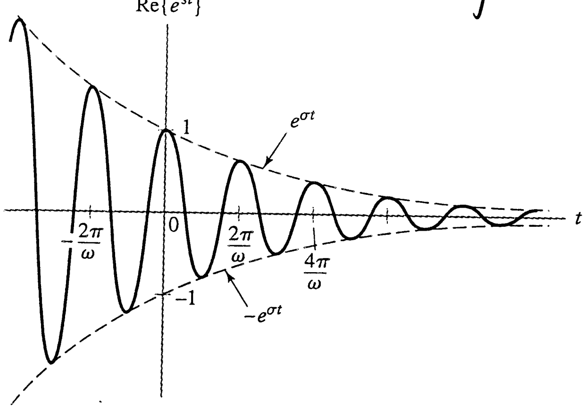
ENE 208 | The Laplace Transform

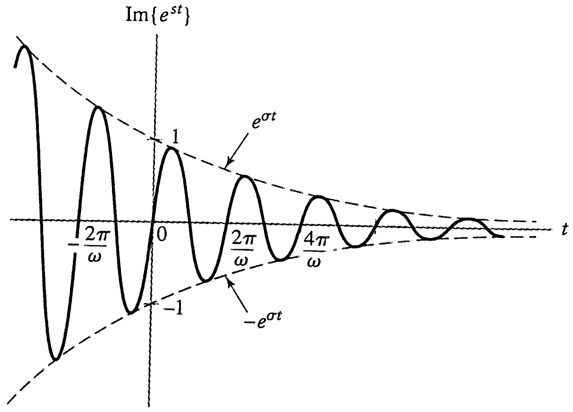
- The Laplace transform provides a broader characterization of continuous-time LTI systems and their interaction with signals than is possible with Fourier methods. For example, the Laplace transform can be used to analyze a large class of continuous-time problems involving signals that are not absolutely integrable, such as the impulse response of an unstable system. The FT does not exist for signals that are not absolutely integrable, so FT-based methods cannot be employed in this class of problems.
- The Laplace transform comes in two varieties: (1) unilateral, or one sided and (2) bilateral, or two sided. The unilateral Laplace transform is a convenient tool for solving differential equations with initial conditions. The bilateral Laplace transform offers insight into the nature of system characteristics such as stability, causality and frequency response.
- The primary role of the Laplace transform in engineering is the transient and stability analysis of causal LTI systems described by differential equations.
- French mathematician: Pierre-Simon Laplace developed a mathematical method for solving differential equations by the use of algebraic techniques.
- British physicist Oliver Heaviside is credited with early usage of operational methods in circuit analysis.

The Laplace Transform

Let e^{st} be a complex exponential with complex freq $s = \sigma + j\omega$. We may write

$$e^{st} = e^{\sigma t}\cos(\omega t) + j e^{\sigma t}\sin(\omega t)$$





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- The real part of e^{st} is an exponentially damped cosine, and the imaginary part is an exponentially damped sine. (In the figure, it is assumed that σ is negative.)
- The real part of s is the exponential damping factor σ, and the imaginary part of s is the frequency of the cosine and sine factor, namely, ω.
- Consider applying an input of the form $x(t) = e^{st}$ to an LTI system with impulse response $h(t)$. The system output is given by

$$y(t) = H\{x(t)\} = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

We use $x(t) = e^{st}$ to obtain

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

We define the transfer function $H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$

So that we may write

$$y(t) = H\{e^{st}\}$$
$$= H(s)e^{st}$$

The action of the system on an input e^{st} is multiplication by the transfer function $H(s)$. If an eigenfunction is defined as a signal that passes through the system without being modified except for multiplication by a scalar. Hence, we identify e^{st} as an eigenfunction of the LTI system and $H(s)$ as the corresponding eigenvalue.

Next, we express the complex-valued transfer function $H(s)$ in polar form as $H(s) = |H(s)|e^{j\phi(s)}$ where $|H(s)|$ and $\phi(s)$ are the magnitude and phase of $H(s)$, respectively. Now we rewrite the LTI system output as

$$y(t) = |H(s)|e^{j\phi(s)}e^{st}$$

We use $s = \sigma + j\omega$ to obtain

$$y(t) = |H(\sigma+j\omega)|e^{\sigma t}e^{j\omega t + \phi(\sigma+j\omega)}$$

$$= |H(\sigma+j\omega)|e^{\sigma t}\cos(\omega t+\phi(\sigma+j\omega)) + j|H(\sigma+j\omega)|e^{\sigma t}\sin(\omega t+\phi(\sigma+j\omega)).$$

$\therefore$ the system changes the amplitude of the input by $|H(\sigma+j\omega)|$ and shifts the phase of the sinusoidal components by $\phi(\sigma+j\omega)$.

