Let's continue on the discussion of the FT

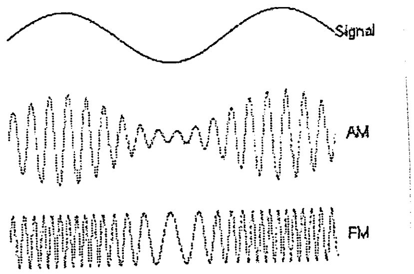
The freq shifting property: $\mathcal{F}(f(t)e^{j2\pi f_0 t}) = \int f(t)e^{j2\pi f_0 t} e^{-j2\pi f t}\,dt$

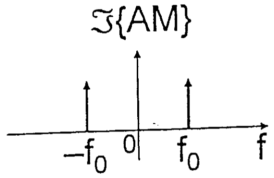
$$= \int f(t)e^{-j2\pi(f-f_0)t}\,dt$$

$$\therefore\ f(t)e^{j2\pi f_0 t} \rightleftharpoons F(f-f_0)$$

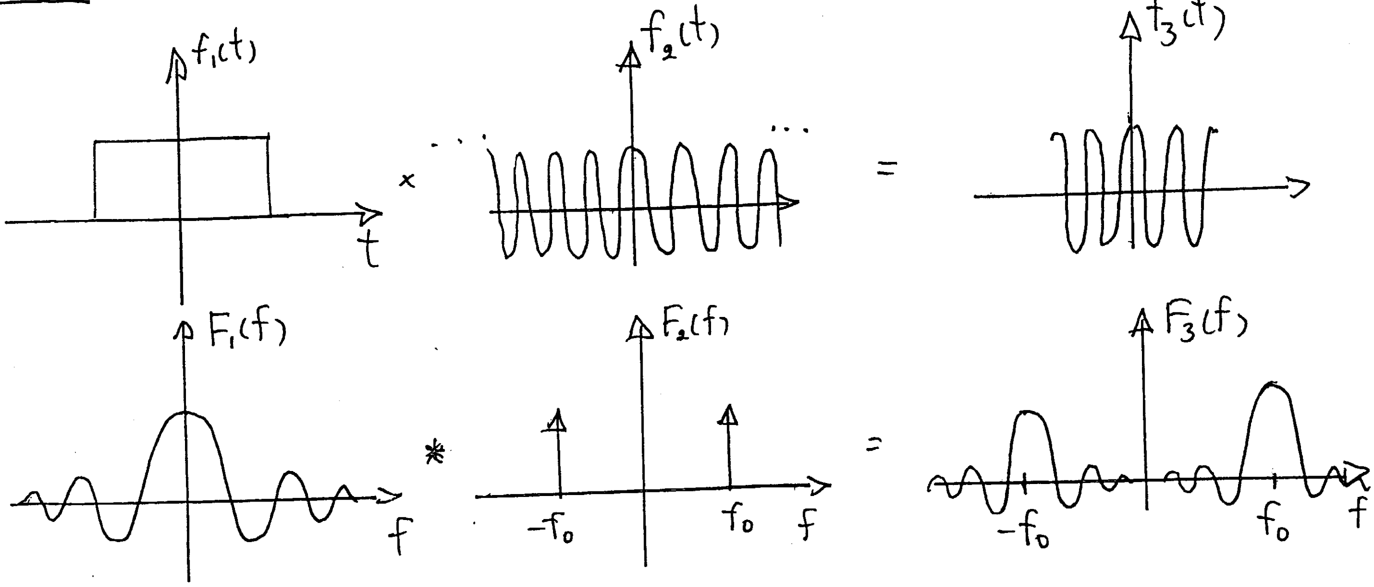
Amplitude modulation (AM): $\mathcal{F}[f(t)\cos(2\pi f_0 t)] = \frac{1}{2}\mathcal{F}[f(t)e^{j2\pi f_0 t}] + \frac{1}{2}\mathcal{F}[f(t)e^{-j2\pi f_0 t}]$

$$= \frac{1}{2}F(f-f_0) + \frac{1}{2}F(f+f_0)$$





Example



$\boxed{\text{Convolution}}$

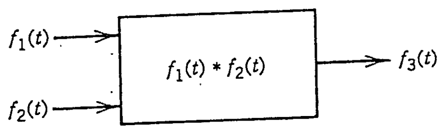
One method used to determine the output of a system (either analog or digital) when the input is known is convolution. This method of solution has existed for a long time but was not popular until the advent of digital computer. We will first introduce convolution by use of continuous systems, then show that this method applied also to discrete system.

Convolution is a binary operation. A binary operation maps an ordered pair of elements from a set into a single element of the set.

A) Continuous-Time Convolution:

$$f_3(t) = f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\lambda) f_2(t-\lambda)\, d\lambda \quad \text{—} (1)$$

$$= \int_{-\infty}^{\infty} f_1(t-\lambda) f_2(\lambda)\, d\lambda \quad \text{—} (2)$$

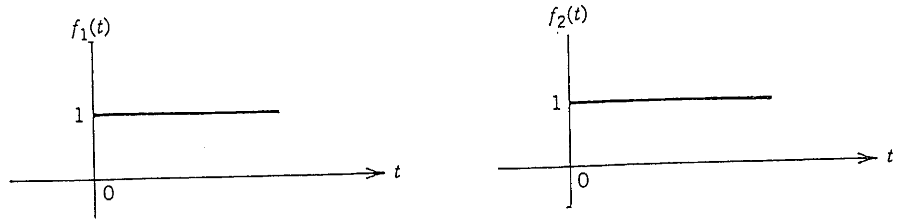


Eq 1 states that in order to find $f_3(t)$, we must perform the following steps:

1. Find $f_1(\lambda)$. This is accomplished by simply substituting λ for t in the expression of $f_1(t)$
2. Find $f_2(t-\lambda)$. Again simply substitute $(t-\lambda)$ for t in the expression of $f_2(t)$
3. Multiply $f_1(\lambda)$ and $f_2(t-\lambda)$ together and integrate over all λ.
4. Repeat steps 1-3 for all possible values of t.

