_i = \frac{2}{h_i+h_{i+1}}\int_{x_{i-1/2}}^{x_{i+1/2}} q(x)dx$$

综上可得微分方程在节点 x_i 处的离散化格式 (差分方程) 为

$$-\left[a_{i+1}\frac{u_{i+1}-u_i}{h_{i+1}}-a_i\frac{u_i-u_{i-1}}{h_i}\right]+\frac{1}{2}r_i\left(u_{i+1}-u_{i-1}\right)+\frac{h_i+h_{i+1}}{2}d_iu_i=\frac{h_i+h_{i+1}}{2}\varphi_i$$

其中

$$\varphi_i=\frac{2}{(h_i+h_{i+1})}\int_{x_{i-1/2}}^{x_{i+1/2}}f(x)dx.$$

因此逼近微分方程 (13) 的差分方程为

$$\begin{cases} -\left[a_{i+1}\frac{u_{i+1}-u_i}{h_{i+1}}-a_i\frac{u_i-u_{i-1}}{h_i}\right]+\frac{1}{2}r_i\left(u_{i+1}-u_{i-1}\right)+\frac{h_i+h_{i+1}}{2}d_iu_i=\frac{h_i+h_{i+1}}{2}\varphi_i \\ \qquad i=1,2,\cdots,N-1 \\ u_0=\alpha,\quad u_N=\beta. \end{cases}$$

$\square$

习题 4 差分方程

$$\begin{cases} L_h u_i \equiv a_0^{(i)} u_i - a_1^{(i)} u_{i-1} - a_2^{(i)} u_{i+1} = f_i, \\ \qquad\qquad\qquad\qquad\qquad\qquad i = 1, \cdots, N-1 \\ u_0 = \alpha, u_N = \beta \end{cases}$$

的系数矩阵 A 的第 i 行非 0 元素形如

$$A = \begin{pmatrix} & \ddots & & & \\ \ddots & \ddots & & & \\ & -a_1^{(i)} & a_0^{(i)} & -a_2^{(i)} & \\ & & & \ddots & \ddots \\ & & & \ddots & \end{pmatrix}$$

且

(1) $a_0^{(i)} > 0, a_1^{(i)} < 0, a_2^{(i)} < 0$;

(2) (弱)对角占优性 $a_0^{(i)} - \left|a_1^{(i)}\right| - \left|a_2^{(i)}\right| \geq 0, i = 2, \cdots, N-2$

(3) 严格对角占优性 $a_0^{(1)} - \left|a_2^{(1)}\right| > 0,\ a_0^{(N-1)} - \left|a_1^{(N-1)}\right| > 0$

试证明其逆矩阵是非负矩阵，即矩阵 A 为 M 矩阵.

定义: 矩阵 $A = (a_{ij} \in R^{n\times n})$ 称为 M 矩阵, 如果它满足:

(i) $a_{ii} > 0,\ i = 1, 2, \cdots, n$;

(ii) $a_{ij} \leq 0, i \neq j, 1 \leq i, j \leq n$;

(iii) A 是非奇异矩阵;

(iv) $A^{-1} \geq 0$.

分析: 注意到 A 是对角占优且不可约矩阵, 故 A 非奇异(证明见"数值计算方法" 教材 pp.138), 因此要证明 A 是 M 矩阵, 只需证明 (iv) 成立即可.

证明: 记 $A^{-1}=(\alpha_1,\cdots\alpha_{N-1})$, 只需证明其任一列向量 $\alpha_i\geqslant 0$. 令 $\alpha_i=(u_1,\cdots,u_{N-1})^T$, 因为 $AA^{-1}=E$, 所以 $A\alpha_i=e_i$, 即

$$A\alpha_i=e_i=(0,\cdots,0,1,0,\cdots 0)^T$$

$\Leftrightarrow$

$$L_h u_i\geqslant 0,\quad i=1,\cdots,N-1$$

$$u_0=0,\quad u_N=0$$

所以由极值定理的推论 2 可知 $u_i\geqslant 0$, $i=1,\cdots,N-1$.

(推论 2 若 $L_h u_i=f_i\geq 0\ (\leq 0),\forall x_i\in I_h$, 且 $u_0\geq 0(\leq 0)$, $u_N\geq 0(\leq 0)$, 则 $u_i\geq 0\ (\leq 0)$, $\forall x_i\in I_h$.)