$\rightarrow$ The system does not change the damping factor σ or the sinusoidal frequency ω of the input.

<u>Laplace transform representation</u>

we want to find a representation of arbitrary signals as a weighted superposition of eigenfunctions e^{st}. Substituting $s = \sigma + j\omega$, and using t as the variable of integration:

$$H(\sigma + j\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-(\sigma + j\omega)t}\, dt = \int_{-\infty}^{\infty} [h(t)\, e^{-\sigma t}]\, e^{-j\omega t}\, dt$$

This indicates that $H(\sigma + j\omega)$ is the Fourier transform of $h(t) e^{-\sigma t}$.
Hence, the inverse Fourier transform of $H(\sigma+j\omega)$ must be $h(t) e^{-\sigma t}$;

$$h(t) e^{-\sigma t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\sigma + j\omega)\, e^{j\omega t}\, d\omega$$

$$h(t) = e^{\sigma t} \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\sigma + j\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\sigma + j\omega)\, e^{(\sigma + j\omega)t}\, d\omega$$

Now, substituting $s = \sigma + j\omega$ and $d\omega = \frac{ds}{j}$, we get

$$h(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} H(s)\, e^{st}\, ds.$$

$\therefore$ The Laplace transform of $x(t)$ is $\boxed{X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt}$

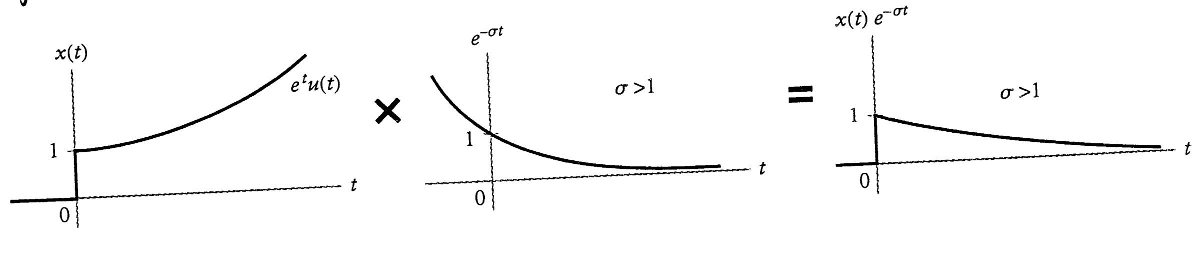
And the inverse Laplace transform

$$\boxed{x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s)\, e^{st}\, ds}$$

$\boxed{\text{Convergence}}$ A necessary condition for convergence of the Laplace transform is the absolute integrability of $x(t) e^{-\sigma t}$.

$$\int_{-\infty}^{\infty} |x(t) e^{-\sigma t}|\, dt < \infty$$

The range of σ for which the Laplace transform converges is termed the region of convergence (ROC). Note that the Laplace transform exists for signals that do not have a Fourier transform. By limiting ourselves to a certain range of σ, we ensure that $x(t) e^{-\sigma t}$ is absolutely integrable, even though $x(t)$ is not absolutely integrable by itself.



The s-plane It's convenient to represent the complex frequency s graphically in terms of a complex plane termed the s-plane. The horizontal axis represents the real part of s (i.e. the exponential damping factor σ), and the vertical axis represents the imaginary part of s (i.e. the sinusoidal freq ω).

If $x(t)$ is absolutely integrable, then we may obtain the Fourier transform from the Laplace transform by setting $\sigma = 0$;

$$\boxed{X(j\omega) = X(s)\Big|_{\sigma=0}}$$

In the s-plane, $\sigma = 0$ corresponds to the imaginary axis. We thus say that the Fourier transform is given by the Laplace transform evaluated along the imaginary axis

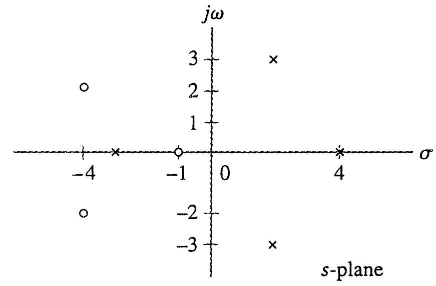


FIGURE 6.3 The s-plane. The horizontal axis is Re$\{s\}$ and the vertical axis is Im$\{s\}$. Zeros are depicted at $s = -1$ and $s = -4 \pm 2j$, and poles are depicted at $s = -3$, $s = 2 \pm 3j$, and $s = 4$.