If we want to use eq (2), we simply interchange f_1 and f_2 in the foregoing steps. The choice between eqs (1) and (2) is based on the functions $f_1(t)$ and $f_2(t)$. If $f_2(t)$ is less complicated function than $f_1(t)$, then eq (1) is used.

Example convolve the two step functions shown below



① Find $f_1(\lambda)$ → simply replace t by λ

② Find $f_2(t-\lambda)$, we can write $f_2(t) = 1 \quad t>0$

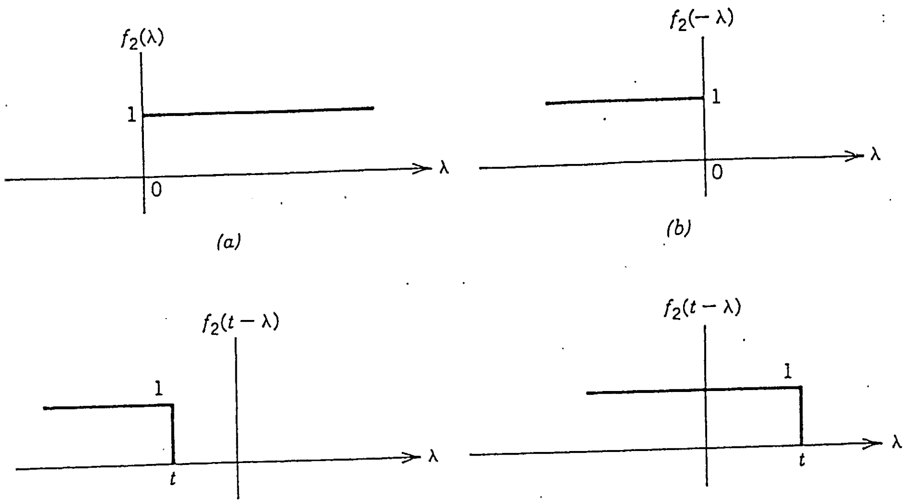
$$= 0 \quad t<0$$

Then substitute $(t-\lambda)$ for t, $f_2(t-\lambda) = 1 \quad t-\lambda > 0$

$$= 0 \quad t-\lambda < 0$$

or $f_2(t-\lambda) = 1 \quad \lambda < t$

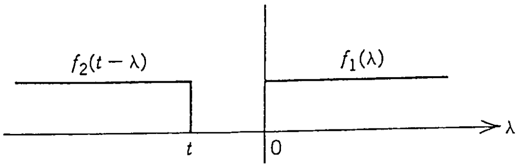
$$= 0 \quad \lambda > t$$



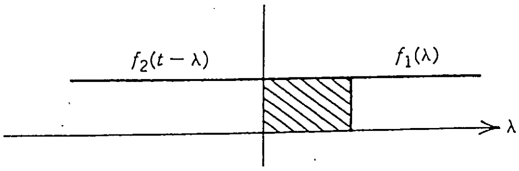
(a) (b) (c) (d)

The same result can be obtained graphically. We begin in Fig ⓐ by plotting $f_2(\lambda)$ versus λ. The function is flipped in Fig ⓑ to obtain $f_2(-\lambda)$. Note that $f_2(-\lambda) = f_2(0-\lambda)$, or this is $f_2(t-\lambda)$ if $t=0$.

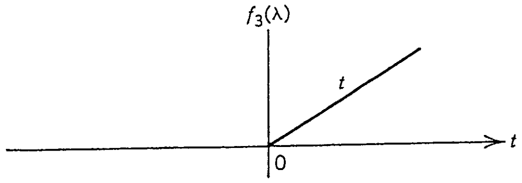
This same function is shown for other values of t in Figs ⓒ and ⓓ, first for a negative value of t and then for a positive value of t. This completes steps ① and ②.



(a)



(b)



(c)

Now, we must multiply $f_1(\lambda)$ by $f_2(t-\lambda)$, step 3 and integrate according to ① This must be done for every value of t in the interval $-\infty < t < \infty$ (step 4).

The solution is continued in Fig. above, where ⓐ the value of t is less than 0 and the product $f_1(\lambda) f_2(t-\lambda)$ is zero for every value of λ.

