ENE 208 circuit analysis by Laplace transforms

① Laplace transform by partial fraction expansion

The Laplace transforms of interest in circuit analysis can, in most cases, be expressed as the ratio of two polynomials. Thus, let us assume a transform given by

$$F(s) = \frac{N(s)}{D(s)}$$

where $N(s)$ is a numerator polynomial of the form:

$$N(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$$

and $D(s)$ is a denominator polynomial of the form

$$D(s) = b_m s^m + b_m s^{m-1} + \dots + b_1 s + b_0$$

Hence, the numerator polynomial contains n roots (zeros) and the denominator polynomial contains m roots (poles)

We now consider $D(s)$ to be factored as

$$F(s) = \frac{N(s)}{b_m(s-p_1)(s-p_2)\dots(s-p_m)}$$

where p's represent the various poles of $F(s)$. It can be shown that for real physical System, $m \geq n$. The m poles of $F(s)$ may be arbitrarily classified into 4 groups:

1 Real poles of first order
2 Complex poles of first order
3 Real poles of multiple order
4 Complex poles of multiple order

Assume that the denominator contains r real poles of first order, we may write $F(s)$ as follows:

$$F(s) = \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \dots + \frac{A_r}{s-p_r} + R(s)$$

where $R(s)$ is the remaining portion of the expansion due to poles belonging to other classification. The method above is called a partial fraction expansion. The first r terms may be inverted on a term-by-term basis by means of

$$e^{-\alpha t} \longleftrightarrow \frac{1}{s+\alpha}$$

So our basic problem at present is the determination of the A coefficients!!

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This task may be achieved as follows: Consider an arbitrary coefficient A_k. Let us multiply both sides by $(s-p_k)$ and rearrange some terms. This results in

$$(s-p_k)F(s) = (s-p_k)\left[\frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \cdots + \frac{A_r}{s-p_r} + R(s)\right] + A_k$$

in which the A_k term has been removed from the brackets. Note that the $(s-p_k)$ multiplier has canceled the denominator of the A_k term. It will also cancel the $(s-p_k)$ factor in the denominator of $F(s)$.

Suppose, then, that we use a "trick" by letting $s = p_k$. As results, all terms on the right-hand side of the eqn vanish except the term A_k. However, the LHS, in general, is nonzero, since the factor will have canceled. Hence,

$$\boxed{A_k = (s-p_k)F(s)\Big|_{s=p_k}}$$

This procedure is then repeated at each of the r real poles of simple order

Once all the A's are determined, the inverse transform of the portion involving the real poles of simple order can be determined.

$$\therefore f_1(t) = A_1 e^{p_1 t} + A_2 e^{p_2 t} + \cdots + A_r e^{p_r t}$$

$$\text{and } f_+(t) = f_1(t) + \mathcal{L}^{-1}[R(s)]$$

In most cases, the real poles will be negative in sign, implying that the time function is a sum of decaying exponential terms!!

Example Determine the inverse transform of $V(s) = \dfrac{s+4}{s^2+3s+2}$

Sol^n: we factor the denominator first: $V(s) = \dfrac{s+4}{(s+1)(s+2)}$

Then, the partial fraction expansion reads: $V(s) = \dfrac{A_1}{s+1} + \dfrac{A_2}{s+2}$

$\therefore$ The constants are determined as follows:

$$A_1 = (s+1)V(s)\Big|_{s=-1} = (s+1)\left[\frac{s+4}{(s+1)(s+2)}\right]_{s=-1} = \frac{s+4}{s+2}\Big|_{s=-1}$$

$$= \frac{-1+4}{-1+2} = 3$$

$$A_2 = (s+2)V(s)\Big|_{s=-2}$$

$$= (s+2)\left[\frac{s+4}{(s+1)(s+2)}\right]_{s=-2} = \frac{s+4}{s+1}\Big|_{s=-2} = \frac{-2+4}{-2+1} = -2$$

Thus $V(s) = \dfrac{3}{s+1} - \dfrac{2}{s+2}$

$$\therefore v(t) = \mathcal{L}^{-1}[V(s)] = 3e^{-t} - 2e^{-2t}$$

Under certain conditions, the situation is occasionally encountered in which the numerator and denominator have the same degree. In this case, the numerator polynomial is first divided by the denominator polynomial, yielding a constant plus a remainder function whose denominator is of higher degree than its numerator. The inverse transform of the constant is an impulse function ($\delta(t)$) and the inverse transform of the remainder function can be determined from the method discussed previously.

