ENE 104
Electric Circuit Theory

Lecture 05:

Capacitors and Inductors

Week #5 : Dejwoot KHAWPARISUTH
office: CB40906 Tel: 02-470-9065
Email: dejwoot.kha@mail.kmutt.ac.th

http://webstaff.kmutt.ac.th/~dejwoot.kha/

Objectives : Ch7

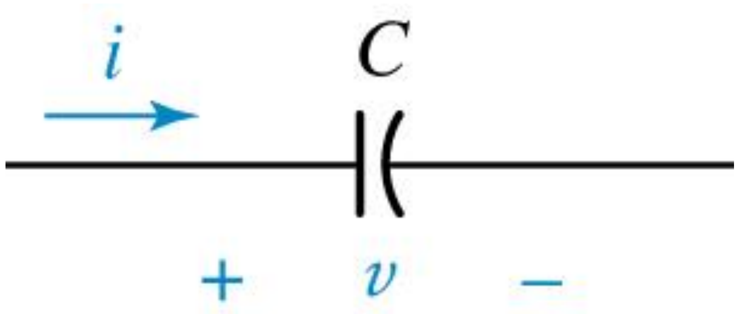
- the voltage-current relationship of ideal **capacitors**
- the voltage-current relationship of ideal **inductors**
- the energy stored
- series/parallel combinations of capacitors
- series/parallel combinations of inductors
- the behavior of op amp circuits with capacitors

Introduction:

Both the inductor and the capacitor are passive elements that are capable of storing and delivering finite amount of energy.

Note: **-** a passive element
- an active element
(furnishing an average power > 0)

The Capacitors:



Electrical symbol and current-voltage conventions for a capacitor.

$$i = C\frac{dv}{dt}$$

Voltage-Current Relationships:

$$i(t) = C\frac{dv(t)}{dt} \quad \Rightarrow dv = \frac{1}{C} i(t)dt$$

Between t_0 and t;
$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(t')dt' + v(t_0)$$

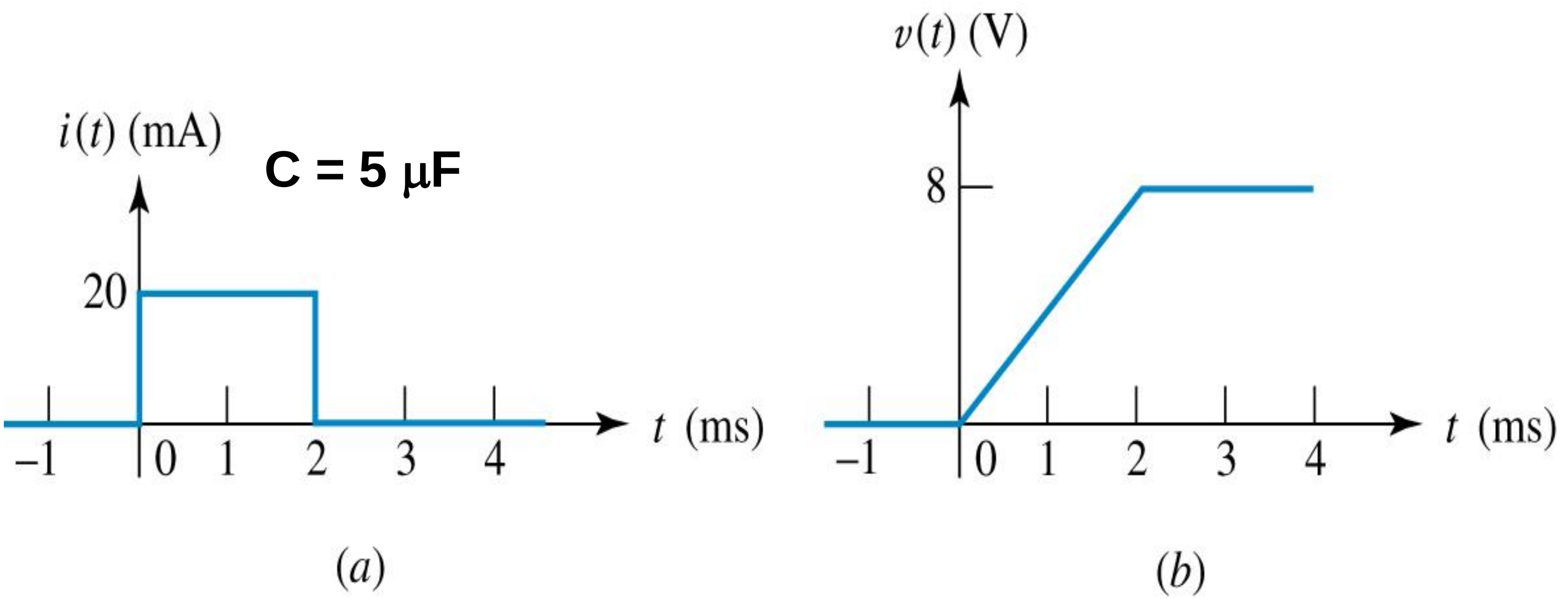
Or
$$v(t) = \frac{1}{C}\int i dt \; + v(t_0)$$

Or
$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(t')dt'$$

Or
$$v(t) = \frac{q(t)}{C}$$

Example 7.1:

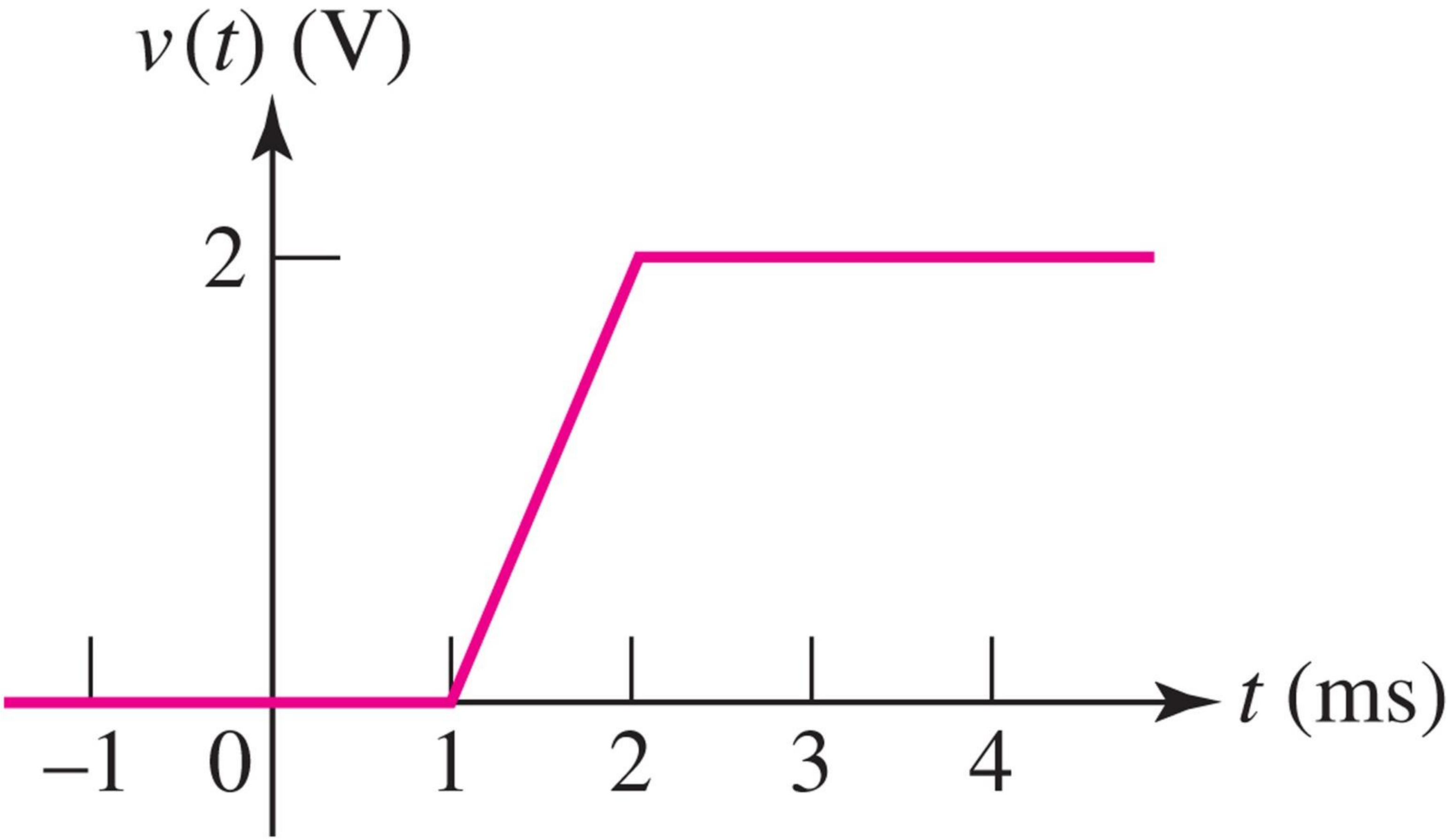
Find the capacitor voltage that is associated with the current shown graphically below.



$$dv = \frac{1}{C} i(t) dt$$

Practice: 7.1

Determine the current through a 100-pF capacitor if its voltage as a function of time is given by the figure



Energy Storage:

The power delivered to a capacitor is

$$p = vi = v \cdot C\frac{dv}{dt}$$

The energy stored in its electric field is therefore

$$\int_{t_0}^{t} p \cdot dt' = C\int_{t_0}^{t} v\frac{dv}{dt'}dt' = C\int_{v(t_0)}^{v(t)} v \cdot dv = \frac{1}{2}C\left\{v(t)^2 - v(t_0)^2\right\}$$

Energy Storage:

And thus

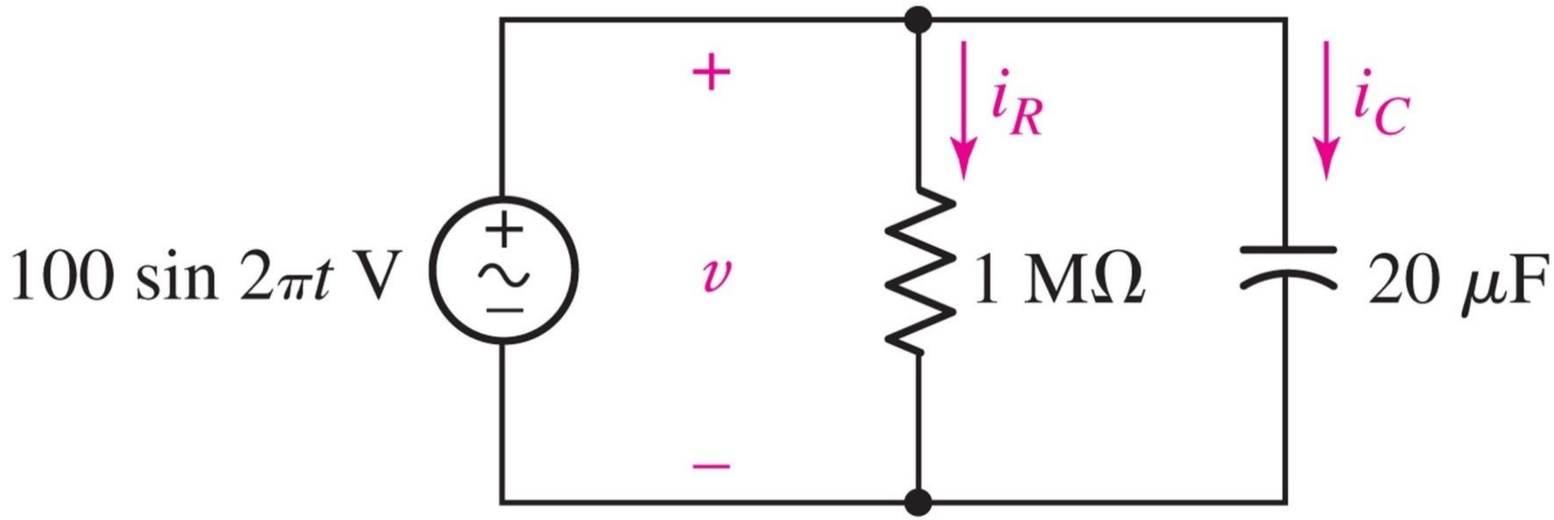
$$w_C(t) - w_C(t_0) = \frac{1}{2}C\left\{v(t)^2 - v(t_0)^2\right\}$$

If we select a zero energy reference at t_0

$$w_C(t) = \frac{1}{2}Cv^2$$

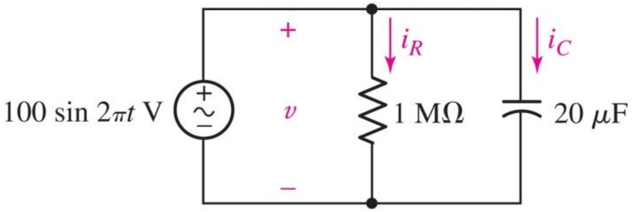
Example: 7.2

Find the maximum energy stored in the capacitor and the energy dissipated in the resistor over the interval $0 < t < 0.5\ s$.



Example: 7.2

Find the maximum energy stored in the capacitor and the energy dissipated in the resistor over the interval $0 < t < 0.5\ s$.