The $j\omega$-axis divides the s-plane in half. The region of the s-plane to the left of the $j\omega$-axis is termed the left half of the s-plane, while the region to the right of the $j\omega$-axis is termed the right half s-plane. The real part of s is negative in the left half and positive in the right half of the plane.

poles and zeros The most commonly encountered form of the Laplace transform in engineering is a ratio of two polynomials in s; that is,

$$X(s) = \frac{b_M s^M + b_{M-1} s^{M-1} + \dots + b_0}{s^N + a_{N-1} s^{N-1} + \dots + a_1 s + a_0}$$

It is useful to factor $X(s)$ as a product of terms involving the roots of the denominator and numerator polynomials:

$$X(s) = \frac{b_M \prod_{k=1}^{M} (s - c_k)}{\prod_{k=1}^{N} (s - d_k)}$$

when c_k are roots of the numerator polynomial, called zeros of $X(s)$, "o" symbol
d_k are roots of the denominator " , called poles of $X(s)$, "x" symbol

EXAMPLE 6.1 LAPLACE TRANSFORM OF A CAUSAL EXPONENTIAL SIGNAL Determine the Laplace transform of

$$x(t) = e^{at}u(t),$$

and depict the ROC and the locations of poles and zeros in the s-plane. Assume that a is real.

Solution: Substitute $x(t)$ into Eq. (6.5), obtaining

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} e^{at}u(t)e^{-st}dt \\ &= \int_{0}^{\infty} e^{-(s-a)t}\,dt \\ &= \frac{-1}{s-a}e^{-(s-a)t}\Big|_0^{\infty}. \end{aligned}$$

To evaluate $e^{-(s-a)t}$ at the limits, we use $s = \sigma + j\omega$ to write

$$X(s) = \frac{-1}{\sigma + j\omega - a}e^{-(\sigma-a)t}e^{-j\omega t}\Big|_0^{\infty}.$$

Now, if $\sigma > a$, then $e^{-(\sigma-a)t}$ goes to zero as t approaches infinity, and

$$\begin{aligned} X(s) &= \frac{-1}{\sigma + j\omega - a}(0 - 1), \quad \sigma > a, \\ &= \frac{1}{s-a}, \quad \mathrm{Re}(s) > a. \end{aligned} \tag{6.8}$$

The Laplace transform $X(s)$ does not exist for $\sigma \leq a$, since the integral does not converge. The ROC for this signal is thus $\sigma > a$, or equivalently, $\mathrm{Re}(s) > a$. The ROC is depicted as the shaded region of the s-plane in Fig. 6.4. The pole is located at $s = a$. ■

The expression for the Laplace transform does not uniquely correspond to a signal $x(t)$ if the ROC is not specified. That is, two different signals may have identical Laplace transforms, but different ROCs. We demonstrate this property in the next example.

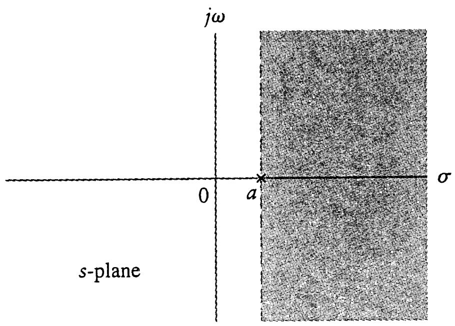


FIGURE 6.4 The ROC for $x(t) = e^{at}u(t)$ is depicted by the shaded region. A pole is located at $s = a$.