Example Determine the inverse transform of $F(s) = \dfrac{2s^2+11s+4}{s(s+1)}$

Sol^n In this prob, $N(s)$ and $D(s)$ have the same degree; hence, we must first divide $N(s)$ by $D(s)$.

$$\begin{array}{r|l} & 2 \\ \hline s^2+s & 2s^2+11s+4 \\ & 2s^2+2s \\ \hline & \quad 9s+4 \end{array}$$

Thus, $\quad F(s) = 2 + \dfrac{9s+4}{s(s+1)}$

The second quantity may be expanded $\rightarrow F(s) = 2 + \dfrac{A_1}{s} + \dfrac{A_2}{s+1}$

$$= 2 + \frac{4}{s} + \frac{5}{s+1}$$

$$f(t) = \mathcal{L}^{-1}[F(s)] = 2\delta(t) + 4 + 5e^{-t}$$

$\boxed{\text{Complex poles of } 1^{st} \text{ order}}$

Example Determine the inverse transform of the function $F(s) = \dfrac{100(s+3)}{(s+1)(s+2)(s^2+2s+5)}$

Sol^n Use partial fraction expansion: $F(s) = \dfrac{A_1}{s+1} + \dfrac{A_2}{s+2} + \dfrac{A_3}{s+1-j2} + \dfrac{A_4}{s+1+j2}$

Find A_1 and A_2 on your own!

A_3 can be found from $\quad (s+1-j2)F(s)\Big|_{s=-1+j2} = (s+1-j2)\times\dfrac{100(s+3)}{(s+1)(s+2)(s^2+2s+5)}\Bigg|_{s=-1+2j}$

$$= \frac{100(s+3)}{(s+1)(s+2)(s+1+2j)}\Bigg|_{s=-1+2j}$$

$$= \frac{100(2+2j)}{(2j)(1+2j)(4j)}$$

$$= 5\sqrt{10}\, e^{j161.6^\circ}$$

Similarly $A_4 = (s+1+j2)F(s)\Big|_{s=-1-2j}$

$$= 5\sqrt{10}\, e^{-j161.6^\circ}$$

Note: A_3 is a complex conjugate of A_4!

$$\therefore \frac{A_3}{s+1-2j} + \frac{A_4}{s+1+j2} = \frac{5\sqrt{10}\,e^{j161.6^\circ}}{s+1-2j} + \frac{5\sqrt{10}\,e^{-j161.6^\circ}}{s+1+2j}$$

And inverse transform would give $5\sqrt{10}\,e^{j161.6^\circ} e^{(-1+2j)t} + 5\sqrt{10}\,e^{-j161.6^\circ} e^{(-1-2j)t}$

$$= 5\sqrt{10}\,e^{-t}\left[e^{j(2t+161.6^\circ)} + e^{-j(2t+161.6^\circ)}\right]$$

$$= 10\sqrt{10}\,e^{-t}\cos(2t+161.6^\circ)$$

(Note: don't forget to do the inverse transform of $\frac{A_1}{s+1}$ and $\frac{A_2}{s+2}$ also!!)