The energy stored in the capacitor is

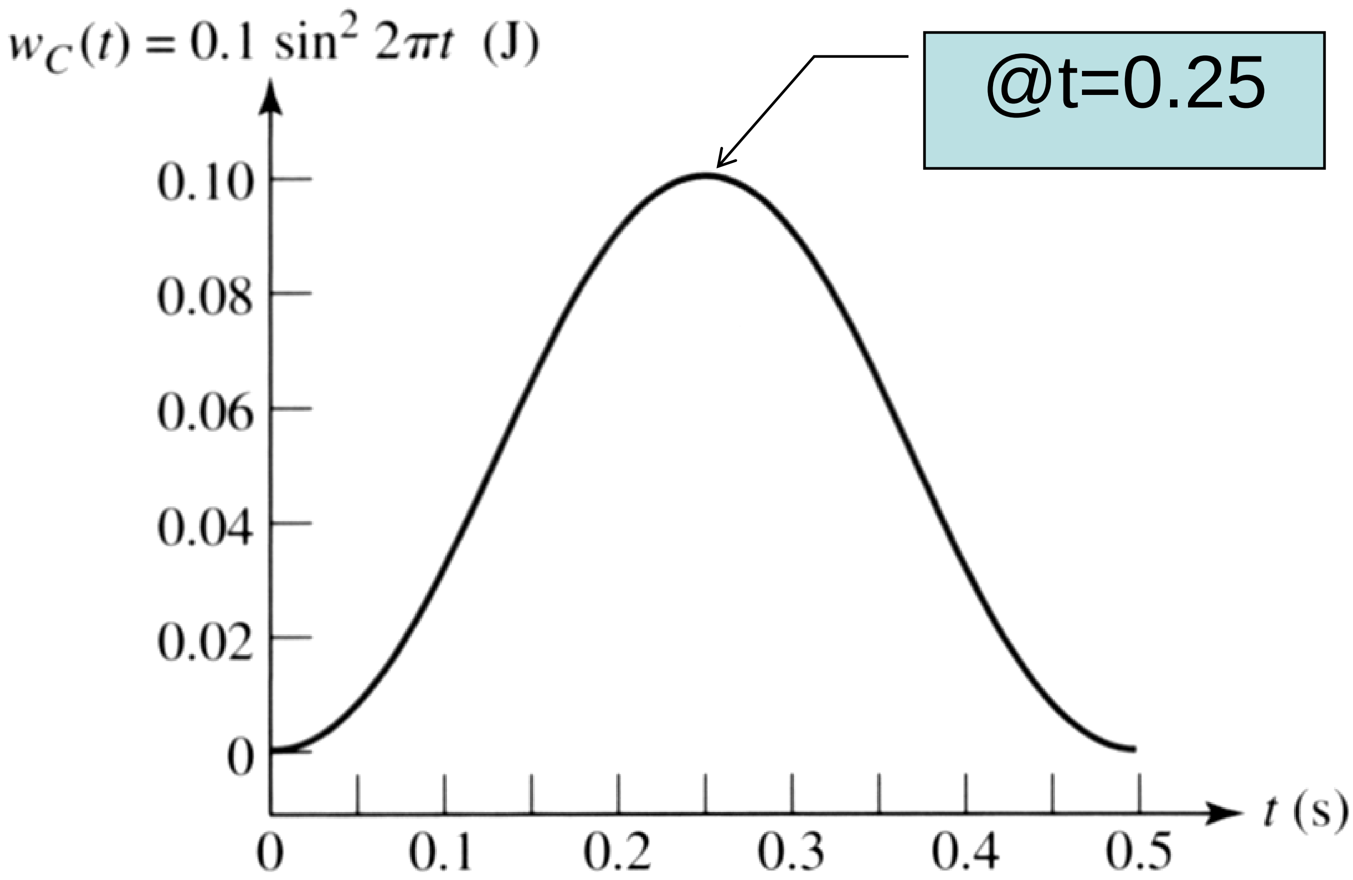
$$w_C(t) = \frac{1}{2}Cv^2 = 0.1\sin^2(2\pi t) \quad \text{Joules}$$

The energy dissipated in the resistor is

$$w_R(t) = \int_0^{0.5} p_R dt = \int_0^{0.5} (v \cdot i_R) dt = \int_0^{0.5} 10^{-2} \sin^2(2\pi t) dt$$

Example: 7.2

The energy stored in the capacitor is



Practice: 7.2

Calculate the energy stored in a 1000-μF capacitor at t = 50 μs if the voltage across it is $1.5cos10^5t$ volts.

Characteristics of an Ideal C.:

1. There is no current through a capacitor if the voltage across it is not changing with time. A capacitor is therefore an *open circuit to dc.*

2. A finite amount of energy can be stored in a capacitor even if the current through the capacitor is zero, such as when the voltage across it is constant.

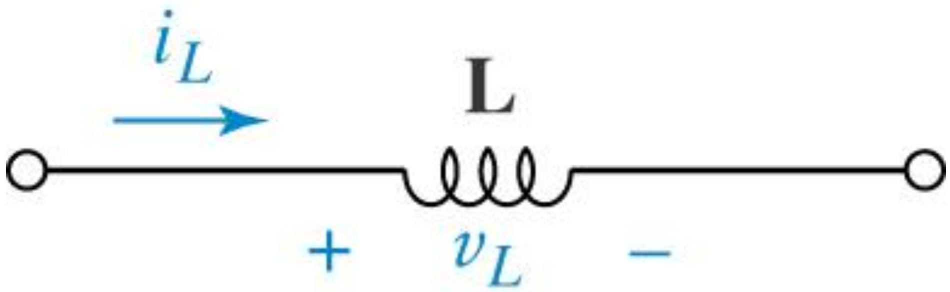
Characteristics of an Ideal C.:

3. It is impossible to change the voltage across a capacitor by a finite amount in zero time, for this requires an infinite current through the capacitor. A capacitor resists an abrupt change in the voltage across it in a manner analogous to the way a spring resists an abrupt change in its displacement.

4. A capacitor never dissipates energy, but only stores it. Although this is true for the *mathematical model,* it is not true for a *physical* capacitor due to finite resistances.

The Inductor:

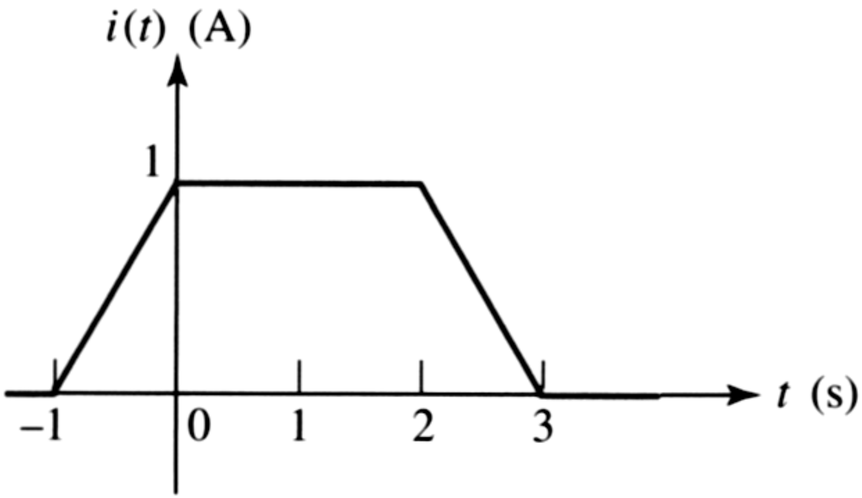
Electrical symbol and current-voltage conventions for an inductor.



$$v = L\frac{di}{dt}$$

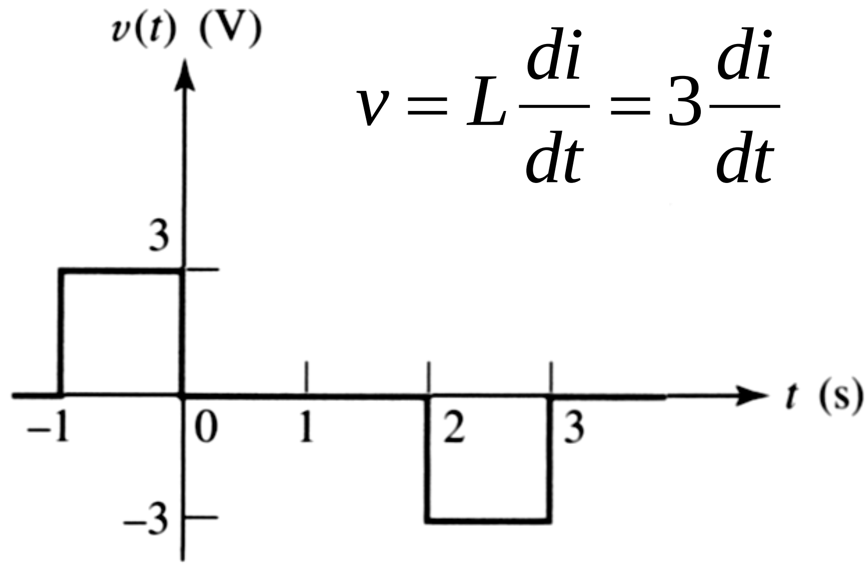
Example: 7.3

Given the waveform of the current in a 3-H inductor as shown, determine the inductor voltage and sketch it.



(a)

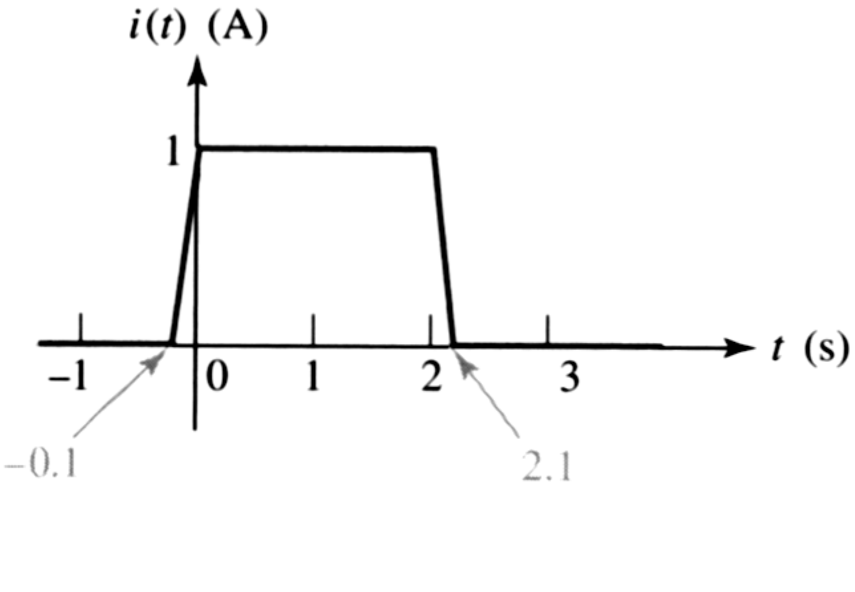
$$v = L\frac{di}{dt} = 3\frac{di}{dt}$$



(b)

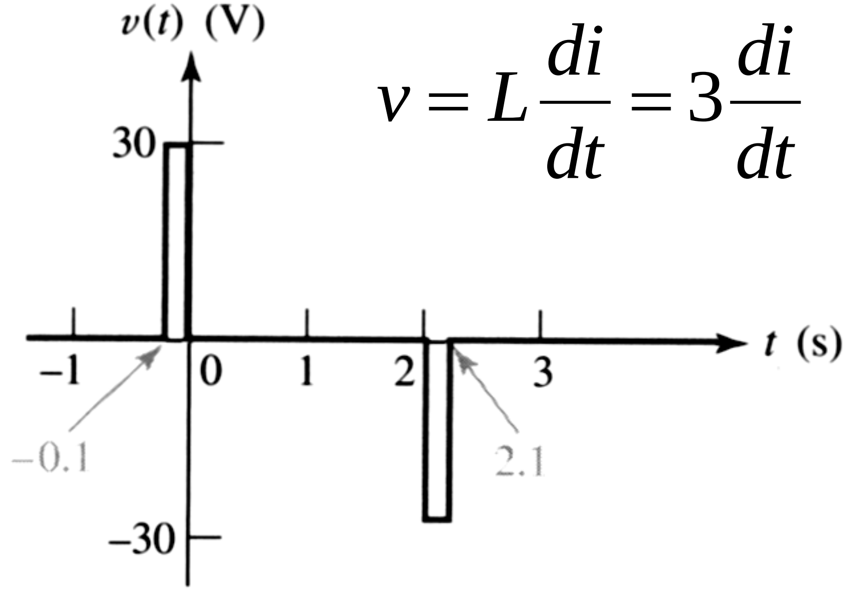
Example: 7.4

Given the waveform of the current in a 3-H inductor as shown, determine the inductor voltage and sketch it.



(a)

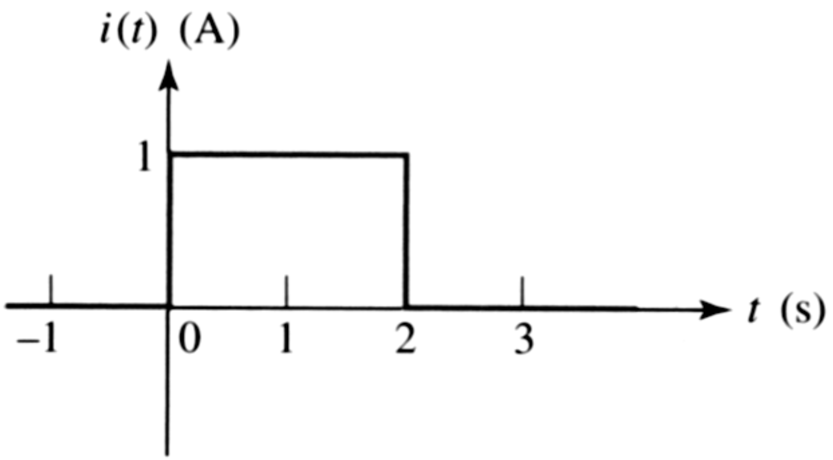
$$v = L\frac{di}{dt} = 3\frac{di}{dt}$$



(b)

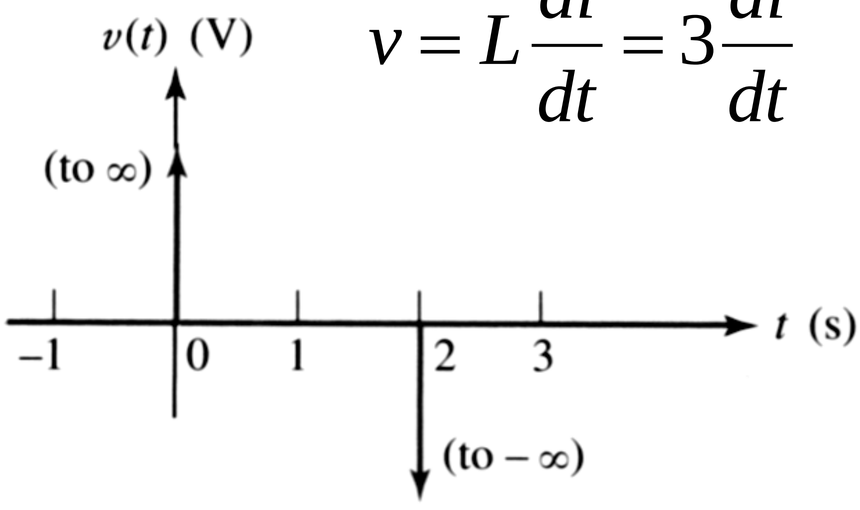
Example: 7.4

Given the waveform of the current in a 3-H inductor as shown, determine the inductor voltage and sketch it.