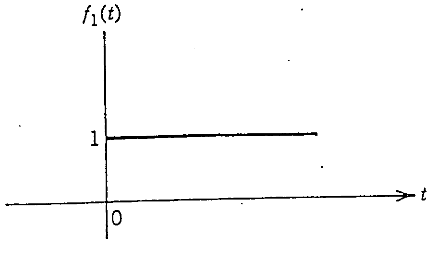
In ⓑ with $t > 0$, the product $f_1(\lambda) f_2(t-\lambda)$ is equal to 1 for $0 < \lambda < t$ and zero everywhere. Thus, $f_3(t)$ is given by

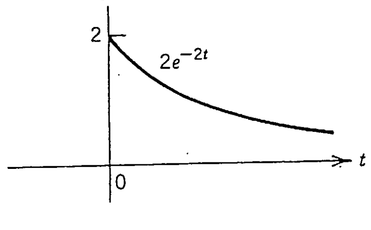
$$f_3(t) = \int_{-\infty}^{0} 0 \, d\lambda + \int_{0}^{t} 1 \, d\lambda + \int_{1}^{\infty} 0 \, d\lambda = t \quad \rightarrow \quad t > 0$$

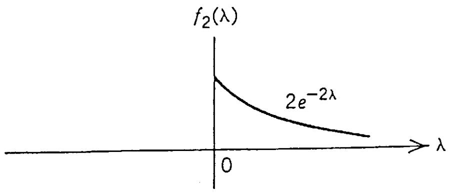
and is plotted in Fig ⓒ, and we see that the convolution of two unit steps results in the unit ramp. ($r(t)$)

$$r(t) = \int_{-\infty}^{t} u(\lambda) d\lambda = t u(t) = \int_{-\infty}^{\lambda_3} \int_{-\infty}^{\lambda_2} \delta(\lambda_1) d\lambda_1 d\lambda_2$$

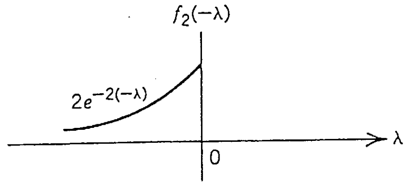
<u>Example</u>: Convolve the two functions shown



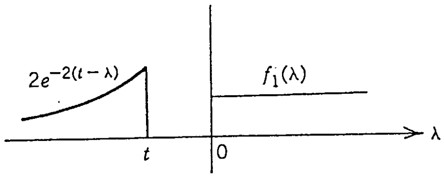




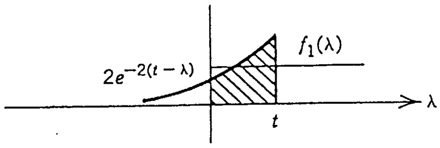
(a)



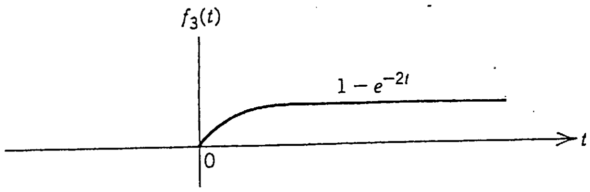
(b)



(c)



(d)



(e)

We will graphically flip and slip $f_2(t)$ to illustrate how this is done. (It would be easier to flip and slip $f_1(t)$ since it is a simpler function than $f_2(t)$ and we would normally choose the easier approach) Therefore, the following formula will be used.

$$f_3(t) = \int_{-\infty}^{\infty} f_1(\lambda) f_2(t-\lambda)\, d\lambda$$

We proceed from $f_2(\lambda)$ in Fig ⓐ to $f_2(-\lambda)$ in Fig ⓑ to $f_2(t-\lambda)$ in Fig ⓒ and Fig ⓓ

To evaluate $f_3(t)$, we note that $f_1(\lambda) f_2(t-\lambda) = 0$ for $t<0$

$\therefore \quad f_3(t) = 0$ for $t<0$

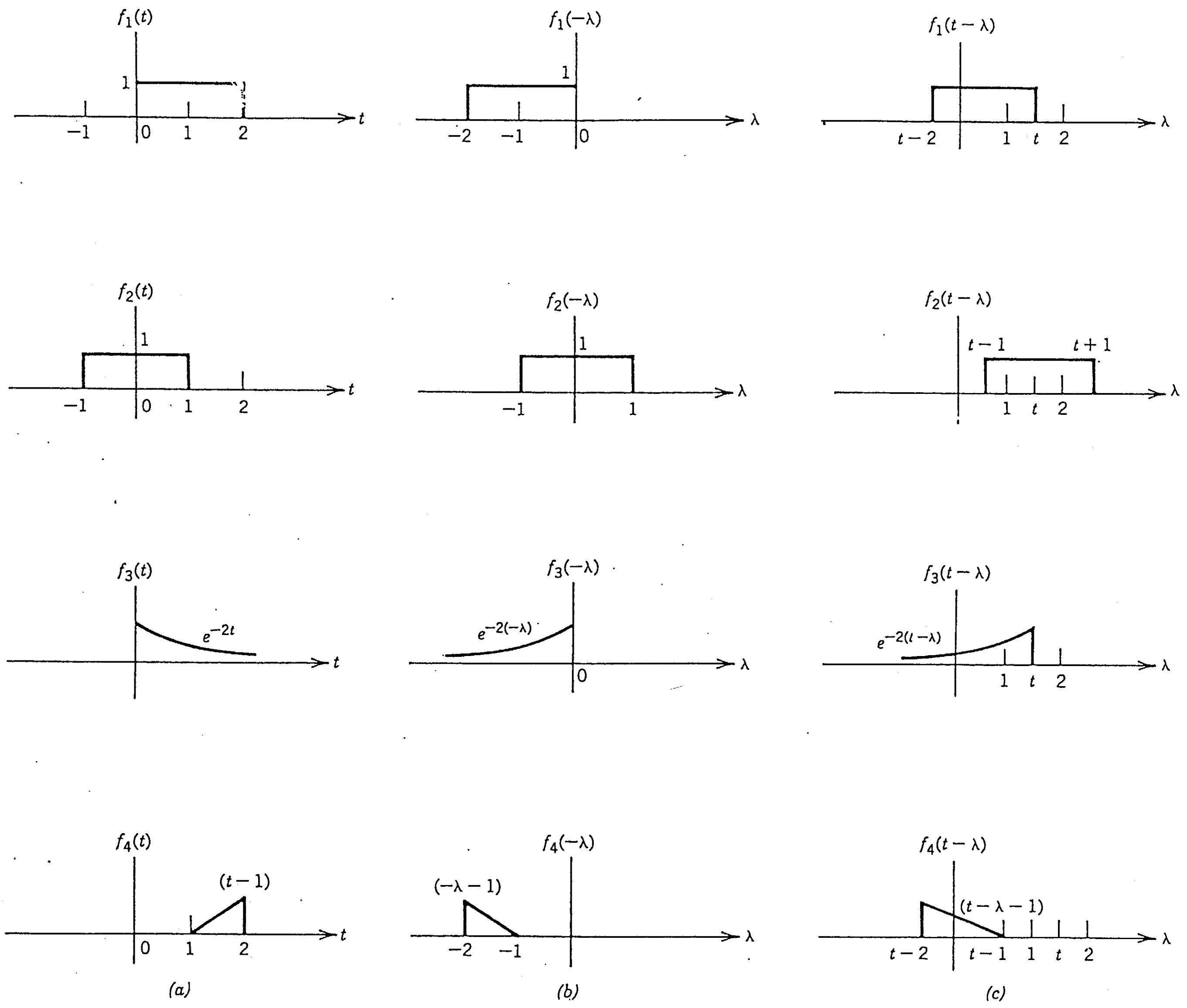
For $t>0$, we have Fig ⓓ $\quad f_3(t) = \int_0^t 2e^{-2(t-\lambda)}\, d\lambda$