EXAMPLE 6.2 LAPLACE TRANSFORM OF AN ANTICAUSAL EXPONENTIAL SIGNAL An anticausal signal is zero for $t > 0$. Determine the Laplace transform and ROC for the anticausal signal

$$y(t) = -e^{at}u(-t).$$

Solution: Using $y(t) = -e^{at}u(-t)$ in place of $x(t)$ in Eq. (6.5), we obtain

$$\begin{aligned} Y(s) &= \int_{-\infty}^{\infty} -e^{at}u(-t)e^{-st}\,dt \\ &= -\int_{-\infty}^{0} e^{-(s-a)t}\,dt \\ &= \frac{1}{s-a}e^{-(s-a)t}\Big|_{-\infty}^{0} \\ &= \frac{1}{s-a}, \quad \mathrm{Re}(s) < a. \end{aligned} \tag{6.9}$$

The ROC and the location of the pole at $s = a$ are depicted in Fig. 6.5. ■

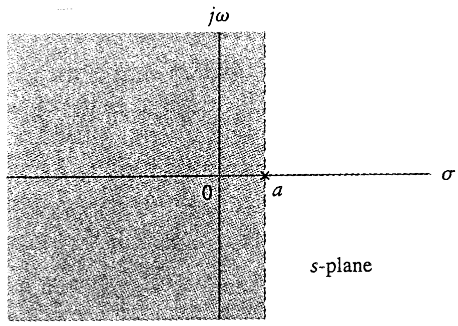


FIGURE 6.5 The ROC for $y(t) = -e^{at}u(-t)$ is depicted by the shaded region. A pole is located at $s = a$.

Examples 6.1 and 6.2 reveal that the Laplace transforms $X(s)$ and $Y(s)$ are equal, even though the signals $x(t)$ and $y(t)$ are different. However, the ROCs of the two signals are different.

$\boxed{\text{The Unilateral Laplace Transform}}$ There are many applications of Laplace transforms in which it is reasonable to assume that the signals involved are causal, that is zero for time $t < 0$. In such problems, it is advantageous to define the unilateral or one-sided Laplace transform, which is based on the nonnegative-time ($t \geq 0$) portion of a signal.

$$\boxed{X(s) = \int_{0^-}^{\infty} x(t)e^{-st}\,dt}$$

The lower limit of 0^- implies that we do include discontinuities and impulses that occur at $t=0$ in the integral. By working with causal signals, we remove the ambiguity inherent in the bilateral transform and thus do <u>not</u> need to consider the ROC.

Properties of the Laplace transform

Linearity	$a_1x_1(t) + a_2x_2(t)$	$a_1\mathbf{X}_1(s) + a_2\mathbf{X}_2(s)$
Differentiation (1)	$x^{(1)}(t)$	$s\mathbf{X}(s) - x(0^-)$
Differentiation (2)	$x^{(2)}(t)$	$s^2\mathbf{X}(s) - sx(0^-) - x^{(1)}(0^-)$
Differentiation (n)	$x^{(n)}(t)$	$s^n\mathbf{X}(s) - s^{n-1}x(0^-) - s^{n-2}x^{(1)}(0^-) - \cdots - x^{(n-1)}(0^-)$
Integration	$\int_0^\infty x(t)\,dt$	$\frac{1}{s}\mathbf{X}(s) + \int_{-\infty}^{0^-} x(t)\,dt \times \frac{1}{s}$
s shift	$x(t)e^{s_0 t}$	$\mathbf{X}(s - s_0)$
Convolution	$x_1(t) * x_2(t) = \int_0^t x_1(\lambda)x_2(t-\lambda)\,d\lambda$	$\mathbf{X}_1(s)\mathbf{X}_2(s)$
Time Delay	$x(t-\tau)u(t-\tau) \quad \tau \geq 0$	$\mathbf{X}(s)e^{-s\tau}$
Time Scale	$x(at) \quad a > 0$	$\frac{1}{a}\mathbf{X}(s/a)$

Table 6CT.1

Laplace Transform Properties

The region of convergence of the Laplace transform of $\delta(t)$ is the entire s-plane.
The region of convergence of every other entry is the half plane to the right of the rightmost pole.