(a)

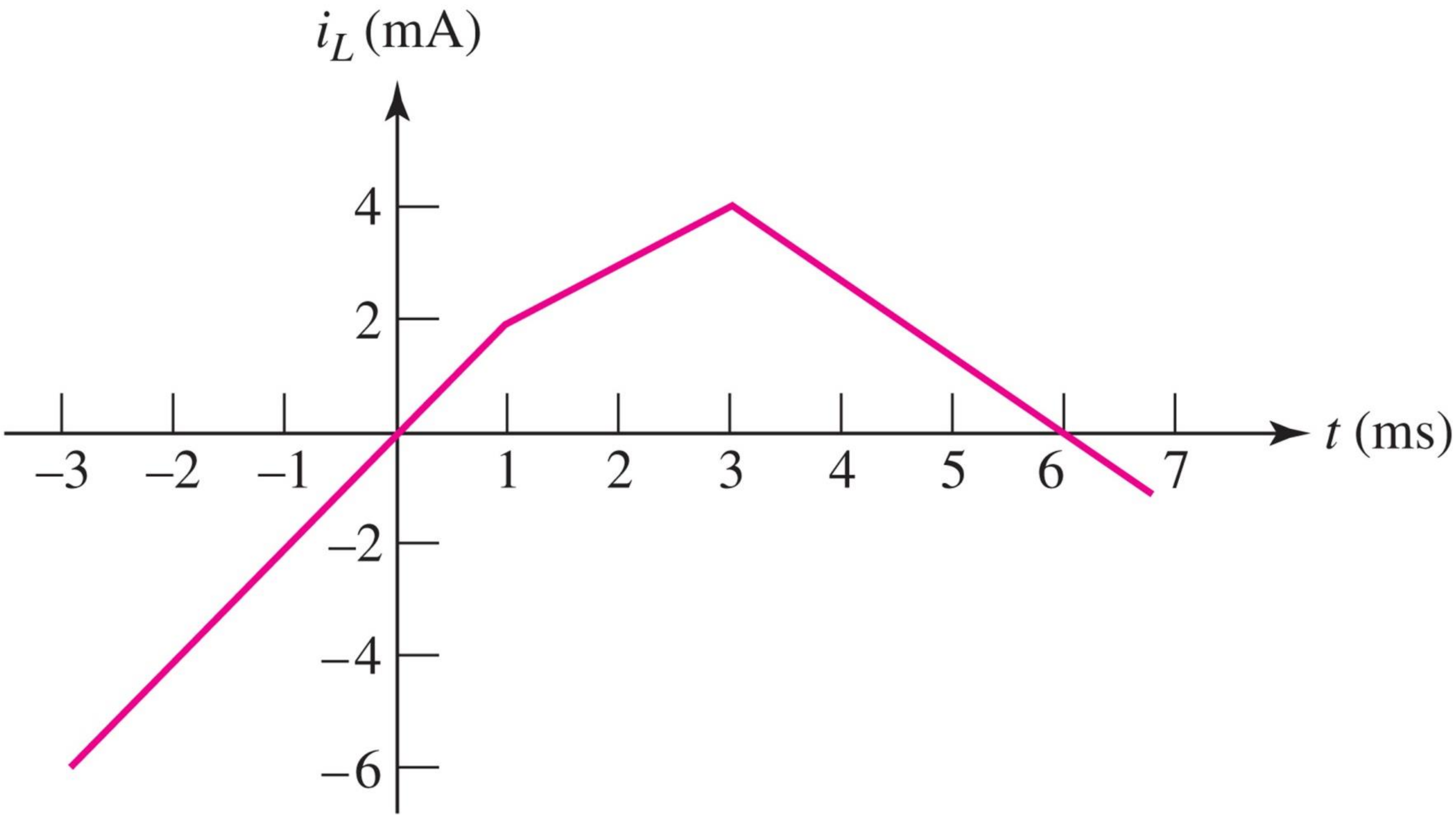
$$v = L\frac{di}{dt} = 3\frac{di}{dt}$$



(b)

Practice: 7.3

The current through a 0.2-H inductor is shown in the figure. Assume the passive sign convention, and find v_L at t equal to: (a) 0; (b) 2 ms; (c) 6 ms.



Voltage-Current Relationships:

From

$$v = L\frac{di}{dt}$$

Rewritten

$$di = \frac{1}{L}v(t)dt$$

Thus

$$\int_{i(t_0)}^{i(t)} di' = \frac{1}{L}\int_{t_0}^{t} v(t')dt'$$

Lead to

$$i(t) - i(t_0) = \frac{1}{L}\int_{t_0}^{t} v dt'$$ or $$\boxed{i(t) = \frac{1}{L}\int_{t_0}^{t} v dt' + i(t_0)}$$

Example: 7.5

The voltage across a 2-H inductor is known to be $6\cos 5t$ V. determine the resulting inductor current if ถ้า $i(t=-\pi/2)=1$ A.

$$i(t)=\frac{1}{L}\int_{t_0}^{t} v\,dt' + i(t_0)$$

$$i(t)=\frac{1}{2}\int_{t_0}^{t} 6\cos 5t'\,dt' + i(t_0)$$

$$\begin{aligned} i(t) \quad &= \frac{1}{2}\left(\frac{6}{5}\right)\sin 5t - \frac{1}{2}\left(\frac{6}{5}\right)\sin 5t_0 + i(t_0) \\ &= 0.6\sin 5t - 0.6\sin 5t_0 + i(t_0) \end{aligned}$$

Energy Storage:

The absorbed power: $$p = vi = i \cdot L \frac{di}{dt}$$

The energy:

$$\int_{t_0}^{t} p \cdot dt' = L \int_{t_0}^{t} i \frac{di}{dt'} dt' = L \int_{i(t_0)}^{i(t)} i \cdot di = \frac{1}{2} L \left\{ i(t)^2 - i(t_0)^2 \right\}$$

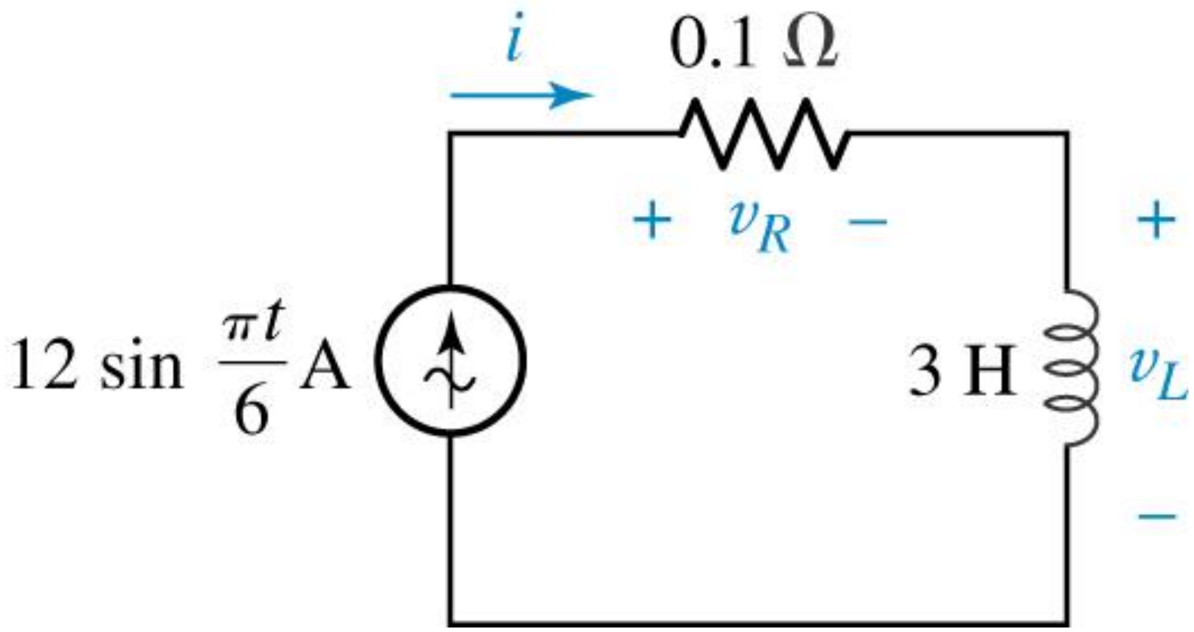
Thus: $$w_L(t) - w_L(t_0) = \frac{1}{2} L \left\{ i(t)^2 - i(t_0)^2 \right\}$$

Assume $i(t_0) = 0$

We simply have: $$\boxed{w_L(t) = \frac{1}{2} L i^2}$$

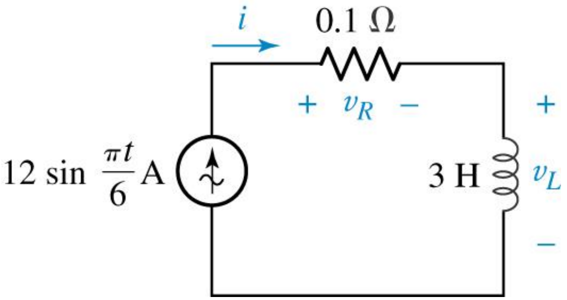
Example: 7.6

find the maximum energy stored in the inductor and calculate how much energy is dissipated in the resistor in the time during which the energy is being stored in and then recovered from the inductor



Example:

find the maximum energy stored in the inductor



$$w_L(t) = \frac{1}{2} L i^2$$

$$= \frac{1}{2}(3)\left(12 \sin \frac{\pi t}{6}\right)^2 = 216 \sin^2 \frac{\pi t}{6}$$

the maximum energy stored in the inductor:

= 216 J. @t=3s.

Example:

calculate how much energy is dissipated in the resistor in the time during which the energy is being stored in and then recovered from the inductor:

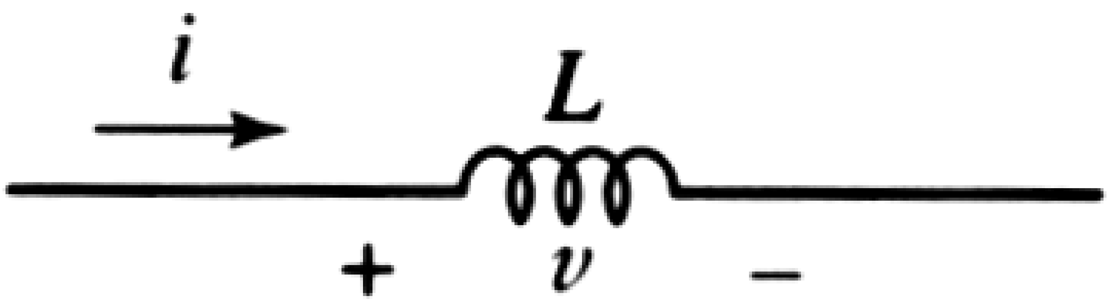
$$P_R = i^2 R = \left(12\sin\frac{\pi t}{6}\right)^2 (0.1) = 14.4\sin^2\frac{\pi t}{6} \quad \text{W.}$$

$$\therefore w_R = \int_0^6 p_R dt = 43.2 \quad \text{J.}$$

$$\approx \frac{43.2}{216} = 20\%$$

Practice: 7.4

Let L = 25 mH for the inductor of the figure (a) find v at $t = 12ms$ if $i = 10te^{-100t}$ A. (b) Find i at $t = 0.1\ s$ if $v = 6e^{-12t}\ V$ and $i(0) = 10\ A$. If $i = 8(1 - e^{-40t})\ mA$, find: (c) the power being delivered to the inductor at $t = 50\ ms$, and (d) the energy stored in the inductor at $t = 40\ ms$.



Characteristics of an Ideal L.:

1. There is no voltage across an inductor *if* the current through it is not changing with time. An inductor is therefore a *short circuit to dc.*

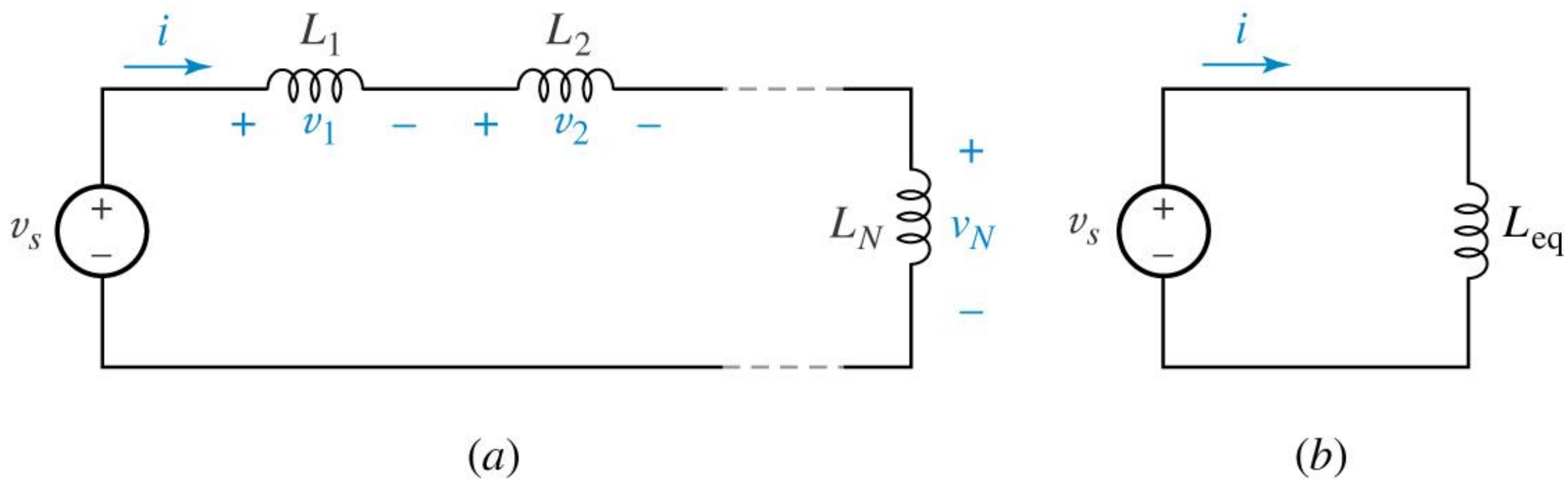
2. A finite amount of energy can be stored in an inductor even if the voltage across the inductor is zero, such as when the current through it is constant.

Characteristics of an Ideal L.:

3. It is impossible to change the current through an inductor by a finite amount in zero time, for this requires an infinite voltage across the inductor. An inductor resists an abrupt change in the current through it in a manner analogous to the way a mass resists an abrupt change in its velocity.

4. The inductor never dissipates energy, but only stores it. Although this *is* true for the *mathematical* model, it is not true for a *physical* inductor due to series resistances.