$\boxed{\text{Multiple-order pole}}$ Now let's consider the case of multiple-order pole

$$F(s) = \frac{Q(s)}{(s-p)^r}$$

The partial fraction expansion of $F(s)$ requires the following form:

$$F(s) = \frac{A_1}{(s-p)^r} + \frac{A_2}{(s-p)^{r-1}} + \cdots + \frac{A_k}{(s-p)^{r-k+1}} + \cdots + \frac{A_r}{(s-p)} + R(s)$$

where $R(s)$ is the expansion due to all other poles. Let $F_1(s)$ represent the expansion of interest at present.

$$F_1(s) = \frac{A_1}{(s-p)^r} + \frac{A_2}{(s-p)^{r-1}} + \cdots + \frac{A_k}{(s-p)^{r-k+1}} + \cdots + \frac{A_r}{(s-p)}$$

It can be shown that the general k^{th} coefficient is given by the formula

$$A_k = \frac{1}{(k-1)!}\frac{d^{k-1}}{ds^{k-1}}Q(s)\Big|_{s=p} \quad \text{and} \quad Q(s) = (s-p)^r F(s)$$

Once the coefficients are known, the inverse transform can be determined by using

$$e^{-\alpha t} t^n \longleftrightarrow \frac{n!}{(s+\alpha)^{n+1}}$$

The general form is hence

$$f_1(t) = \left[\frac{A_1 t^{r-1}}{(r-1)!} + \frac{A_2 t^{r-2}}{(r-2)!} + \cdots + \frac{A_k e^{r-k}}{(r-k)!} + \cdots + A_r\right]e^{pt}$$

Example

Determine the inverse transform of

$$F(s) = \frac{s^2+4}{s(s+1)(s+2)^3} \qquad \textbf{(5–165)}$$

Solution

First we will determine the response due to the third-order pole, $s = -2$. Letting $F_1(s)$ represent this portion of the transform and $f_1(t)$ its inverse, we have

$$F_1(s) = \frac{A_1}{(s+2)^3} + \frac{A_2}{(s+2)^2} + \frac{A_3}{(s+2)} \qquad \textbf{(5–166)}$$

Furthermore,

$$Q(s) = \frac{s^2+4}{s(s+1)} \qquad \textbf{(5–167)}$$

The required derivatives are

$$\frac{dQ(s)}{ds} = \frac{2s(s^2 + s) - (2s + 1)(s^2 + 4)}{(s^2 + s)^2} = \frac{s^2 - 8s - 4}{s^2(s + 1)^2} \quad \textbf{(5–168)}$$

$$\frac{d^2Q(s)}{ds^2} = \frac{s^2(s + 1)^2(2s - 8) - (s^2 - 8s - 4)[2s^2(s + 1) + 2s(s + 1)^2]}{s^4(s + 1)^4} \quad \textbf{(5–169)}$$

There is not much point in simplifying the last expression, since no further differentiation is required, and when the proper value is inserted shortly, it will reduce fairly quickly.

By means of Equation (5–157), the coefficients are

$$A_1 = \frac{4 + 4}{(-2)(-1)} = 4 \quad \textbf{(5–170)}$$

$$A_2 = \frac{4 + 16 - 4}{(4)(1)} = 4 \quad \textbf{(5–171)}$$

$$\begin{aligned} A_3 &= \frac{1}{2}\left\{\frac{(4)(1)(-12) - (4 + 16 - 4)(2)[(4)(-1) + (-2)(1)]}{(16)(1)}\right\} \\ &= \frac{1}{2}\left[\frac{-48 - (16)(-12)}{16}\right] = \frac{9}{2} = 4.5 \end{aligned} \quad \textbf{(5–172)}$$

Thus,

$$F_1(s) = \frac{4}{(s + 2)^3} + \frac{4}{(s + 2)^2} + \frac{4.5}{(s + 2)} \quad \textbf{(5–173)}$$

and from Equation (5–158);

$$f_1(t) = [2t^2 + 4t + 4.5]\varepsilon^{-2t} \quad \textbf{(5–174)}$$

Let us designate the remainder of the time function as $f_2(t)$. It is left as an exercise for the reader to show that