习题 1* 考虑二维 Poisson 方程边值问题

$$
\begin{aligned}
-\Delta u &= f, \quad (x,y)\in\Omega &(17)\\
u|_\Gamma &= \alpha &(18)
\end{aligned}
$$

其中单位圆域 $\Omega=\{(x,y)|x^2+y^2<1\}$. 引入极坐标变换

$$
\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \end{cases} \quad (r,\theta)\in\hat{\Omega} \tag{19}
$$

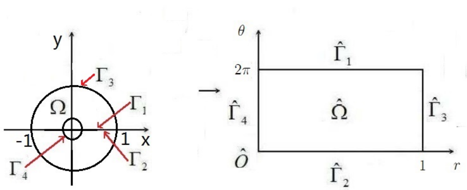
其中 $\hat{\Omega}=\{(r,\theta)|0<r\le 1; 0\le\theta\le 2\pi\}$.

试证明边值问题 (17), (18) 在极坐标系下可改写为 (下面简记 $u := u(x(r,\theta),y(r,\theta)))$)

$$
-\Delta_{r,\theta}u = -\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\right] = f(r,\theta),\ (r,\theta)\in\hat{\Omega} \tag{20}
$$

$$
u(1,\theta)=\alpha(\cos\theta,\sin\theta), u(r,0)=u(r,2\pi), r\frac{\partial u}{\partial r}|_{(0,\theta)}=0 \tag{21}
$$

证明：首先证明边值条件 (21) 成立.
为此给出从物理区域 Ω 到计算区域 $\hat{\Omega}$ 变换的示意图.



由上图可见:

(1) 由于边界 Γ_1 和 Γ_2 在物理域 是重合的, 故极坐标下的解函数在 Γ_1 和 Γ_2 所对应的计算区域边界 $\hat{\Gamma}_1$ 和 $\hat{\Gamma}_2$ 上的函数值应该相等, 即满足如下周期边界条件 $u(r,0)=u(r,2\pi)$.

(2) 由于边界 $\Gamma_3:\{(x,y)|x^2+y^2=1\}$ 所对应的计算区域边界为 $\hat{\Gamma}_3:\{(1,\theta)|\theta\in(0,2\pi)\}$, 所以极坐标下的解函数在$\hat{\Gamma}_3$ 应满足边界条件 $u(1,\theta)=\alpha(\cos\theta,\sin\theta)$.

(3) 由于计算域边界 $\hat{\Gamma}_4$ 是物理域边界 Γ_4 的极限情形 $(r \to 0)$.
由 $u(0,\theta)$ 在 (r,θ) 处的 Taylor 展开式

$$u(0,\theta) = u(r,\theta) - r u_r(r,\theta) + O(r^2)$$

可知：当 $r \to 0$ 时 (设解函数 u 在原点处连续),

$$r u_r(r,\theta) = u(r,\theta) - u(0,\theta) + O(r^2) \to 0$$

因此极坐标下的解函数在边界$\hat{\Gamma}_4$ 上应满足边界条件

$$r\frac{\partial u}{\partial r}\Big|_{(0,\theta)} = 0$$

接着证明方程 (20)成立. 为此先给出变换 (19) 的逆变换.

当 $x>0,y>0$ 时, $\theta\in(0,\frac{\pi}{2})$, 变换 (19) 的逆变换为

$$\begin{cases} r=\sqrt{x^2+y^2}, \\ \theta=\arctan(\frac{y}{x}), \end{cases} \tag{22}$$

当 $x<0,y\geq 0$ 时, $\theta\in(\frac{\pi}{2},\pi]$, 变换 (19) 的逆变换为

$$\begin{cases} r=\sqrt{x^2+y^2}, \\ \theta=\pi+\arctan(\frac{y}{x}), \end{cases}$$

当 $x<0,y<0$ 时, $\theta\in(\pi,\frac{3\pi}{2})$, 变换 (19) 的逆变换为

$$\begin{cases} r=\sqrt{x^2+y^2}, \\ \theta=\pi+\arctan(\frac{y}{x}), \end{cases}$$

当 $x>0,y<0$ 时, $\theta\in(\frac{3\pi}{2},2\pi)$, 变换 (19) 的逆变换为

$$\begin{cases} r=\sqrt{x^2+y^2}, \\ \theta=2\pi+\arctan(\frac{y}{x}), \end{cases}$$