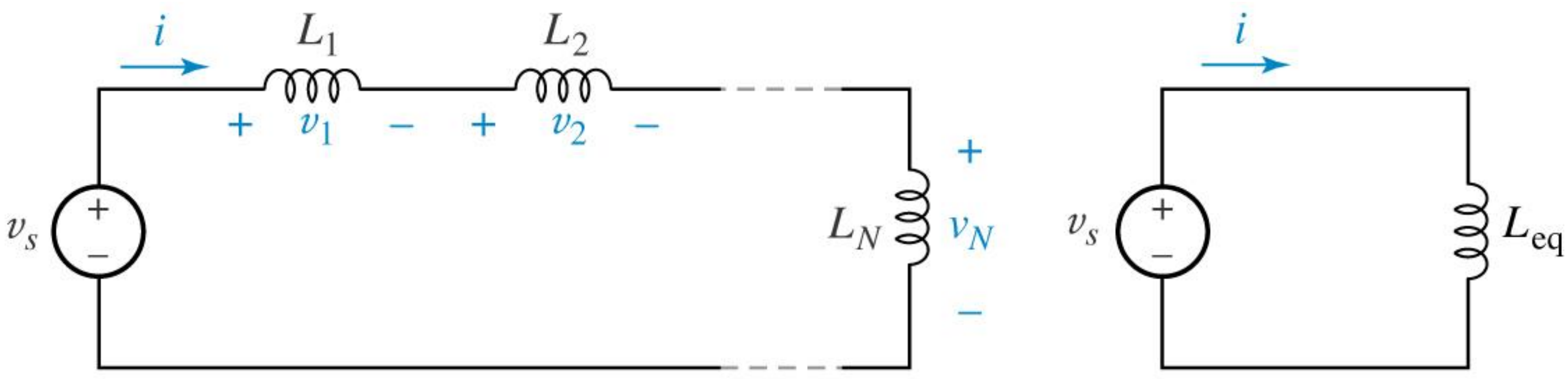
Inductance Combinations:



(a) N inductors connected in series; (b) equivalent circuit;

$$
\begin{aligned}
v_s &= v_1 + v_2 + \ldots + v_N \\
&= L_1 \frac{di}{dt} + L_2 \frac{di}{dt} + \ldots + L_N \frac{di}{dt} \\
&= (L_1 + L_2 + \ldots + L_N) \frac{di}{dt}
\end{aligned}
$$

Inductance Combinations:



(a) (b)

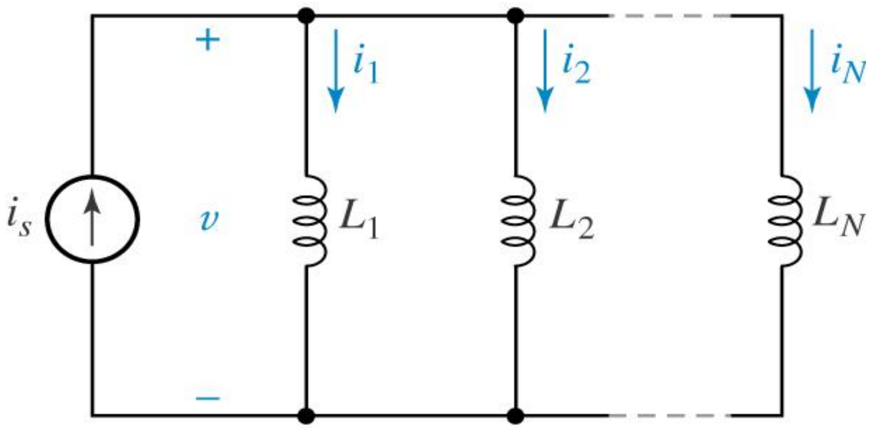
(a) N inductors connected in series; (b) equivalent circuit;

$$L_{eq} = L_1 + L_2 + \ldots + L_N$$

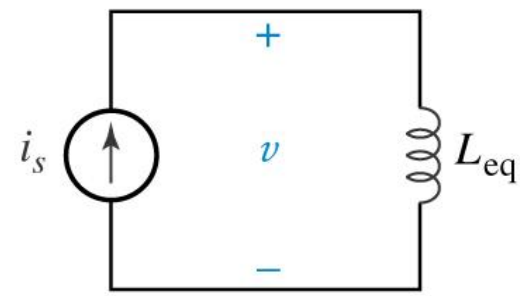
Or

$$L_{eq} = \sum_{n=1}^{N} L_n$$

Inductance Combinations:



N inductors connected in parallel;



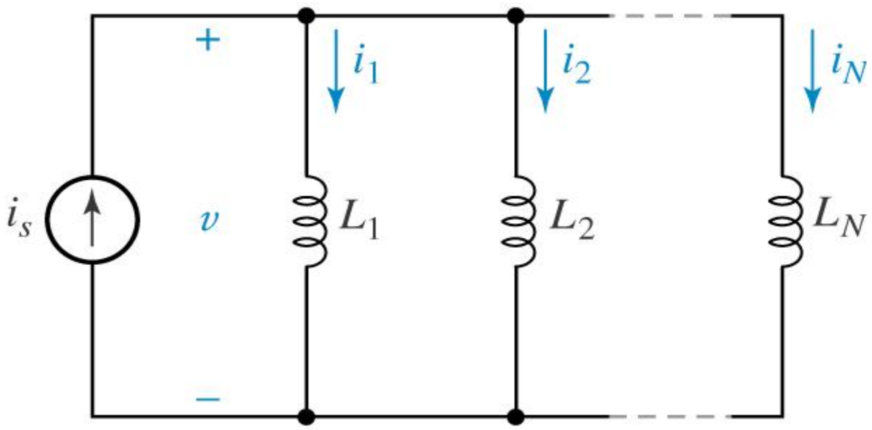
equivalent circuit for circuit in

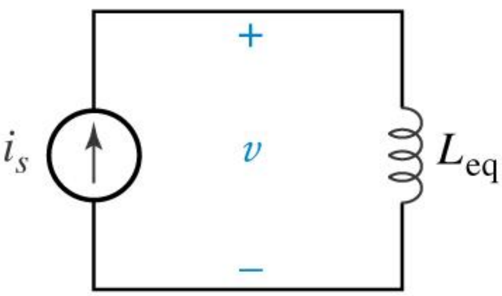
$$i_s(t) = \frac{1}{L_{eq}} \int_{t_0}^{t} v\,dt' + i_s(t_0)$$

$$i_s = \sum_{n=1}^{N} i_n = \sum_{n=1}^{N} \left[\frac{1}{L_n} \int_{t_0}^{t} v\,dt' + i_n(t_0) \right]$$

$$= \left(\sum_{n=1}^{N} \frac{1}{L_n} \right) \int_{t_0}^{t} v\,dt' + \sum_{n=1}^{N} i_n(t_0)$$

Inductance Combinations:



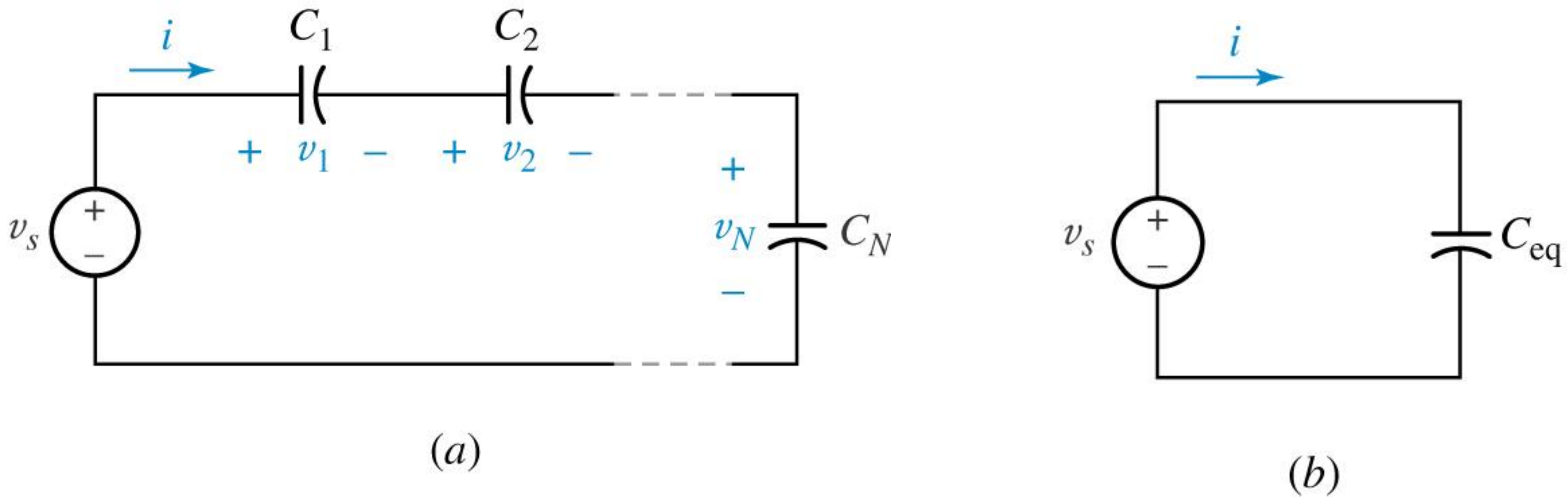


equivalent circuit for circuit in

N inductors connected in parallel;

$$L_{eq} = \frac{1}{1/L_1 + 1/L_2 + \ldots + 1/L_N}$$

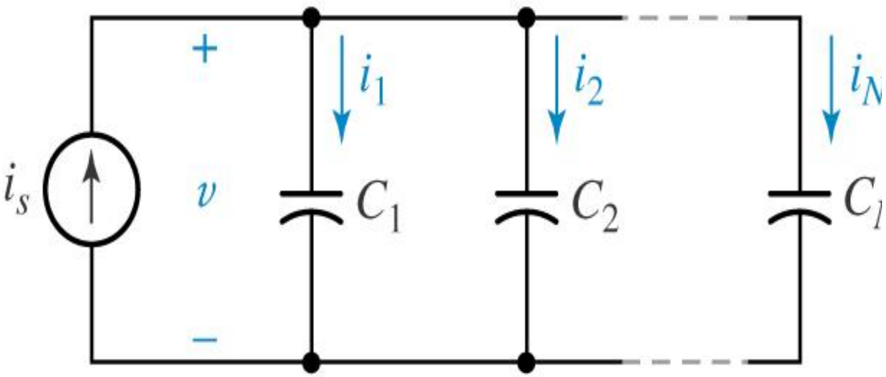
Capacitance Combinations:



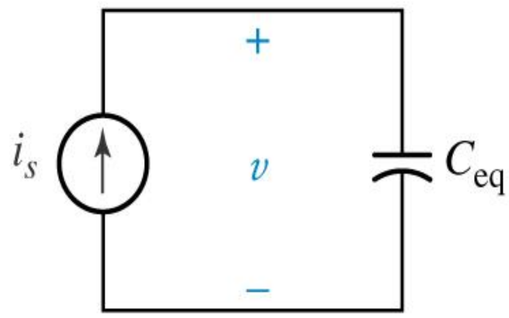
$$C_{eq} = \frac{1}{1/C_1 + 1/C_2 + \ldots + 1/C_N}$$

(a) N capacitors connected in series; (b) equivalent circuit;

Capacitance Combinations:



(c)



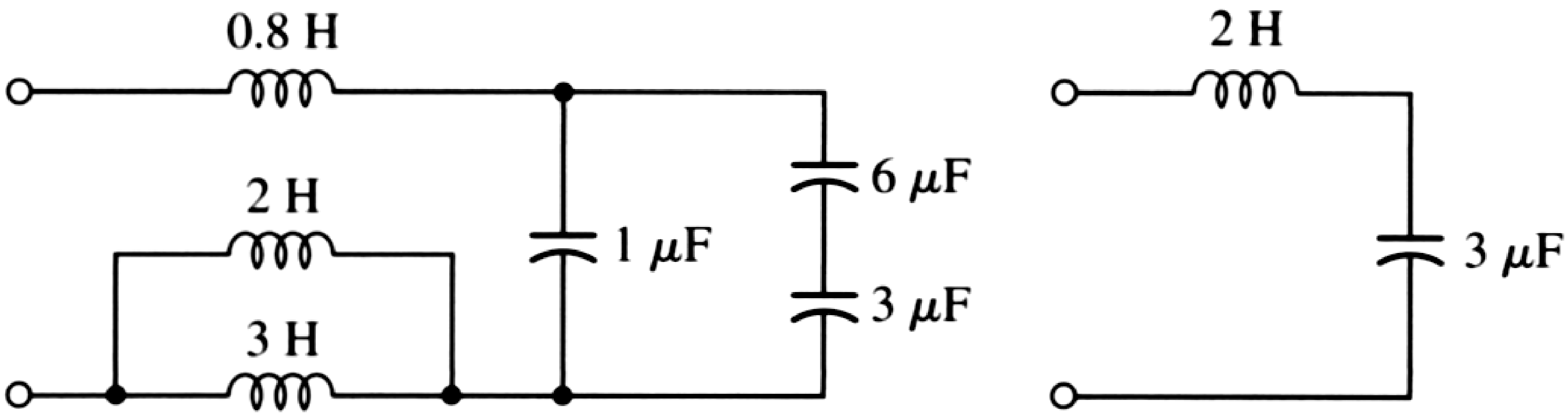
(d)

$$C_{eq} = C_1 + C_2 + \ldots + C_N$$

(c) N capacitors connected in parallel; (d) equivalent circuit to (c).

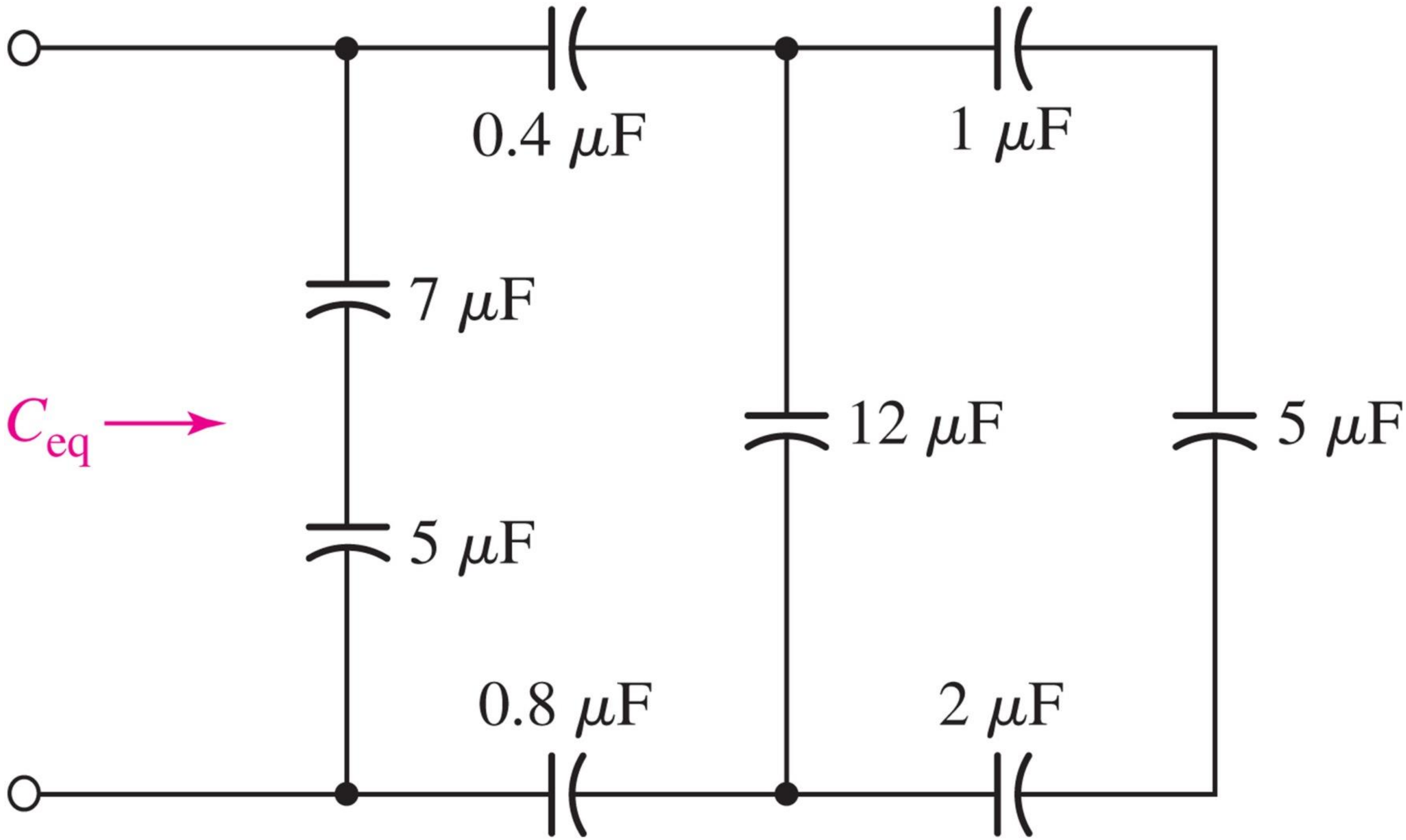
Example: 7.7

simplify the network using series/parallel combinations.