$$= 2e^{-2t} \int_0^t e^{2\lambda}\, d\lambda = 1 - e^{-2t}, \quad t>0$$

$\therefore$ $f_3(t)$ is plotted in Fig ⓓ

Note: you must be able to plot and write the expression for $f(t-\lambda)$ so that the limits of integration can be established and integral evaluated.

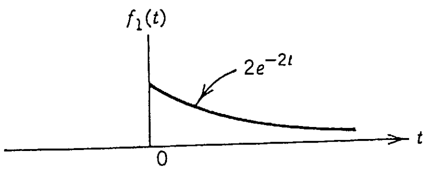
Example: For each function $f(t)$ shown, plot $f(t-\lambda)$ versus λ for a value of t given by $t = 1.5$. Also label the figures with the correct equation.



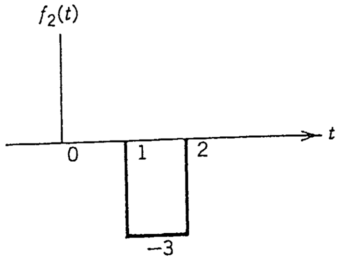
Example: convolve f_1 and f_2 shown below:

sol^n since $f_2(t)$ is less complex than $f_1(t)$, we choose to flip and slip f_2!

so we write
$$f_2(t) = -3 \qquad 1 < t < 2$$
$$f_2(t-\lambda) = -3 \qquad 1 < t-\lambda < 2$$
$$f_2(t-\lambda) = -3 \qquad -2 < \lambda - t < -1$$
$$\therefore \; f_2(t-\lambda) = -3 \qquad t-2 < \lambda < t-1$$



(a)



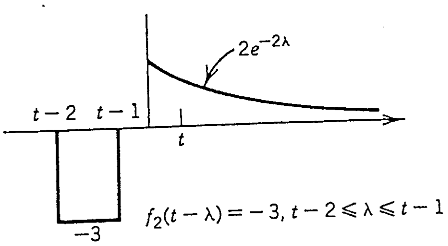
(b)

Fig(c) plots $f_1(\lambda)$ and $f_2(t-\lambda)$ for $t<1$. The functions do not overlap so their product, and the integral of their product are zero

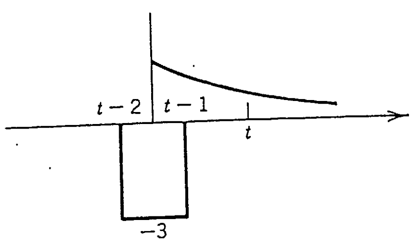
For $1<t<2$, the situation shown in Fig(d). We can write
$$f_3(t) = \int_0^{t-1} (-3)\,2e^{-2\lambda}\,d\lambda \qquad 1<t<2$$
$$= 3\left[e^{2(1-t)} - 1\right] \qquad 1<t<2$$

For the case $t>2$, the situation is in Fig(e), we write
$$f_3(t) = \int_{t-2}^{t-1} (-3)\,2e^{-2\lambda}\,d\lambda \qquad t>2$$
$$= 3\left[e^{2} - e^{4}\right]e^{-2t} \qquad t>2$$

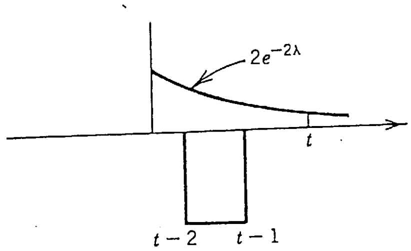
The complete function $f_3(t)$ is plotted in Fig(f)