下面以 $x>0, y>0$ 为例证明, 其他情况的结果是一样的.
由 (22) 可知

$$\frac{\partial r}{\partial x}=\frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}}=\frac{x}{r}=\cos\theta.$$

$$\frac{\partial\theta}{\partial x}=\frac{1}{1+(\frac{y}{x})^2}\cdot\frac{-y}{x^2}=\frac{-y}{r^2}=\frac{-\sin\theta}{r}.$$

由以上两式, 有

$$\frac{\partial w}{\partial x}=\cos\theta\frac{\partial w}{\partial r}+(\frac{-\sin\theta}{r})\frac{\partial w}{\partial\theta}. \tag{23}$$

同理,

$$\frac{\partial r}{\partial y}=\frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}}=\frac{y}{r}=\sin\theta.$$

$$\frac{\partial\theta}{\partial y}=\frac{1}{1+(\frac{y}{x})^2}\cdot\frac{1}{x}=\frac{x}{r^2}=\frac{\cos\theta}{r}.$$

由上两式可得

$$\frac{\partial w}{\partial y} = \sin\theta \frac{\partial w}{\partial r} + (\frac{\cos\theta}{r})\frac{\partial w}{\partial \theta}. \tag{24}$$

利用 (23) (这里令 $w = f_1$) 和 (24) (这里令 $w = f_2$) 可得

$$\begin{aligned} \nabla \cdot \vec{f} &= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} \\ &= \cos\theta \frac{\partial f_1}{\partial r} + (-\frac{\sin\theta}{r})\frac{\partial f_1}{\partial \theta} + \sin\theta \frac{\partial f_2}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f_2}{\partial \theta} \end{aligned}$$

特别的, 取 $\vec{f} = \nabla u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})$, 并利用 (23) 和 (24) 有

$$
\begin{aligned}
\nabla\cdot\nabla u &= \cos\theta\frac{\partial\frac{\partial u}{\partial x}}{\partial r}+(-\frac{\sin\theta}{r})\frac{\partial\frac{\partial u}{\partial x}}{\partial\theta}+\sin\theta\frac{\partial\frac{\partial u}{\partial y}}{\partial r}+\frac{\cos\theta}{r}\frac{\partial\frac{\partial u}{\partial y}}{\partial\theta}\\
&= \cos\theta\frac{\partial(\cos\theta\frac{\partial u}{\partial r}+(\frac{-\sin\theta}{r})\frac{\partial u}{\partial\theta})}{\partial r}+(-\frac{\sin\theta}{r})\frac{\partial(\cos\theta\frac{\partial u}{\partial r}+(\frac{-\sin\theta}{r})\frac{\partial u}{\partial\theta})}{\partial\theta}\\
&\quad+\sin\theta\frac{\partial(\sin\theta\frac{\partial u}{\partial r}+(\frac{\cos\theta}{r})\frac{\partial u}{\partial\theta})}{\partial r}+\frac{\cos\theta}{r}\frac{\partial(\sin\theta\frac{\partial u}{\partial r}+(\frac{\cos\theta}{r})\frac{\partial u}{\partial\theta})}{\partial\theta}\\
&= \cos\theta\left(\cos\theta\frac{\partial^2 u}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial u}{\partial\theta}+(\frac{-\sin\theta}{r})\frac{\partial^2 u}{\partial r\partial\theta}\right)\\
&\quad+(-\frac{\sin\theta}{r})\left(-\sin\theta\frac{\partial u}{\partial r}+\cos\theta\frac{\partial^2 u}{\partial r\partial\theta}+(-\frac{\cos\theta}{r})\frac{\partial u}{\partial\theta}+(-\frac{\sin\theta}{r})\frac{\partial^2 u}{\partial\theta^2}\right)\\
&\quad+\sin\theta\left(\sin\theta\frac{\partial^2 u}{\partial r^2}+(-\frac{\cos\theta}{r^2})\frac{\partial u}{\partial\theta}+(\frac{\cos\theta}{r})\frac{\partial^2 u}{\partial r\partial\theta}\right)\\
&\quad+\frac{\cos\theta}{r}\left(\cos\theta\frac{\partial u}{\partial r}+\sin\theta\frac{\partial^2 u}{\partial r\partial\theta}+(-\frac{\sin\theta}{r})\frac{\partial u}{\partial\theta}+(\frac{\cos\theta}{r})\frac{\partial^2 u}{\partial\theta^2}\right)\\
&= \frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}
\end{aligned}
$$

至此就证得了 (17) 的极坐标形式为 (18).