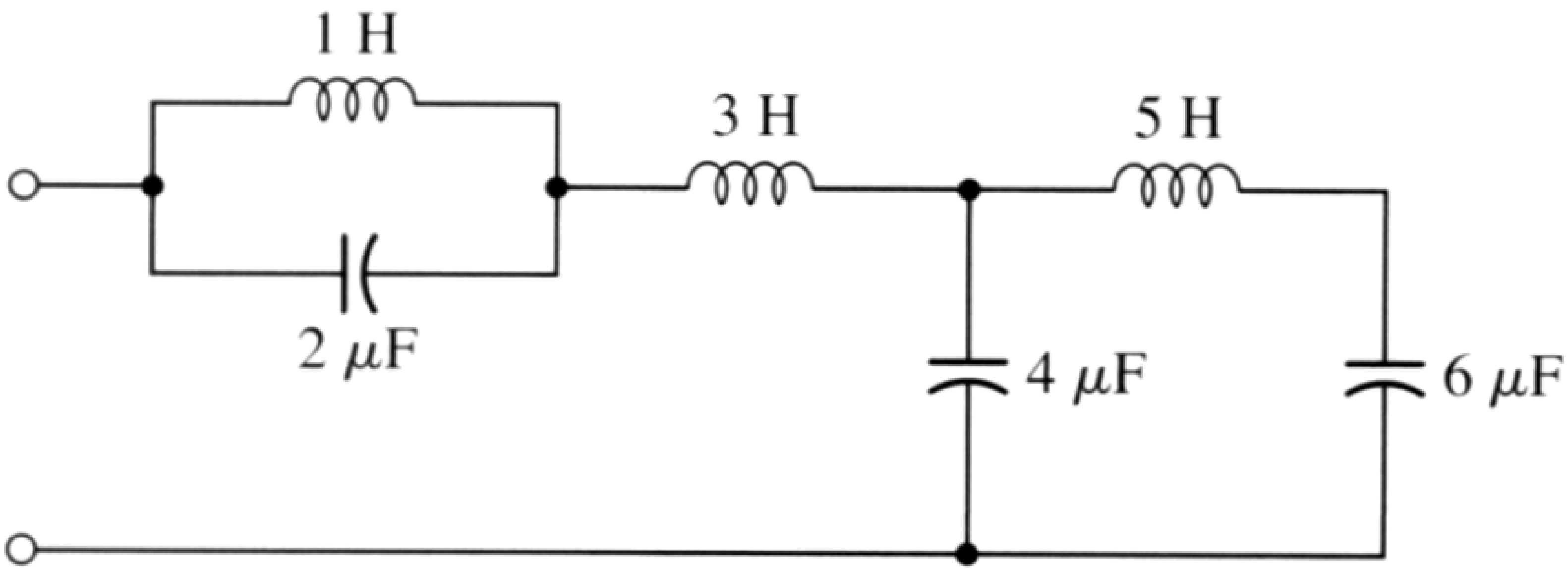
(a) (b)

Practice: 7.5



note:

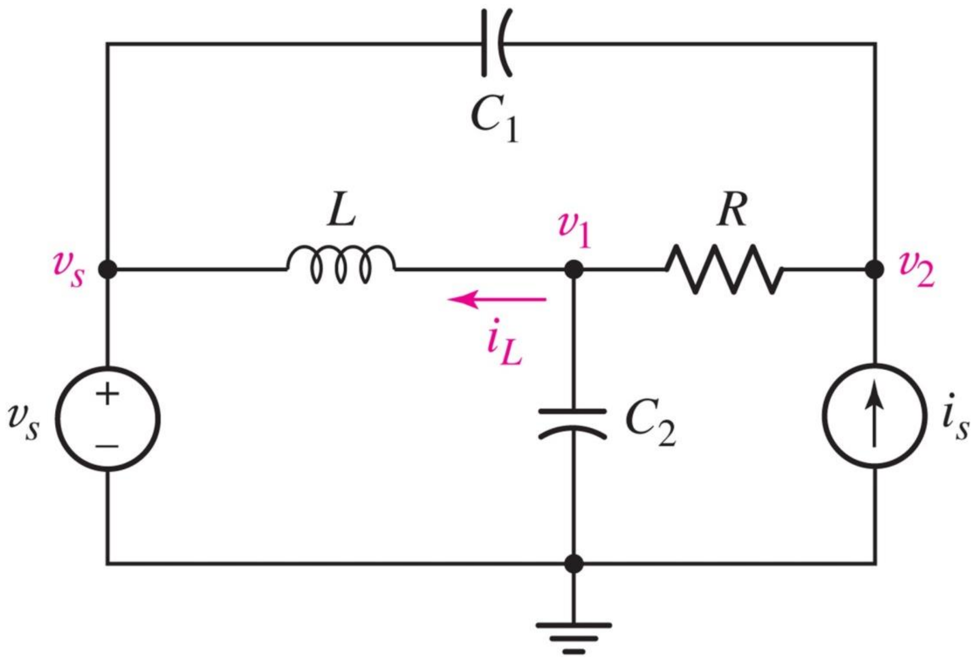
No series or parallel combinations of either the inductors or the capacitors can be made



Consequences of Linearity:

There will be constant-coefficient linear integrodifferential equations

Example: write appropriate nodal equations for the circuit



$$i_C(t) = C\frac{dv_C(t)}{dt}$$

$$i_L(t) = \frac{1}{L}\int_{t_0}^{t} v_L\,dt' + i_L(t_0)$$

Consequences of Linearity:

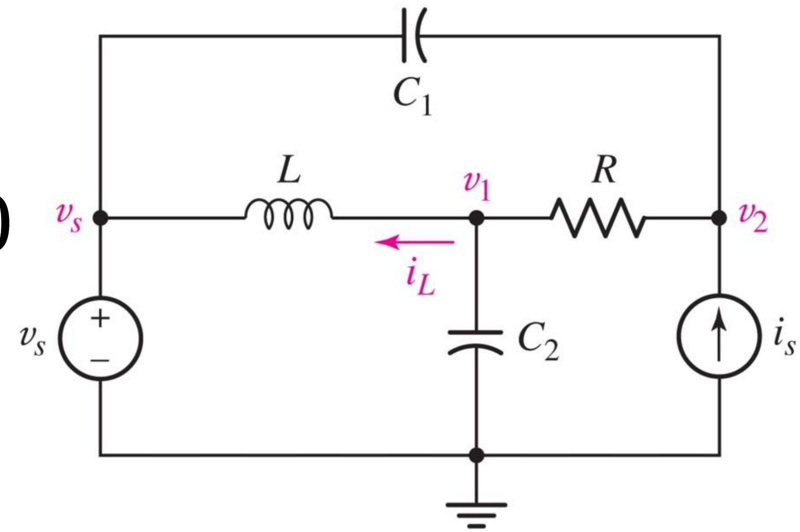
Example: write appropriate nodal equations for the circuit

Node 1:

$$\frac{1}{L}\int_{t_0}^{t}(v_1 - v_s)dt' + i_L(t_0) + \frac{v_1 - v_2}{R} + C_2\frac{dv_1}{dt} = 0$$

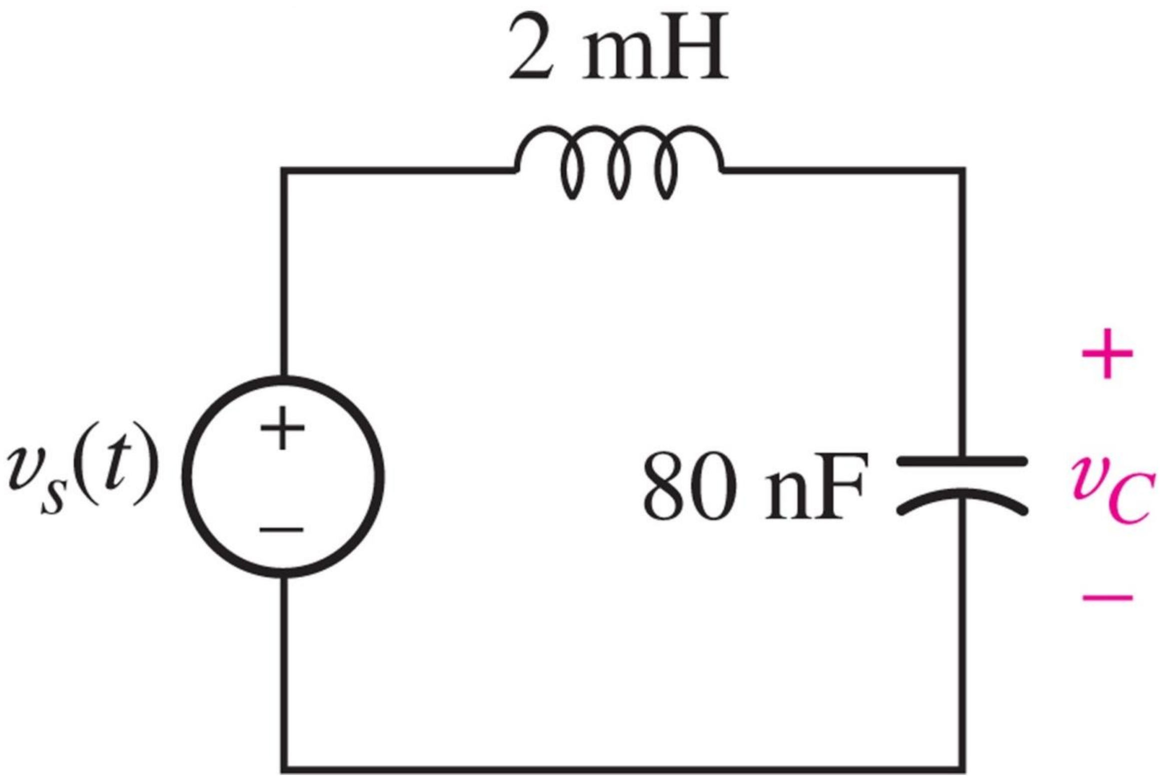
Node 2:

$$C_1\frac{d(v_2 - v_s)}{dt} + \frac{v_2 - v_1}{R} - i_s = 0$$



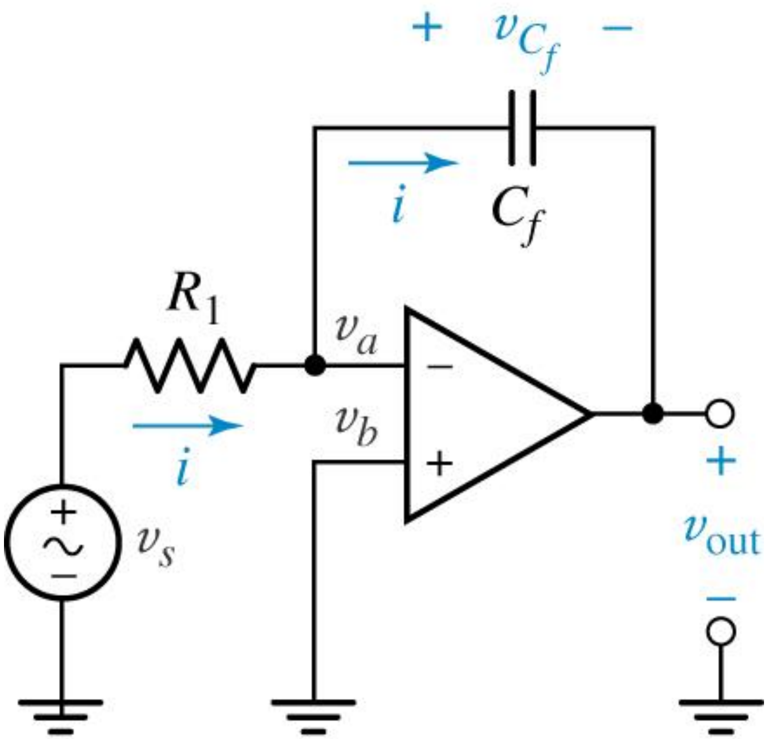
Practice: 7.6

If $v_c(t) = 4cos10^5t$ V in the circuit, find $v_s(t)$.



Simple Op Amp C/T with C.:

An ideal op amp connected as an integrator.



At the inverting input:

$$0 = \frac{v_a - v_s}{R_1} + i$$

Note:

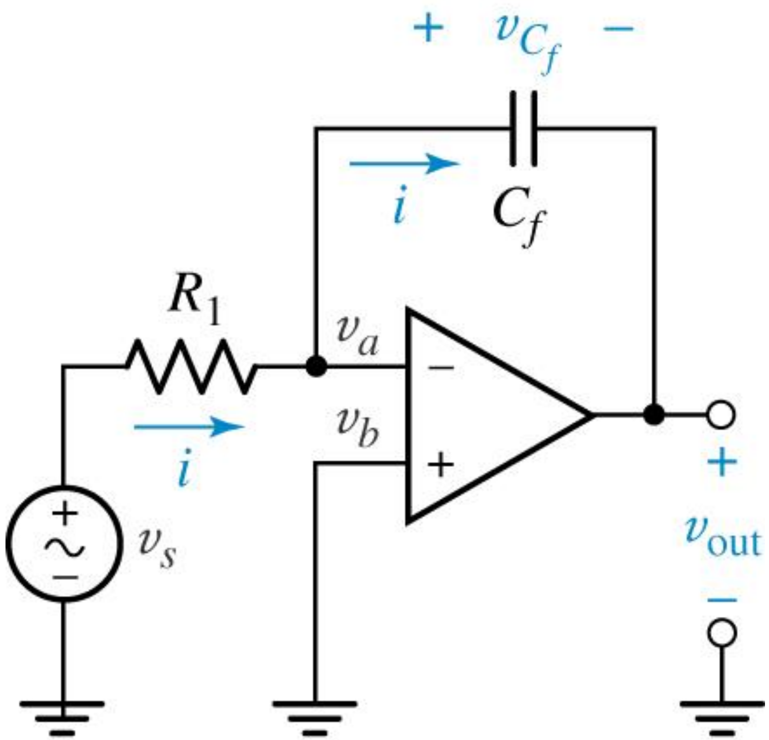
$$i = C_f \frac{dv_{C_f}}{dt}$$

Resulting in:

$$0 = \frac{v_a - v_s}{R_1} + C_f \frac{dv_{C_f}}{dt}$$

Simple Op Amp C/T with C.:

An ideal op amp connected as an integrator.