	Waveform	**Transform**
6CT.LT1	$\delta(t)$	1
6CT.LT2	1	$\frac{1}{s}$
6CT.LT3	t	$\frac{1}{s^2}$
6CT.LT4	$\frac{1}{(n-1)!}t^{(n-1)}$	$\frac{1}{s^n}$
6CT.LT5	$e^{s_0 t}$	$\frac{1}{s - s_0}$
6CT.LT6	$te^{s_0 t}$	$\frac{1}{(s-s_0)^2}$
6CT.LT7	$\frac{1}{(n-1)!}t^{(n-1)}e^{s_0 t}$	$\frac{1}{(s-s_0)^n}$
6CT.LT8	$e^{j\Omega_0 t}$	$\frac{1}{s - j\Omega_0}$
6CT.LT9	$e^{\sigma_0 t}$	$\frac{1}{s - \sigma_0}$
6CT.LT10	$\cos(\Omega_0 t)$	$\frac{s}{s^2 + \Omega_0^2} = \frac{s}{(s - j\Omega_0)(s + j\Omega_0)} = \frac{\frac{1}{2}}{s - j\Omega_0} + \frac{\frac{1}{2}}{s + j\Omega_0}$
6CT.LT11	$\sin(\Omega_0 t)$	$\frac{\Omega_0}{(s - j\Omega_0)(s + j\Omega_0)}$
6CT.LT12	$Ae^{\sigma_0 t}\cos(\Omega_0 t + \phi) = \mathcal{R}\{2Xe^{s_0 t}\}$	$\frac{X}{s - s_0} + \frac{X^*}{s - s_0^*} \quad X = \frac{1}{2}Ae^{j\phi}, s_0 = \sigma_0 + j\Omega_0$
6CT.LT13	$\frac{1}{(n-1)!}t^{(n-1)}Ae^{\sigma_0 t}\cos(\Omega_0 t + \phi) = \frac{1}{(n-1)!}t^{(n-1)}\mathcal{R}\{2Xe^{s_0 t}\}$	$\frac{X}{(s - s_0)^n} + \frac{X^*}{(s - s_0^*)^n} \quad X = \frac{1}{2}Ae^{j\phi}, s_0 = \sigma_0 + j\Omega_0$

Table 6CT.2

Laplace Transform Pairs

D.1 *Basic Laplace Transforms*

Signal $x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$	Transform $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$	ROC
$u(t)$	$\frac{1}{s}$	$\text{Re}\{s\} > 0$
$tu(t)$	$\frac{1}{s^2}$	$\text{Re}\{s\} > 0$
$\delta(t-\tau), \quad \tau \geq 0$	$e^{-s\tau}$	for all s
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\text{Re}\{s\} > -a$
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$	$\text{Re}\{s\} > -a$
$[\cos(\omega_1 t)]u(t)$	$\frac{s}{s^2+\omega_1^2}$	$\text{Re}\{s\} > 0$
$[\sin(\omega_1 t)]u(t)$	$\frac{\omega_1}{s^2+\omega_1^2}$	$\text{Re}\{s\} > 0$
$[e^{-at}\cos(\omega_1 t)]u(t)$	$\frac{s+a}{(s+a)^2+\omega_1^2}$	$\text{Re}\{s\} > -a$
$[e^{-at}\sin(\omega_1 t)]u(t)$	$\frac{\omega_1}{(s+a)^2+\omega_1^2}$	$\text{Re}\{s\} > -a$

D.1.1 Bilateral Laplace Transforms for Signals That Are Nonzero for $t < 0$

Signal	*Bilateral Transform*	*ROC*
$\delta(t-\tau), \tau < 0$	$e^{-s\tau}$	for all s
$-u(-t)$	$\frac{1}{s}$	$\text{Re}\{s\} < 0$
$-tu(-t)$	$\frac{1}{s^2}$	$\text{Re}\{s\} < 0$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\text{Re}\{s\} < -a$
$-te^{-at}u(-t)$	$\frac{1}{(s+a)^2}$	$\text{Re}\{s\} < -a$

D.2 *Laplace Transform Properties*

Signal	Unilateral Transform $x(t) \overset{\mathcal{L}_u}{\longleftrightarrow} X(s)$ $y(t) \overset{\mathcal{L}_u}{\longleftrightarrow} Y(s)$	Bilateral Transform $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$ $y(t) \overset{\mathcal{L}}{\longleftrightarrow} Y(s)$	ROC $s \in R_x$ $s \in R_y$
$ax(t) + by(t)$	$aX(s) + bY(s)$	$aX(s) + bY(s)$	At least $R_x \cap R_y$
$x(t-\tau)$	$e^{-s\tau}X(s)$ if $x(t-\tau)u(t) = x(t-\tau)u(t-\tau)$	$e^{-s\tau}X(s)$	R_x
$e^{s_o t}x(t)$	$X(s-s_o)$	$X(s-s_o)$	$R_x + \text{Re}\{s_o\}$
$x(at)$	$\frac{1}{a}X\left(\frac{s}{a}\right), \quad a > 0$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	$\frac{R_x}{\lvert a\rvert}$
$x(t) * y(t)$	$X(s)Y(s)$ if $x(t) = y(t) = 0 \quad$ for $t < 0$	$X(s)Y(s)$	At least $R_x \cap R_y$
$-tx(t)$	$\frac{d}{ds}X(s)$	$\frac{d}{ds}X(s)$	R_x
$\frac{d}{dt}x(t)$	$sX(s) - x(0^-)$	$sX(s)$	At least R_x
$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\frac{1}{s}\int_{-\infty}^{0^-} x(\tau)\,d\tau + \frac{X(s)}{s}$	$\frac{X(s)}{s}$	At least $R_x \cap \{\text{Re}\{s\} > 0\}$