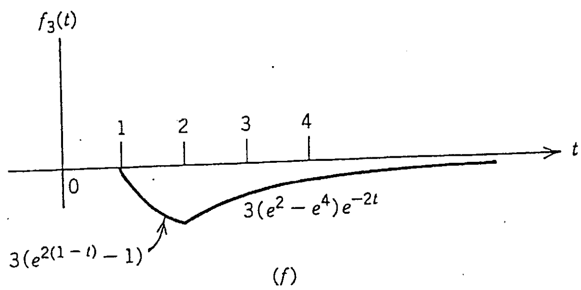
(c)



(d)

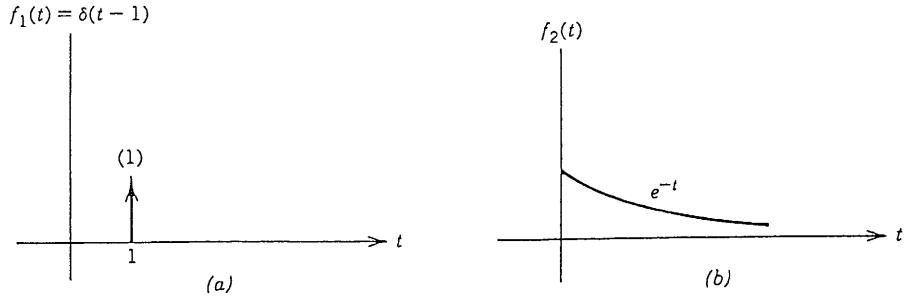


(e)



(f)

Example Convolve the two signals shown



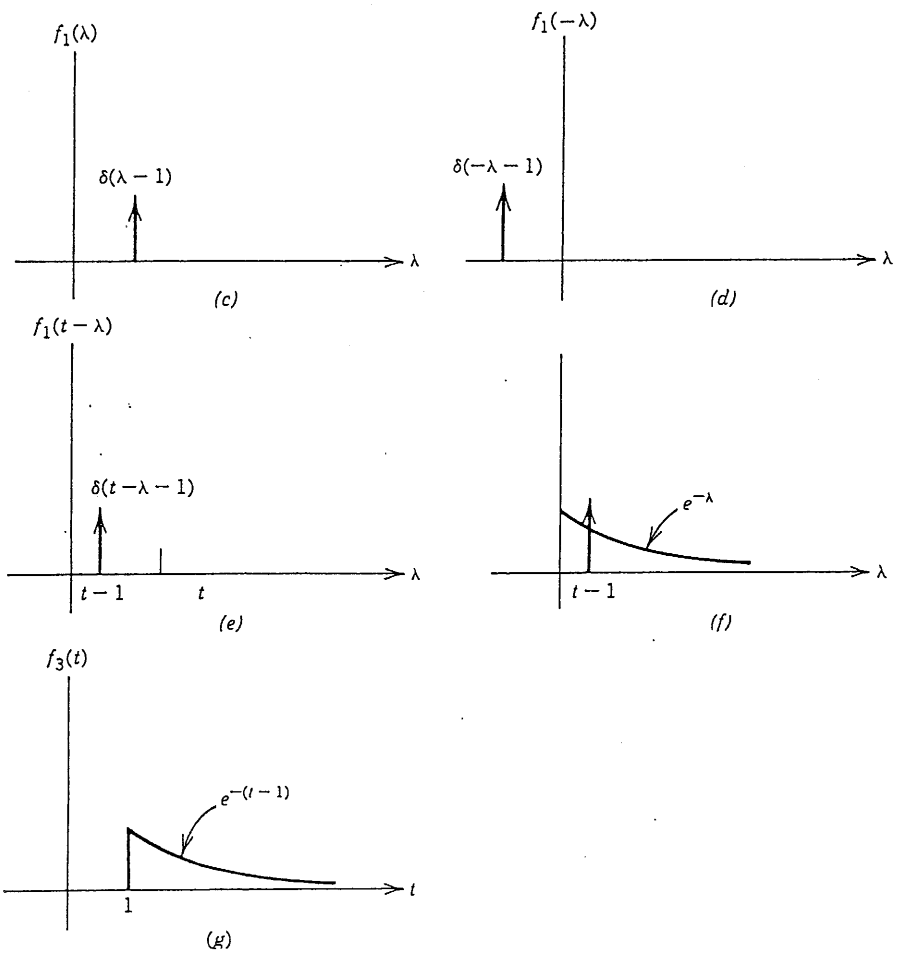
We will find $f_1(t-\lambda)$. The process of flipping and slipping the δ function is shown in Figs (c) and (d) and (e).

For $t<1$, the product is zero

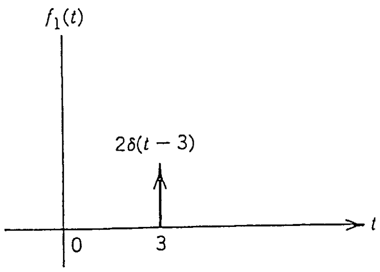
For $t>1$, (shown in Fig (f)): $f_3(t) = \int_1^{\infty} \delta(t-\lambda-1) e^{-\lambda} \, d\lambda$

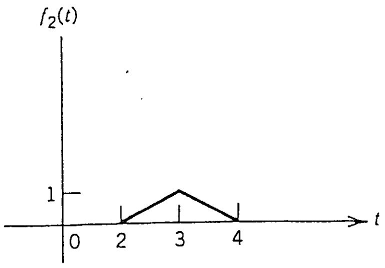
$$= e^{-(t-1)} \qquad t>1$$

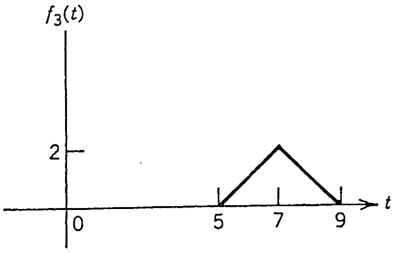
The result is plotted in Fig (g)