$$0 = \frac{v_a - v_s}{R_1} + C_f \frac{dv_{C_f}}{dt}$$

Ideal op amp: $v_a = v_b = 0$

$$0 = \frac{-v_s}{R_1} + C_f \frac{dv_{C_f}}{dt}$$

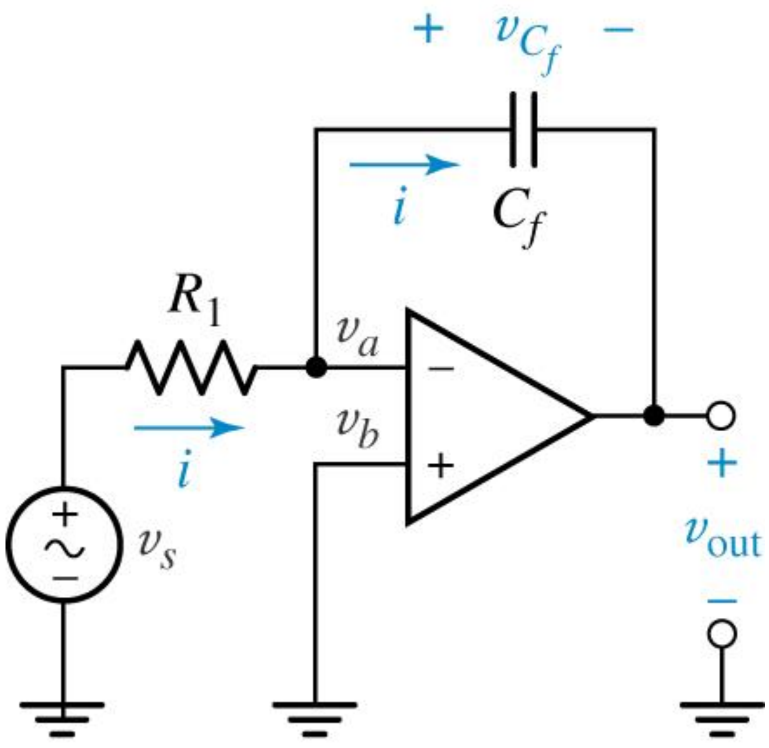
Solving for v_{out}

$$v_{C_f} = v_a - v_{out} = 0 - v_{out} = \frac{1}{R_1 C_f} \int_0^t v_s dt' + v_{C_f}(0)$$

Simple Op Amp C/T with C.:

An ideal op amp connected as an integrator.

$$v_{C_f} = v_a - v_{out} = 0 - v_{out} = \frac{1}{R_1 C_f}\int_0^t v_s dt' + v_{C_f}(0)$$

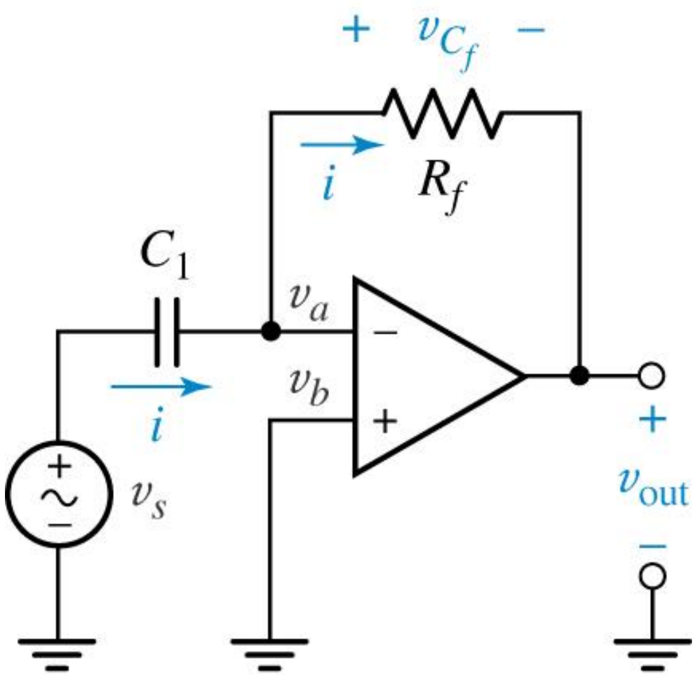


Or

$$v_{out} = -\frac{1}{R_1 C_f}\int_0^t v_s dt' - v_{C_f}(0)$$

Simple Op Amp C/T with C.:

An ideal op amp connected as a differentiator.



$$0 = C_1 \frac{dv_{C_1}}{dt} + \frac{v_a - v_{out}}{R_f}$$

Ideal op amp: $v_a = v_b = 0$

$$C_1 \frac{dv_{C_1}}{dt} = \frac{v_{out}}{R_f}$$

Solving for v_{out}

$$v_{out} = R_f C_1 \frac{dv_{C_1}}{dt}$$

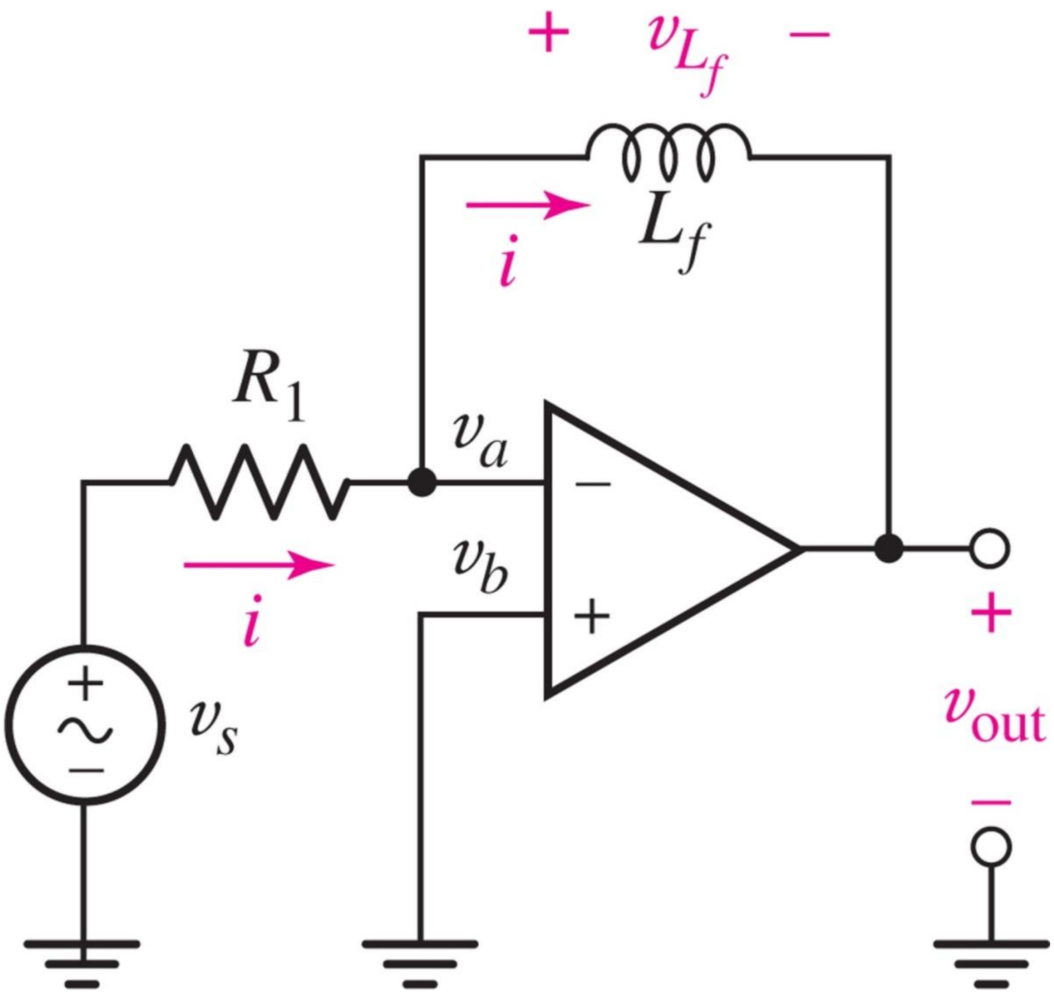
$$\therefore v_{out} = -R_f C_1 \frac{dv_s}{dt}$$

Note:

$$v_{C_1} = v_a - v_s = -v_s$$

Practice: 7.7

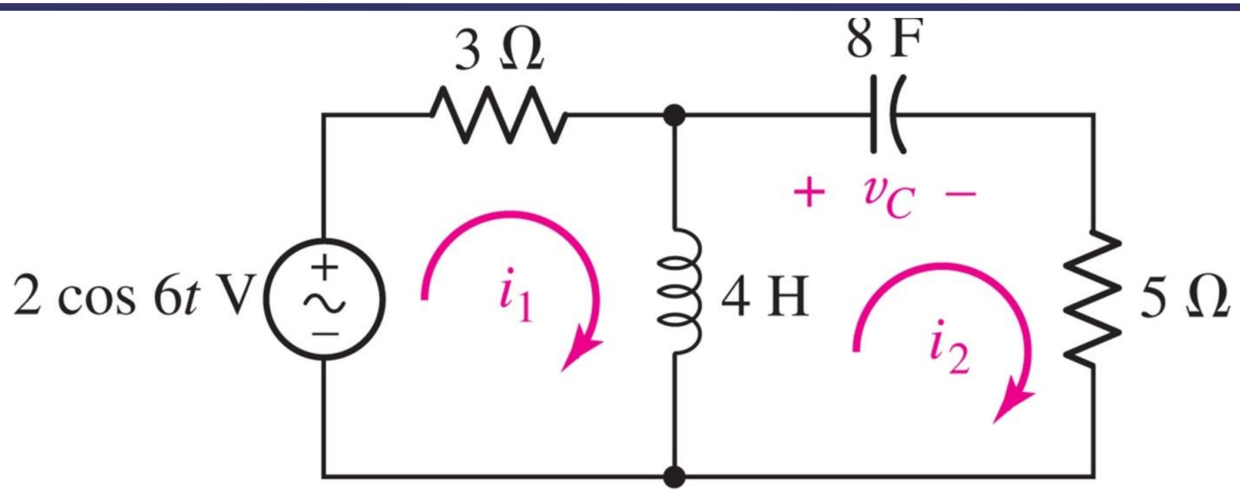
Derive an expression for v_{out} in term of v_{in}



Duality:

Two circuits are "duals" if the mesh equations that characterize one of them have the same mathematical form as the nodal equations that characterize the other.

Duality:



The mesh equations are:

$$3i_1 + 4\frac{di_1}{dt} - 4\frac{di_2}{dt} = 2\cos 6t$$

$$-4\frac{di_1}{dt} + 4\frac{di_2}{dt} + \frac{1}{8}\int_0^t i_2 dt' + 5i_2 = -10$$

Note: the capacitor voltage v_C is assumed to be 10V. at t=0

Duality:

By replacing the mesh current i1 and i2 with two node-to-reference voltage v1 and v2.

$$3i_1 + 4\frac{di_1}{dt} - 4\frac{di_2}{dt} = 2\cos 6t$$

$$-4\frac{di_1}{dt} + 4\frac{di_2}{dt} + \frac{1}{8}\int_0^t i_2 dt' + 5i_2 = -10$$

We obtain:

$$3v_1 + 4\frac{dv_1}{dt} - 4\frac{dv_2}{dt} = 2\cos 6t$$

$$-4\frac{dv_1}{dt} + 4\frac{dv_2}{dt} + \frac{1}{8}\int_0^t v_2 dt' + 5v_2 = -10$$

Duality:

$$3v_1 + 4\frac{dv_1}{dt} - 4\frac{dv_2}{dt} = 2\cos 6t$$

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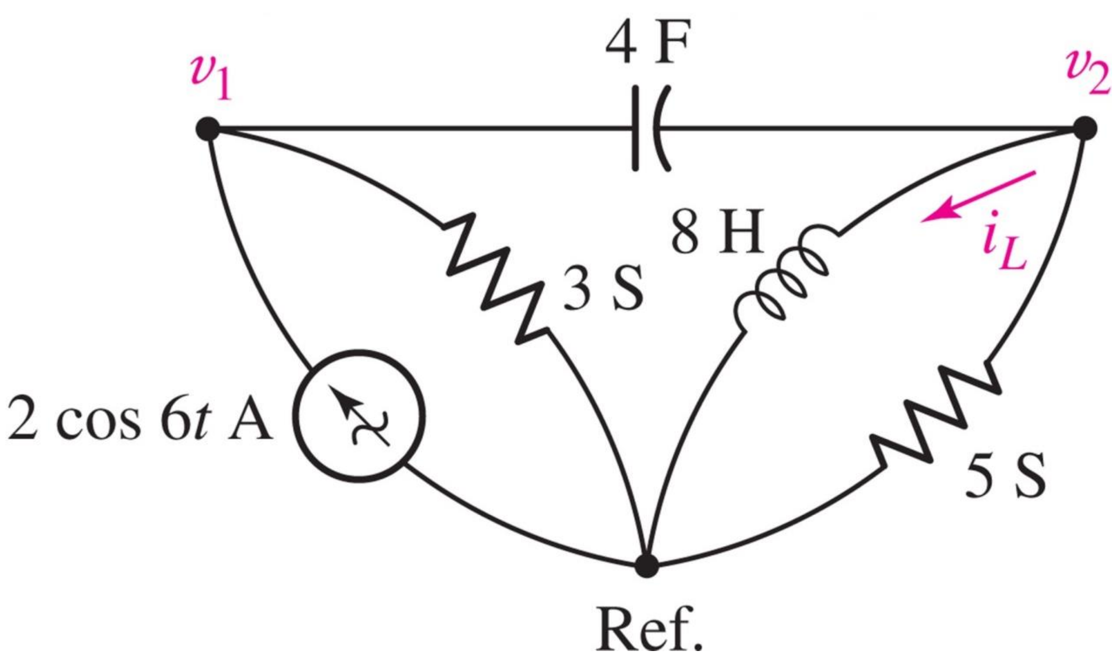
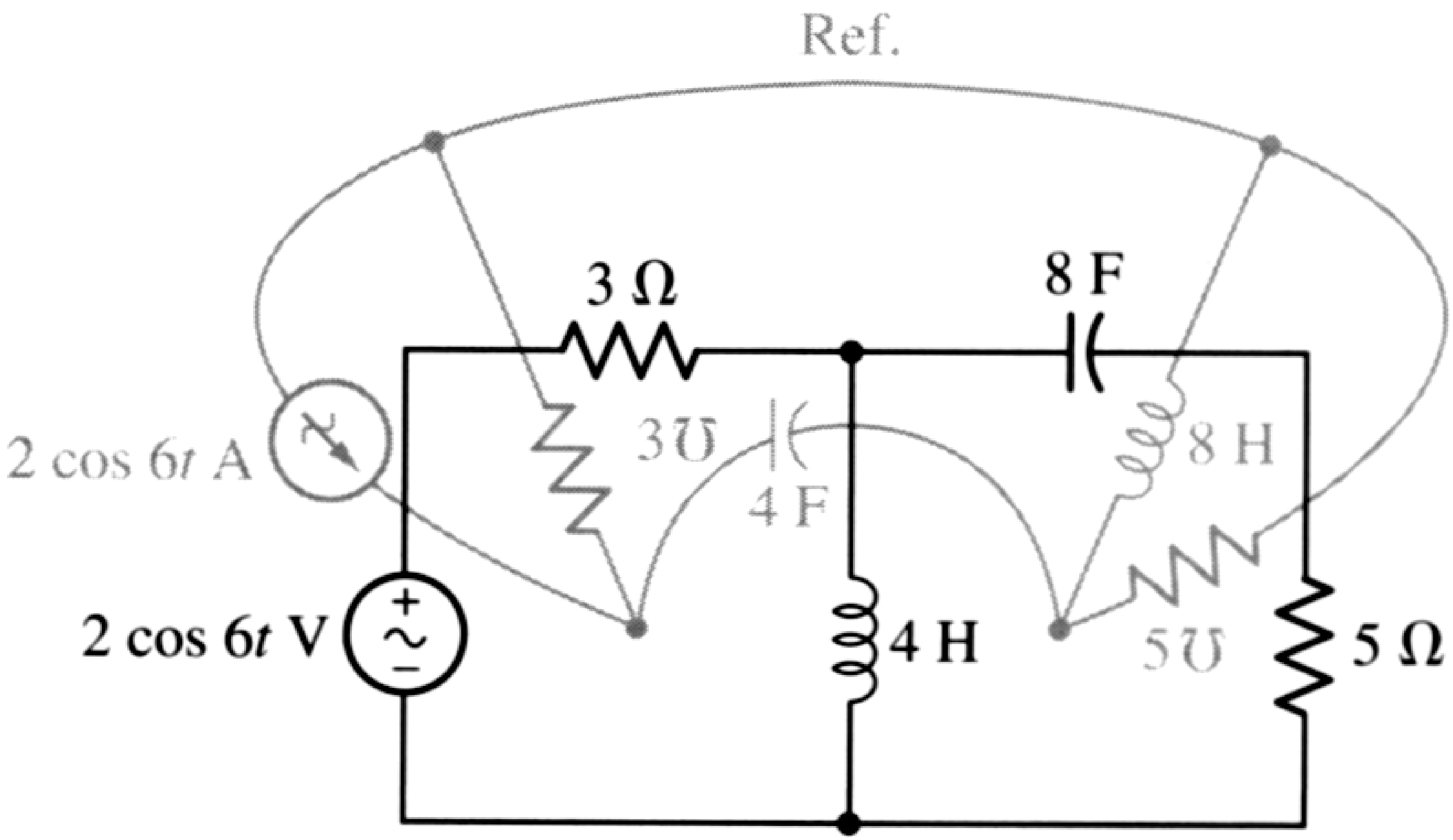
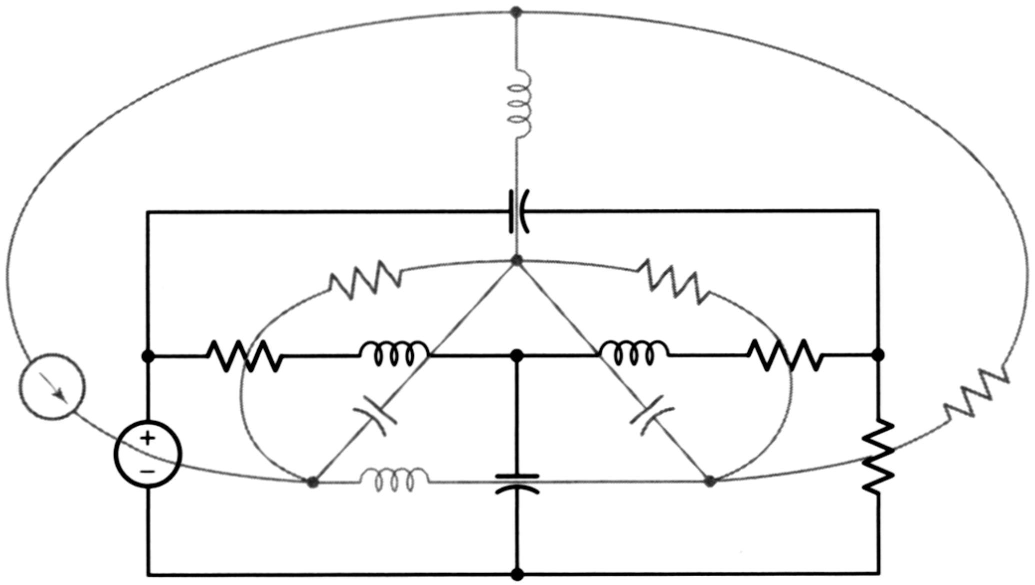


Duality:

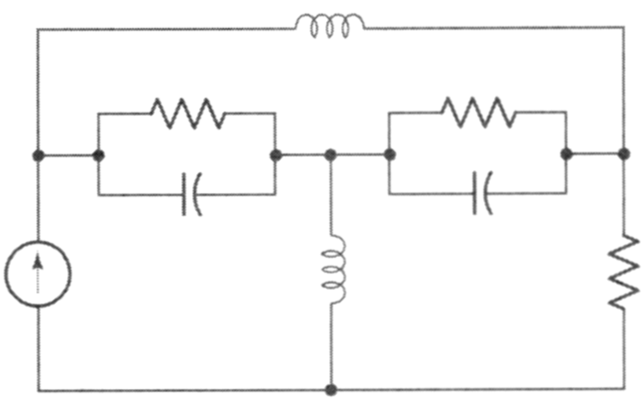
We can constructed the dual of the circuit directly from the circuit diagram



Example:



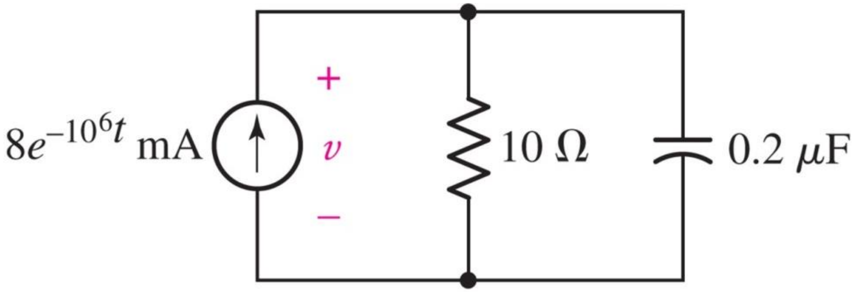
(a)



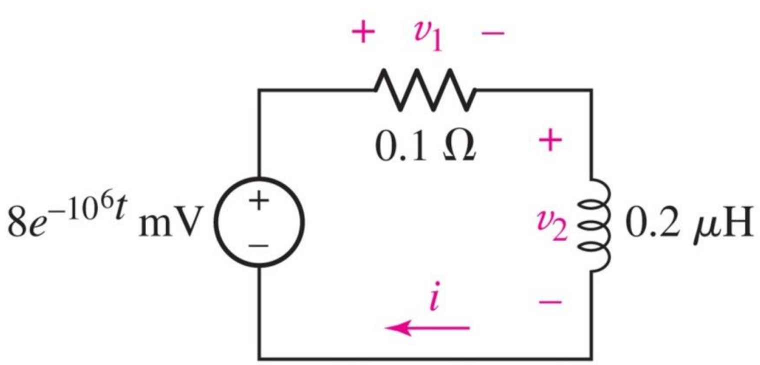
(b)

Practice: 7.8

Write the single nodal equation for the circuit of Fig. a, and show, by direct substitution, that $v = -80e^{-10^6 t}\, mV$ is a solution. Knowing this, find (a) v_1; (b) v_2; and (c) i for the circuit of Fig. b.



(a)



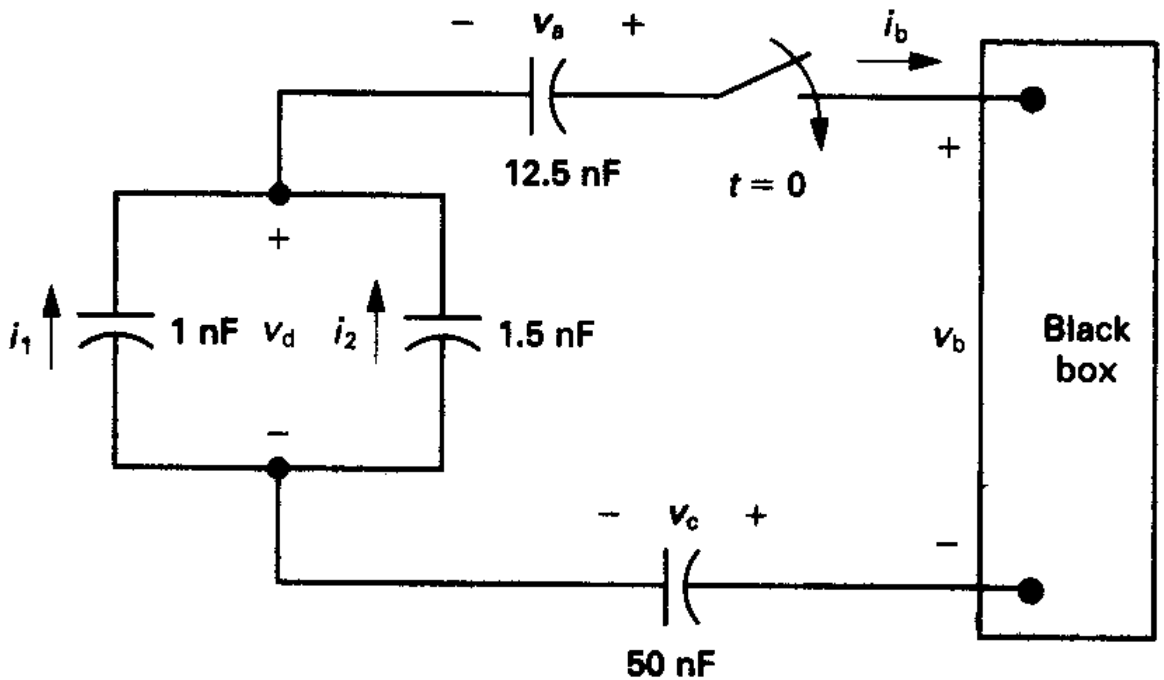
(b)

Example:

The four capacitors in the circuit are connected across the terminals of a black box at t=0. The resulting current i_b for t >= 0 is:

$$i_b = 50e^{-250t}\,\mu A.$$

If $v_a(0)$=15 V.
$v_c(0)$=-45 V.
and
$v_d(0)$=40 V.



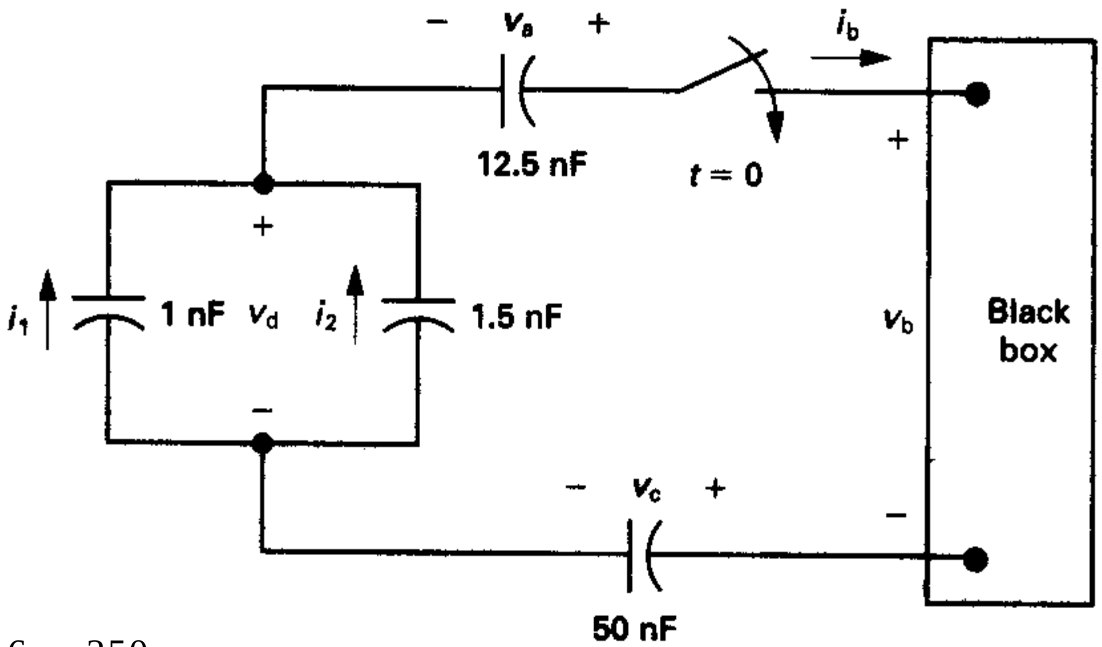
Find the following:

$$t \geq 0 : (a) v_b(t), (b) v_a(t), (c) v_c(t), (d) v_d(t), (e) i_1(t), and (f) i_2(t)$$

Example:

Find

$$(a)v_b(t), t \geq 0$$



$$C_1 = 1 + 1.5 = 2.5nF.$$

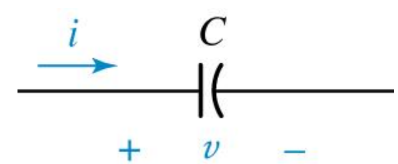
$$\frac{1}{C_2} = \frac{1}{2.5} + \frac{1}{12.5} + \frac{1}{50} = \frac{1}{2}$$

$$v_b(t) = -\frac{10^9}{2}\int_0^t 50\times 10^{-6} e^{-250x} dx + 100$$

$$= -25000 \left.\frac{e^{-250x}}{-250}\right|_0^t + 100$$

$$= 100\left(e^{-250t} - 1\right) + 100$$

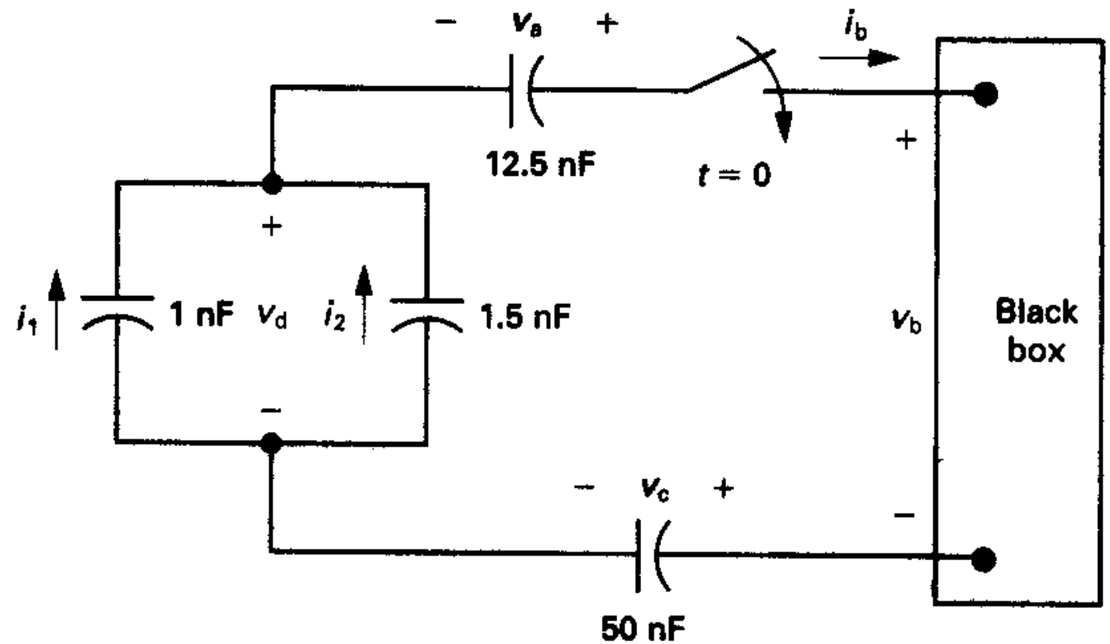
$$= 100e^{-250t} V.$$



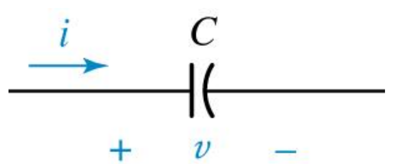
$$i(t) = C\frac{dv(t)}{dt}$$

$$v(t) = \frac{1}{C}\int_{t_0}^t i(t')dt' + v(t_0)$$

Example:

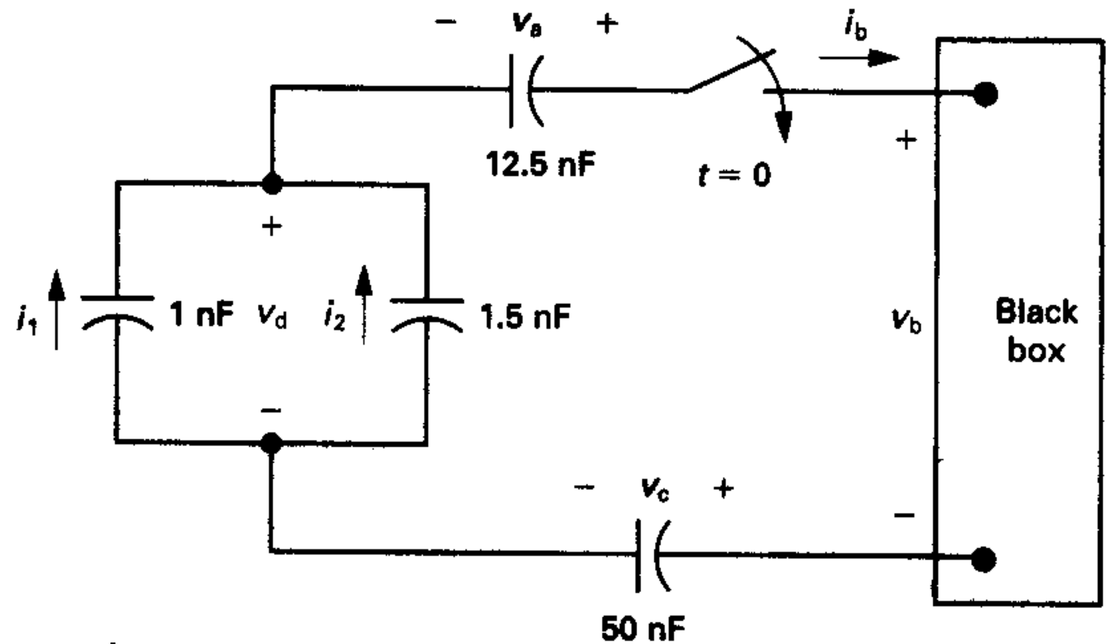


[b]
$$v_a = -\frac{10^9}{12.5}\int_0^t 50 \times 10^{-6} e^{-250x}\,dx + 15$$
$$= -4000\frac{e^{-250x}}{-250}\Big|_0^t + 15$$
$$= 16(e^{-250t} - 1) + 15$$
$$= 16e^{-250t} - 1\text{ V}$$

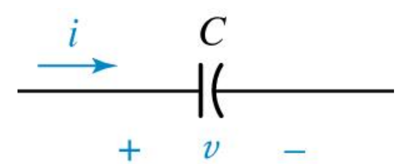


$$i(t) = C\frac{dv(t)}{dt}$$
$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(t')dt' + v(t_0)$$

Example:



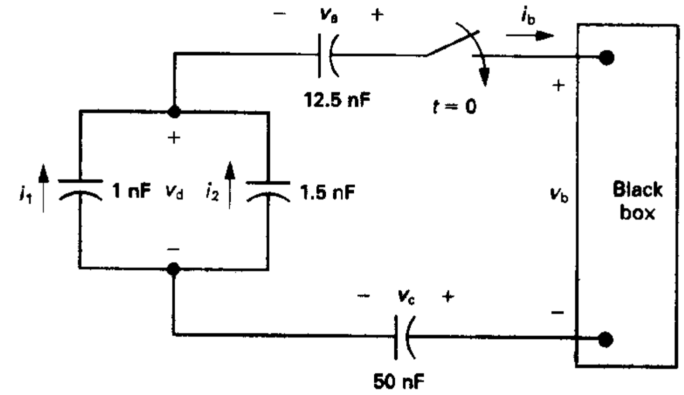
$$
\begin{aligned}
\text{[c]} \quad v_c &= \frac{10^9}{50}\int_0^t 50 \times 10^{-6} e^{-250x}\, dx - 45 \\
&= 1000 \frac{e^{-250x}}{-250}\Bigg|_0^t - 45 \\
&= -4(e^{-250t} - 1) - 45 \\
&= -4e^{-250t} - 41 \text{ V}
\end{aligned}
$$



$$i(t) = C\frac{dv(t)}{dt}$$

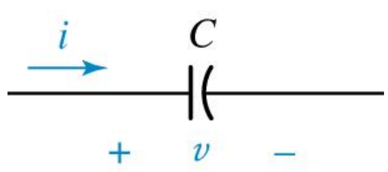
$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(t')dt' + v(t_0)$$

Example:



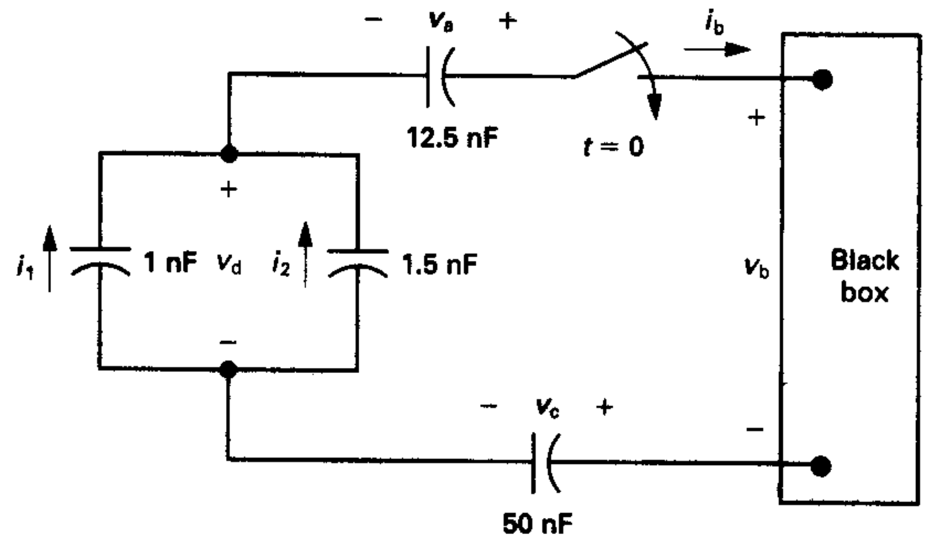
[d]
$$v_d = -\frac{10^9}{2.5}\int_0^t 50\times10^{-6}e^{-250x}\,dx + 40$$
$$= -20{,}000\frac{e^{-250x}}{-250}\Big|_0^t + 40$$
$$= 80(e^{-250t} - 1) + 40$$
$$= 80e^{-250t} - 40\text{ V}$$

CHECK:
$$v_b = v_d + v_a - v_c$$
$$= 80e^{-250t} - 40 + 16e^{-250t} - 1 + 4e^{-250t} + 41$$
$$= 100e^{-250t}\text{ V} \quad \text{(checks)}$$



$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(t')dt' + v(t_0)$$

Example:



[e] $$i_1 = -10^{-9}\frac{d}{dt}[80e^{-250t} - 40]$$

$$= -10^{-9}(-20{,}000e^{-250t})$$

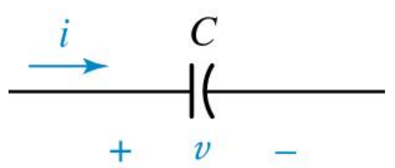
$$= 20e^{-250t}\ \mu A$$

[f] $$i_2 = -1.5 \times 10^{-9}\frac{d}{dt}[80e^{-250t} - 40]$$

$$= -1.5 \times 10^{-9}(-20{,}000e^{-250t})$$

$$= 30e^{-250t}\ \mu A$$

CHECK: $i_1 + i_2 = 50e^{-250t}\ \mu A = i_b$

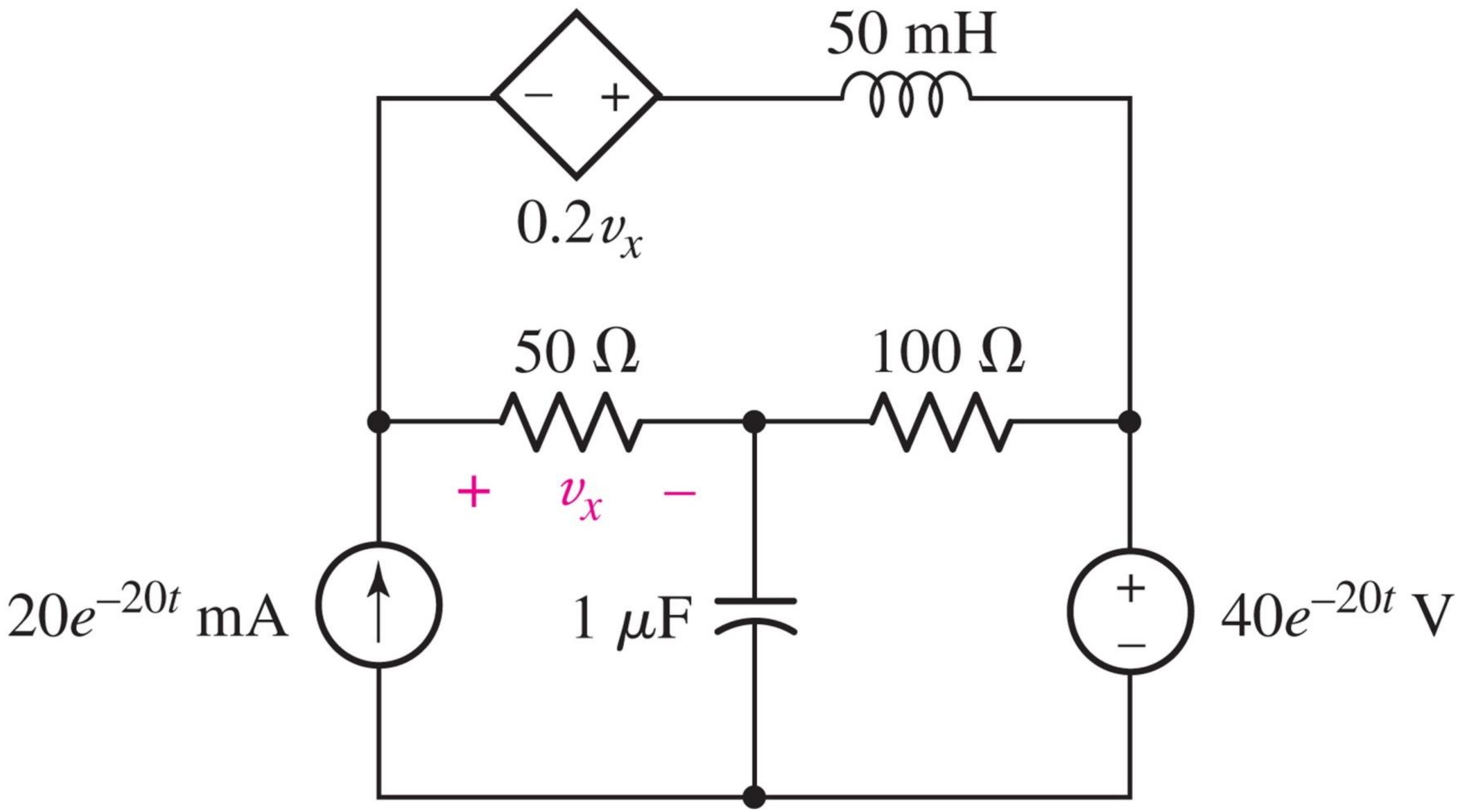


$$i(t) = C\frac{dv(t)}{dt}$$

$$v(t) = \frac{1}{C}\int_{t_0}^{t} i(t')dt' + v(t_0)$$

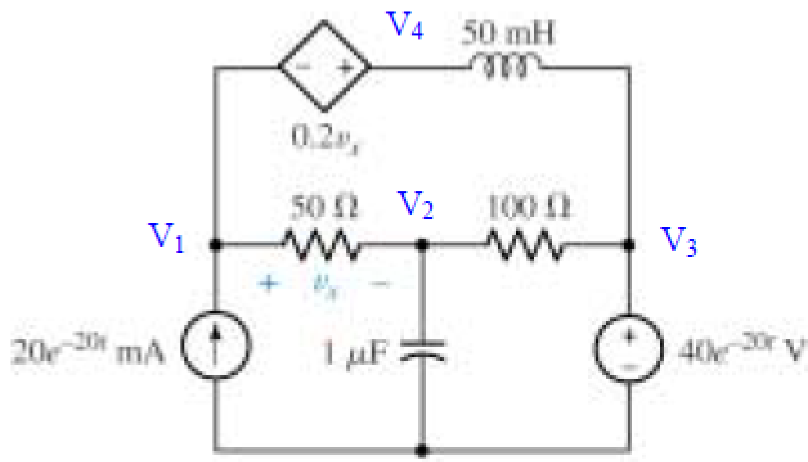
Ex:

Ch7-40, page 208, Sixth Edition: write a complete set of nodal equation



Ex:

We begin by selecting the bottom node as the reference and assigning four nodal voltages:



1, 4 Supernode: $$\boxed{20\times10^{-3}\, e^{-20t} = \frac{V_1 - V_2}{50} + 0.02\times10^{3}\int_0^t \left(V_4 - 40e^{-20t'}\right)dt'} \qquad [1]$$

and: $V_1 - V_4 = 0.2\, V_x$ or $$\boxed{0.8V_1 + 0.2\, V_2 - V_4 = 0} \qquad [2]$$

Node 2: $$\boxed{0 = \frac{V_2 - V_1}{50} + \frac{V_2 - 40e^{-20t}}{100} + 10^{-6}\frac{dV_2}{dt}} \qquad [3]$$

Homework:

Reference:

W.H. Hayt, Jr., J.E. Kemmerly, S.M. Durbin, Engineering Circuit Analysis, Sixth Edition.