For comparison with the Fourier transform

A Short Table of Fourier Transforms

	$f(t)$	$F(\omega)$	
1	$e^{-at}u(t)$	$\dfrac{1}{a+j\omega}$	$a>0$
2	$e^{at}u(-t)$	$\dfrac{1}{a-j\omega}$	$a>0$
3	$e^{-a\lvert t\rvert}$	$\dfrac{2a}{a^2+\omega^2}$	$a>0$
4	$te^{-at}u(t)$	$\dfrac{1}{(a+j\omega)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\dfrac{n!}{(a+j\omega)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	
9	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
10	$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	
11	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$	
12	$\operatorname{sgn} t$	$\frac{2}{j\omega}$	
13	$\cos\omega_0 t\, u(t)$	$\frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\frac{j\omega}{\omega_0^2-\omega^2}$	
14	$\sin\omega_0 t\, u(t)$	$\frac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\frac{\omega_0}{\omega_0^2-\omega^2}$	
15	$e^{-at}\sin\omega_0 t\, u(t)$	$\frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$	$a>0$
16	$e^{-at}\cos\omega_0 t\, u(t)$	$\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$
17	$\operatorname{rect}\left(\frac{t}{\tau}\right)$	$\tau\operatorname{sinc}\left(\frac{\omega\tau}{2}\right)$	
18	$\frac{W}{\pi}\operatorname{sinc}(Wt)$	$\operatorname{rect}\left(\frac{\omega}{2W}\right)$	
19	$\Delta\left(\frac{t}{\tau}\right)$	$\frac{\tau}{2}\operatorname{sinc}^2\left(\frac{\omega\tau}{4}\right)$	
20	$\frac{W}{2\pi}\operatorname{sinc}^2\left(\frac{Wt}{2}\right)$	$\Delta\left(\frac{\omega}{2W}\right)$	
21	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$	$\omega_0=\frac{2\pi}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$	

more about properties...

Differentiation in the s-Domain $\boxed{-t x(t) \xleftrightarrow{\mathcal{L}_u} \frac{d}{ds} X(s)}$

Initial and Final value Theorem:

$$\boxed{\lim_{s \to \infty} s X(s) = x(0^+)}$$

- The initial value theorem does not apply to rational functions $X(s)$ in which the order of the numerator polynomial is greater than or equal to that of the denominator polynomial order.

$$\boxed{\lim_{s \to 0} s X(s) = x(\infty)}$$

- The final-value theorem applies only if all the poles of $X(s)$ are in the left half of the s-plane, with at most a single pole at $s=0$

EXAMPLE 6.6 APPLYING THE INITIAL- AND FINAL-VALUE THEOREMS Determine the initial and final values of a signal $x(t)$ whose unilateral Laplace transform is

$$X(s) = \frac{7s + 10}{s(s + 2)}.$$

Solution: We may apply the initial-value theorem, Eq. (6.21), to obtain

$$\begin{aligned} x(0^+) &= \lim_{s \to \infty} s \frac{7s + 10}{s(s + 2)} \\ &= \lim_{s \to \infty} \frac{7s + 10}{s + 2} \\ &= 7. \end{aligned}$$

The final value theorem, Eq. (6.22), is applicable also, since $X(s)$ has only a single pole at $s = 0$ and the remaining poles are in the left half of the s-plane. We have

$$\begin{aligned} x(\infty) &= \lim_{s \to 0} s \frac{7s + 10}{s(s + 2)} \\ &= \lim_{s \to 0} \frac{7s + 10}{s + 2} \\ &= 5. \end{aligned}$$

