PROPOSITIONAL LOGIC

{ sentential logic OR statement logic }

Proposition

- declarative sentence
- declaring a fact that can either be true or false

e.g. $1+1=2$
- Delhi is the capital of India
- It is raining

} Propositions ✓

- $x+1=2$
- Fetch my umbrella

} Not propositions ✗

Logic

- science of reasoning
- understand & reason diff. mathematical statements
- construct valid arguments (proofs)
- mathematical statement $\xrightarrow{\text{proved true}}$ Theorem

6 logical Operators :
1. Negation ($\neg$, NOT, !, ~)
2. Conjunction (AND, $\wedge$, &, $\cdot$)
3. Disjunction (OR, $\vee$, +)
4. Exclusive OR (XOR, $\oplus$)
5. Implication $p \longrightarrow q$ ← Conditional statement
6. Biconditional $\longleftrightarrow$

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

{ Converse : $q \to p$
Inverse : $\bar{p} \to \bar{q}$
Contrapositive : $\bar{q} \to \bar{p}$ }

① Negation ($\neg$, NOT, !, ~)

P	$\neg$P
T	F
F	T

- **Well - Formed Formula (wff)**

Rule 1 – Every STATEMENT VARIABLE is a wff itself.

Rule 2 – If p is wff. then $\bar{p}$ is also wff

Rule 3 – If p & q are wff then $p \vee q$, $p \wedge q$, $p \to q$ & $p \leftrightarrow q$ are also wff.

Rule 4 – string of symbols consisting of statement variables, connectives and paranthesis is called a wff iff it can be produced using rules 1, 2 & 3 finitely many times.

Rules of Inference in Propositional Logic {criteria for finding validity of an argument}

(Inference → conclusion reached on the basis of evidence)

"if premise then conclusion"

- **Premise** (facts/axioms) : - evidence or assumption
  - a proposition on the basis of which we draw a conclusion.

- **Conclusion** : - result of the assumptions made in an argument.
  - a proposition reached from a given set of premises.

- **Argument** - sequence of statements that ends with a conclusion.
  - set of one or more premises and a conclusion.

- **VALID ARGUMENT** - an argument is valid if and only if it is NOT possible to make all premises true and a conclusion false.

e.g. I

P_1 : "If I love cat then I love dog" (p = I love cat, q = I love dog)

P_2 : "I love cat"

C : Therefore "I love dog"

$$\begin{array}{ll} \overset{T}{p} \rightarrow \overset{T}{q} & T \\ p & T \\ \hline \therefore q & T \end{array} \qquad ((\overbrace{p \rightarrow q}^{P_1}) \wedge \overbrace{p}^{P_2}) \rightarrow \overbrace{q}^{C}$$

VALID Argument

e.g. II

P_1 : "If I love cat then I love dog" (p = I love cat, q = I love dog)

P_2 : "I love dog"

C : Therefore "I love cat"

$$\begin{array}{ll} \overset{F/T}{p} \rightarrow \overset{T}{q} & T \\ q & T \\ \hline \therefore p & F \end{array}$$

INVALID Argument

- C is a valid conclusion of ~~of~~ set of premises $\{A_1, A_2 \ldots A_n\}$ if and only if $\{A_1 \wedge A_2 \wedge A_3 \wedge A_4 \ldots \wedge A_n \rightarrow C$ is a tautology.

- **RULES OF INFERENCE :**

1. RULE P : – A premise can be inserted at any point in the derivation.

2. RULE T : – If q is tautologically implied by one or more of the previous formula in a derivation then q can be inserted in the derivation.

3. RULE CP : – If formula q can be derived ~~po~~ from p and a set of premises then $p \to q$ can be derived from the set of premises alone.
(rule of conditional proof / deduction theorem)

E.g. 1.31 Show $\bar{p}$ is tautologically implied by $\overline{(p \wedge \bar{q})}$, $\bar{q} \vee r$, $\bar{r}$.

Sol

1	$\overline{(p \wedge \bar{q})}$	Rule P
2	$\bar{p} \vee \bar{\bar{q}}$	Rule T, 1, De Morgan's law
3	$\bar{p} \vee q$	Rule T, 2, Double negation law
4	$q \vee \bar{p}$	Rule T, 3, Commutative law
5	$\bar{q} \vee r$	Rule P
6	$\bar{p} \vee r$	Rule T, 4, 5, Resolution
7	$r \vee \bar{p}$	Rule T, 6, commutative law
8	$\bar{r}$	Rule P
$\therefore$ 9	$\bar{p}$	Rule T, 7, 8, disjunctive syllogism

E.g. 1.32 Derive $(\bar{p} \vee q, \bar{q} \vee r, r \to s) \to (p \to s)$

without C.P rule

1	$\bar{p} \vee q$	Rule P
2	$q \vee \bar{p}$	Rule T, 1, Commutative law
3	$\bar{q} \vee r$	Rule P
4	$\bar{p} \vee r$	Rule T, 2, 3, resolution
5	$p \to r$	Rule T, 4, $(\bar{p} \vee q \equiv p \to q)$
6	$r \to s$	Rule P
$\therefore$ 7	$p \to s$	Rule T, 5, 6, Hypothetical syllogism

1·32 example with C.P rule used

$(\bar{p} \vee q, \bar{q} \vee r, r \to s) \longrightarrow (p \to s)$

Acc. to C.P Rule we'll take 'P' as an additional premise and show s..