One more example: $f_1(t) * f_2(t)$







$\boxed{\text{Observation}}$ convolution with an impulse $\rightarrow$ a time translation of the function that is convolved with impulse, and multiply by the area under the impulse

So, in general, to convolve $f(t)$ with $A\delta(t-t_0)$, we simply multiply $f(t)$ by A and shift it by t_0 units.

step-by-step procedure for evaluating convolution integrals:

1. Plot f_1 and f_2 as functions of λ rather than t.
2. Select one function to flip and slip, say f_2.
3. Flip $f_2(\lambda)$ to obtain $f_2(-\lambda)$.
4. Slip f_2 to left or right until the point originally at the origin coincides with the present value of t.
5. The area under the product $f_1(\lambda)f_2(t-\lambda)$ is the value of $f_3(t)$ for that one value of t.
6. Vary t from $-\infty$ to $+\infty$.

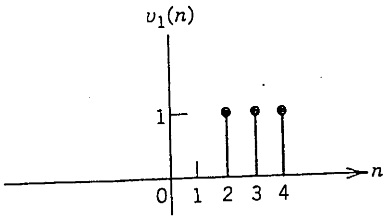
B. Discrete-Time convolution:

Given two discrete-time signals $v_1[n]$ and $v_2[n]$, their convolution is the binary operation given by

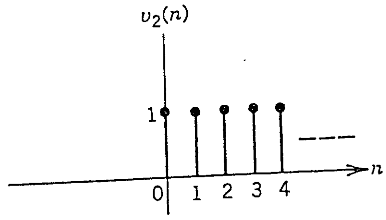
$$V_3[n] = V_1[n] * V_2[n]$$
$$= \sum_{k=-\infty}^{\infty} V_1[k]\, V_2[n-k]$$
$$= \sum_{k=}^{\infty} V_1[n-k]\, V_2[k]$$

The evaluation of this summation is identical in principle to the evaluation of the continuous-time integral.

Example Find $V_3[n] = V_1[n] * V_2[n]$



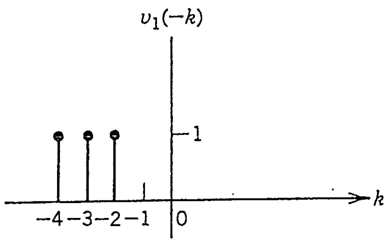
(a)



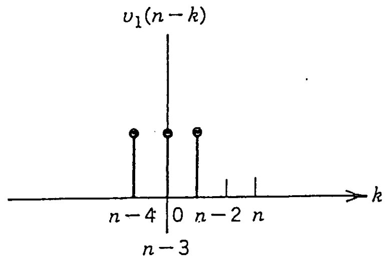
(b)

To convolve these functions, we choose to flip and slip $V_1[n]$ as in Figs c and d, with $n=3$ in Fig d. In Fig e, the two functions are shown together, so that $n=3$, the convolution summation is 2.

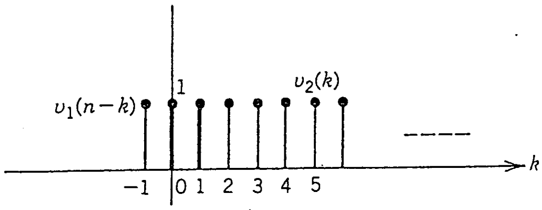
$$\therefore\quad V_3[n] = \begin{cases} 0, & n<2 \\ 1, & n=2 \\ 2, & n=3 \\ 3; & n>4 \end{cases}$$



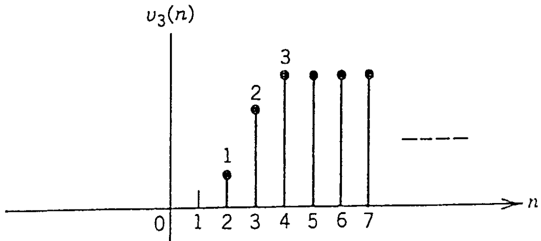
(c)



(d)



(e)



(f)

There are several closed-form identities that are useful in evaluating convolution summations. The most frequently used is the finite geometric series, given by

$$\boxed{\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}}, \quad a \neq 1$$

This can be shown by writing

$$S = \sum_{k=0}^{n} a^k = 1 + a + a^2 + \dots + a^n$$

now multiply both sides by "a" to obtain

$$aS = a + a^2 + a^3 + \dots + a^{n+1}$$

Next subtract aS from S to obtain

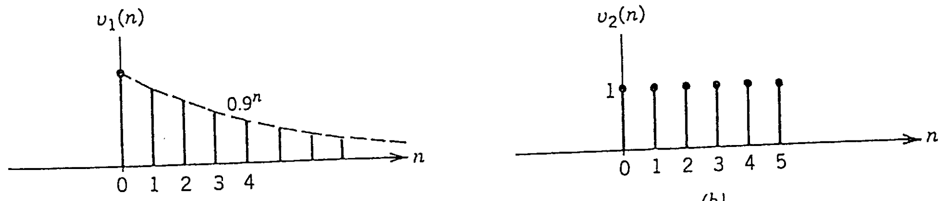
$$S - aS = 1 - a^{n+1}$$

$$\therefore \quad S = \frac{1-a^{n+1}}{1-a}$$

Here is another frequently used identity, which can be derived by similar procedures.