The reader may verify these results by showing that $X(s)$ is the Laplace transform of $x(t) = 5u(t) + 2e^{-2t}u(t)$. ■

EXAMPLE 6CT.9 A Ramp

Consider

$$x(t) = r(t) = tu(t)$$

We showed earlier that

$$u(t) \overset{\text{LT}}{\longleftrightarrow} \frac{1}{s} \quad \mathcal{R}\{s\} > 0$$

The "multiply by t" property yields

$$r(t) \overset{\text{LT}}{\longleftrightarrow} -\frac{d}{ds}\frac{1}{s} = \frac{1}{s^2} \quad \mathcal{R}\{s\} > 0 \tag{46}$$

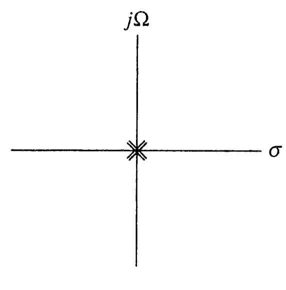
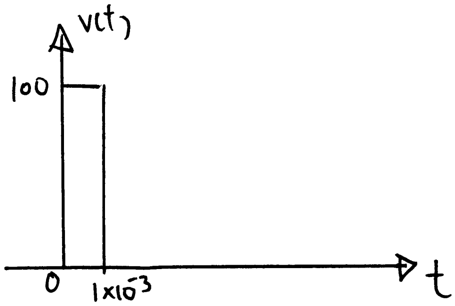


Figure 6CT.5 Pole-zero plot for the unit ramp.

The pole-zero plot is shown in Figure 6CT.5, where the pole is indicated by overlapping crosses. This pole is called a *second-order* pole because the degree of the denominator s^2 is 2.

Laplace transform by using table

Ex



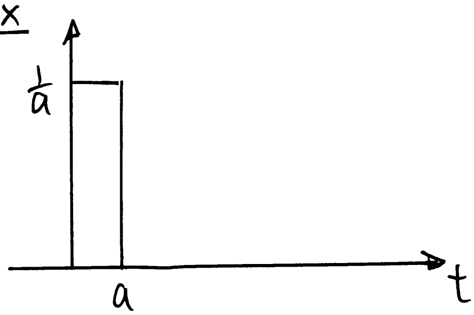
The area of the pulse is $100\,V \times 10^{-3} s = 0.1\,V\cdot s$

$\therefore$ we may approximate $v(t)$ as $v(t) \approx 0.1\,\delta(t)$

$\therefore V(s) = \mathcal{L}[v(t)] = \mathcal{L}[0.1\,\delta(t)]$

$= 0.1\,\mathcal{L}[\delta(t)] = 0.1$

Ex



$f(t) = \frac{1}{a}[u(t) - u(t-a)]$

$F(s) = \mathcal{L}[f(t)]$

$\mathcal{L}[u(t)] = \frac{1}{s}$; $\mathcal{L}[u(t-a)] = \frac{1}{s}e^{-as}$

$\therefore F(s) = \frac{1}{a}[\frac{1}{s} - \frac{1}{s}e^{-as}]$

Ex Find $I(s)$ when $i(t) = 10\cos(20t + 30°)$

First, we decompose $i(t)$ into the sum of a sine function and a cosine function

$\therefore i(t) = 10[\cos 20t \cos 30° - \sin 20t \sin 30°] = 5\sqrt{3}\cos 20t - 5\sin 20t$

$(\because \cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)$

$\therefore I(s) = 5\sqrt{3}\,\mathcal{L}[\cos 20t] - 5\,\mathcal{L}[\sin 20t]$

$$= \frac{5\sqrt{3}s}{s^2+400} - \frac{100}{s^2+400}$$

Inverse Laplace Transform by using table

Ex given $I(s) = \frac{5}{s^2}[1 - 2e^{-3s} + 2e^{-6s} - 2e^{-9s} + 2e^{-12s} - \dots]$, find $i(t)$

since $\mathcal{L}^{-1}[\frac{1}{s^2}] = tu(t)$ and $\mathcal{L}[f(t-a)u(t-a)] = e^{-as}F(s)$

$\therefore i(t) = \mathcal{L}^{-1}[I(s)] = 5[tu(t) - 2(t-3)u(t-3) + 2(t-6)u(t-6)$

$- 2(t-9)u(t-9) + 2(t-12)u(t-12) - \dots]$