①	$\bar{p} \vee q$	Rule P.
②	$p \to q$	Rule T, ①, $(\bar{p} \vee q \equiv p \to q)$
③	p	Rule P, (~~additional~~ assumed premise)
④	q	Rule T, ②③, modus ponens
⑤	$\bar{q} \vee r$	Rule P
⑥	$q \to r$	Rule T, ⑤, $(\bar{p} \vee q \equiv p \to q)$
⑦	r	Rule T, ④, ⑥, modus ponens
⑧	$r \to s$	Rule P
⑨	s	Rule T, ⑦, ⑧, modus ponens
∴ ⑩	$p \to s$	Rule C.P

• CONSISTENCY OF PREMISES : – A set of formulas $A_1, A_2 \ldots A_m$ is INCONSISTENT if their conjuction implies a contradiction.

$$\underbrace{A_1 \wedge A_2 \wedge A_3 \wedge \ldots \wedge A_m}_{\text{incononsistent}} \longrightarrow \underbrace{B \wedge \bar{B}}_{\text{contradiction}}$$

Ex. 1.33 Show inconsistency of $(p \to q, p \to r, q \to \bar{r}, p)$ system.

①	$p \to q$	Rule P
②	$q \to \bar{r}$	Rule P & include premise 'p' also
③	$p \to \bar{r}$	Rule T, ①, ②, Hypothetical syllogism
④	$p \to r$	Rule P
⑤	$(p \to r) \wedge (p \to \bar{r})$	Rule T, ③, ④, Conjunction

∵ ⑤ is a contradiction

∴ Given system is inconsistent.

①	$p \to q$	Rule P
②	$q \to \bar{r}$	Rule P
③	$p \to \bar{r}$	Rule T, ①②, H.S.
④	p	Rule P
⑤	$\bar{r}$	Rule T, ③④, M.P
⑥	$p \to r$	Rule P
⑦	r	Rule T, ④⑥, M.P
⑨	$r \wedge \bar{r}$	Rule T, ⑤⑦, Conju

- **PREDICATE CALCULUS** (first order propositional logic)
- if 'P' is an n-place predicate letter & $x_1, x_2, x_3 \ldots x_n$ are names of objects then $P(x_1, x_2, \ldots x_n)$ denotes STATEMENT / (n-place predicate formula) $\boxed{\text{ATOMIC FORMULA}}$

- **STATEMENT FUNCTION** (simple): an expression ~~wich~~ which consists of a PREDICATE symbol and an individual VARIABLE.
  - e.g. $M(y)$ "y is a Man"
  - Now if 'a' is an object name then $M(a)$ is called SUBSTITUTION INSTANCE

(compound) S.F. – one or more simple statement functions combined through logical connectives.

e.g. $M(y)$ y is man
$V(y)$ y is vegiterian
$M(y) \wedge V(y)$ is Compound S.F,

- **QUANTIFIERS**:

① **UNIVERSAL QUANTIFIER**
- (x) or $(\forall x)$
- for all x
- Every x
- Implication ($\rightarrow$) used in statements

All dogs are animal { $D(x)$: x is dog, $A(x)$: x is Animal.

$\boxed{\forall x\,(D(x) \rightarrow A(x))}$

② **EXISTENTIAL QUANTIFIER**
- $(\exists x)$
- there exists some
- there exists atleast one
- Conjunction ($\wedge$) used in statements

– some roses are red

$\exists x\,(C(x) \wedge R(x))$ — $R(x)$: x is red, $C(x)$: x is rose

e.g. Symbolize, "Everyone has exactly one favorite language".

Sol. $M(y, x)$: y is favorite lang. of x

statement – $(\forall x)(\exists y)\,(M(y,x) \wedge ((\forall z)(z \neq y) \rightarrow \bar{M}(z,x)))$

- **FREE & BOUND variables** –

e.g. $\boxed{M(x) \wedge (\exists x)\,N(x)}$

{SCOPE OF QUANTIFIER – logical expression that follows the quantifier}

∵ it doesn't falls inside any QUANTIFIER { FREE occurance of x

BOUND occurance of x

Here, $N(x)$ is scope of $(\exists x)$ quantifier.

- **UNIVERSE of discourse** – A specific set or class or domain to which variables in a statement function are restricted.

e.g. "all dogs are animals" $(\forall x)(D(x) \rightarrow A(x))$ BUT $(\forall x)\,A(x)$ — here universe is SET OF DOGS