$$\boxed{\sum_{k=0}^{n} k a^k = \frac{a}{(1-a)^2}\left[1-(n+1)a^n + n a^{n+1}\right]}, \quad a \neq 1$$

<u>Example</u> : Convolve the two functions $V_1[n]$ and $V_2[n]$



(a) (b)

we flip and slip V_1 in Figs c and d. For $n<0$, Fig e, $V_3[n] = 0$

For $0<n<5$, Fig f, $V_3[n] = \sum_{k=0}^{n} V_1[n-k]V_2[k] = \sum_{k=0}^{n} 0.9^{(n-k)}$

Now make a change of variable $j = n-k$, then $V_3[n] = \sum_{j=0}^{n} 0.9^j = \frac{1-0.9^{(n+1)}}{1-0.9}$

In Fig g, $n>5$, $V_3[n] = \sum_{k=0}^{5} 0.9^{(n-k)} = \sum_{j=n-5}^{n} 0.9^j$ (set $j=n-k$)

In order to obtain a closed-form expression for $v_3[n]$, let's be defined by

$$S = \sum_{j=n-5}^{n} 0.9^j = 0.9^{n-5} + 0.9^{n-4} + \dots + 0.9^n$$

multiply both sides by 0.9

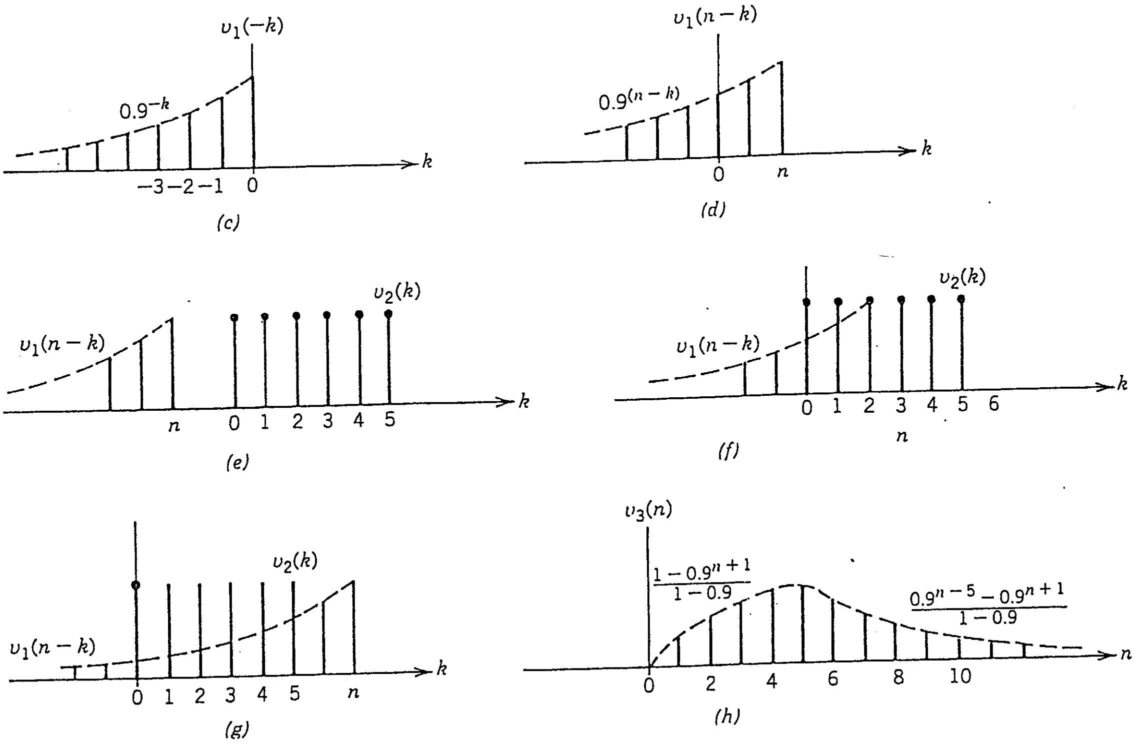
$$0.9S = 0.9^{n-4} + 0.9^{n-3} + \dots + 0.9^n + 0.9^{n+1}$$

subtract 0.9S from S to obtain

$$S - 0.9S = 0.9^{n-5} - 0.9^{n+1}$$

Finally, replace the left side by $(1-0.9)S$ and divide by $(1-0.9)$ to obtain

$$S = \sum_{j=n-5}^{n} 0.9^j = \frac{0.9^{n-5} - 0.9^{n+1}}{1-0.9}$$



<u>Conclusion</u>: The procedure for evaluating convolution summation is as follows:

1) Plot v_1 and v_2 as functions of k rather than n.
2) select one function to flip and slip, say v_2
3) Flip $v_2[k]$ to obtain $v_2[-k]$
4) slip v_2 to left or right until the point originally at the origin coincides with the present value of n.
5) The summation of the product $v_1[k]\, v_2[n-k]$ is the value of $v_3[n]$ for that one value of n.

What's the significance of convolution? It is used to determine the output of a system when the input is known.

$$\text{input} \longrightarrow \boxed{\text{system}} \longrightarrow \text{output}$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = h(t) * x(t)$$

$$\delta(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = h(t) * \delta(t) = h(t)$$

impulse response

The convolution operation $(*)$ has the following properties:

1) commutativity: $x(t) * h(t) = h(t) * x(t)$

2) superposition:
$$[a_1 x_1(t) + a_2 x_2(t)] * h(t) = a_1 x_1(t) * h(t) + a_2 x_2(t) * h(t)$$

3) Associativity:
$$x(t) * [h_1(t) * h_2(t)] = [x(t) * h_1(t)] * h_2(t)$$

$\boxed{\text{SLTI systems}}$ This course deals almost exclusively with stable, linear, time-invariant (SLTI) systems. A system is an entity that manipulates one or more signals to accomplish a function, thereby yielding new signals.

① stable means that output variable does not "blow up" (go to infinity) if the input does not "blow up." A system is said to be bounded-input, bounded output (BIBO) stable if and only if every bounded input results in a bounded output. The output signal satisfies the condition $|y(t)| \leq M_y < \infty$ for all t whenever the input signal $x(t)$ satisfies $|x(t)| \leq M_x < \infty$ for all t

② memory: output signal depends on past or future values of the input signal.
In contrast: a system is said to be memoryless if its output signal depends only on the present value of the input signal.

For example: a resistor (memoryless): $i(t) = \frac{1}{R} v(t)$

an inductor (memory): $i(t) = \frac{1}{L} \int_{-\infty}^{t} v(\tau)\, d\tau$

a moving-average (memory): $y[n] = \frac{1}{3}(x[n] + x[n-2] + x[n-4])$

③ causality : the present value of the output signal depends only on the present or the past values of the input signal. In contrast, the output signal of a non causal system depends on one or more future values of input signal.

Examples : causal : $y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$

non causal : $y[n] = \frac{1}{3}(x[n+1] + x[n] + x[n-1])$

④ Time invariance A system is said to be time invariance if a time delay or time advance of input signal leads to an identical time shift in the output signal. This implies that a time-invariant system responds identically no matter when the input signal is applied.

$$x(t) \rightarrow y(t)$$
$$x(t-\tau) \rightarrow y(t-\tau)$$

⑤ Linearity For any two-input signals $x_1(t)$, $x_2(t)$ and real constant a, the system responses satisfy

$$x_1(t) + x_2(t) \longrightarrow y_1(t) + y_2(t)$$
$$a x_1(t) \longrightarrow a y_1(t)$$

Examples: system A : $y(t) = t \cdot x(t)$

system B : $y(t) : 10 \cdot x(t)$

system C : $y(t) = x(10 \cdot t)$

Ⓐ explicitly depends on t outside $x(t) \rightarrow y(t)$ is not time invariant

Ⓑ does not depend explicitly on $t \rightarrow$ time invariant

Ⓒ not time invariant since a time shift will result in a scaled shift.

Example : derivative operator is an LTI

$\therefore$ ① $\frac{d}{dt}(c_1 x_1(t) + c_2 x_2(t)) = c_1 x_1'(t) + c_2 x_2'(t)$ (linear)

② $\frac{d}{dt} x(t-\tau) = x'(t-\tau)$ (time Invariant)

Homework

1. Convolve the two continuous-time functions shown in Fig. 8.37.

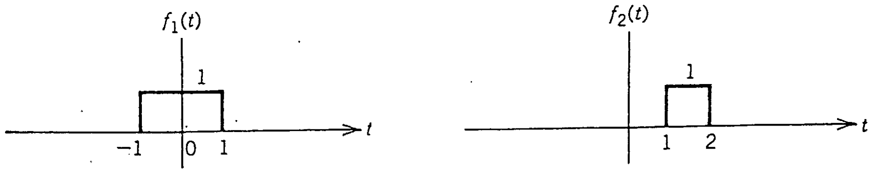


Figure 8.37

2. Convolve the two discrete-time functions shown in Fig. 8.38.

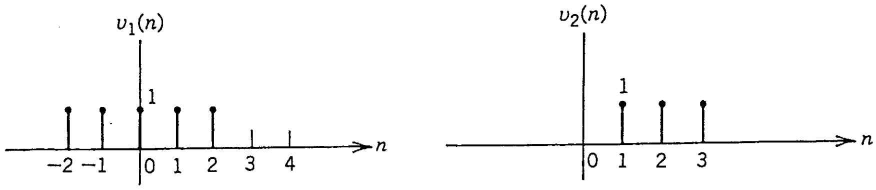
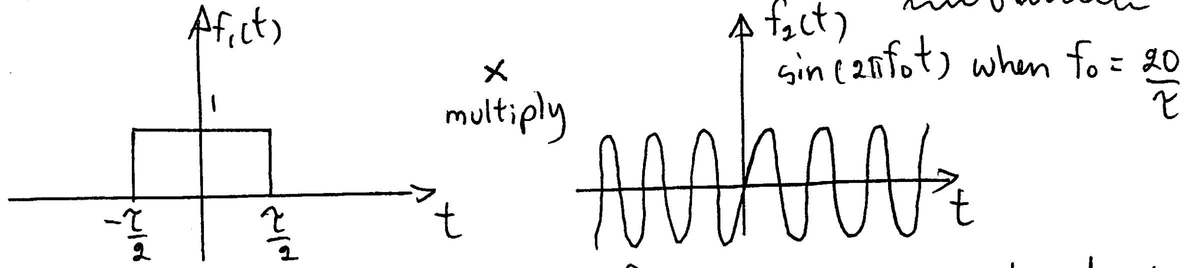


Figure 8.38

3. convolve the two continuous-time functions shown. in freq domain



4. convolve the two continuous-time functions shown in time domain