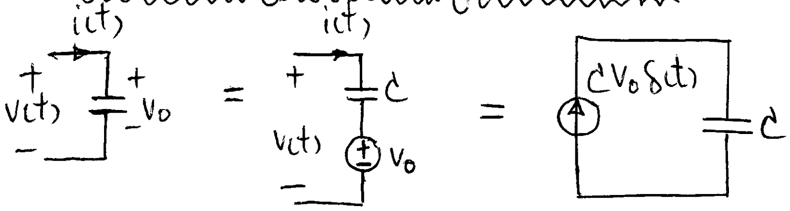
$$f_2(t) = 0.5 - 5\varepsilon^{-t} \quad \textbf{(5–175)}$$

The total time function is, of course, given by

$$f(t) = f_1(t) + f_2(t) \quad \textbf{(5–176)}$$

Circuit analysis by Laplace transform

① Transform equivalent of capacitance



$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)dt$$
$$= \frac{1}{C}\left[\int_{0}^{t} i(t)dt + \int_{-\infty}^{0} i(t)dt\right]$$
$$= \frac{1}{C}\int_{0}^{t} i(t)dt + V_0$$

A charged capacitor is equivalent, for $t>0$, to an uncharge capacitor in series with a dc or step voltage source

$$V(s) = \mathcal{L}[v(t)]$$
$$I(s) = \mathcal{L}[i(t)]$$
$$V(s) = \frac{1}{sC} I(s) + \frac{V_0}{s}$$

$\frac{1}{sC}$ called "transform impedance of a capacitor ($Z(s)$), has dimension as resistance which is ohm

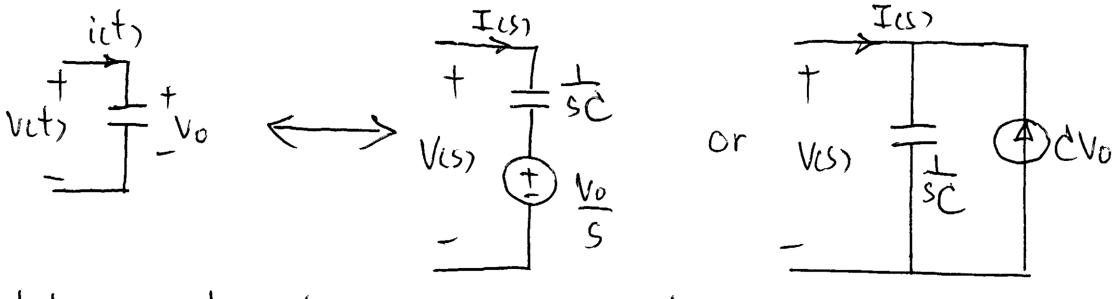
From Ohm's law $\rightarrow$ $V(s) = Z(s) I(s)$

or $I(s) = \frac{V(s)}{Z(s)}$

Transform admittance = reciprocal of $Z(s) = \frac{1}{Z(s)} = Y(s)$

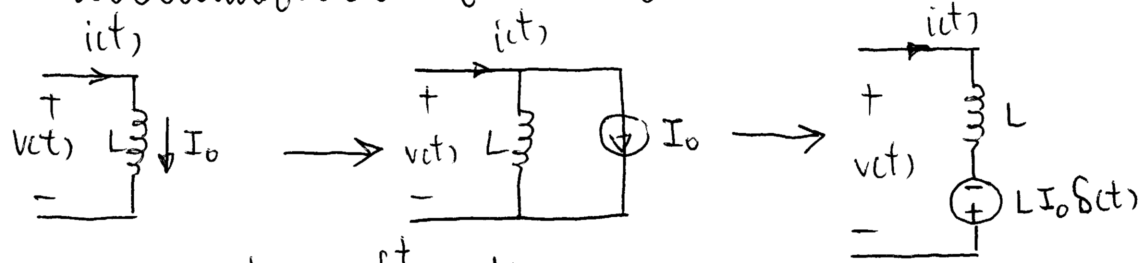
For capacitor, $Y(s) = sC$

Therefore,



In steady state, capacitor acts like an open-circuit.

② Transform equivalent of inductance



$$i(t) = \frac{1}{L}\int_0^t v(t)\,dt + I_0$$

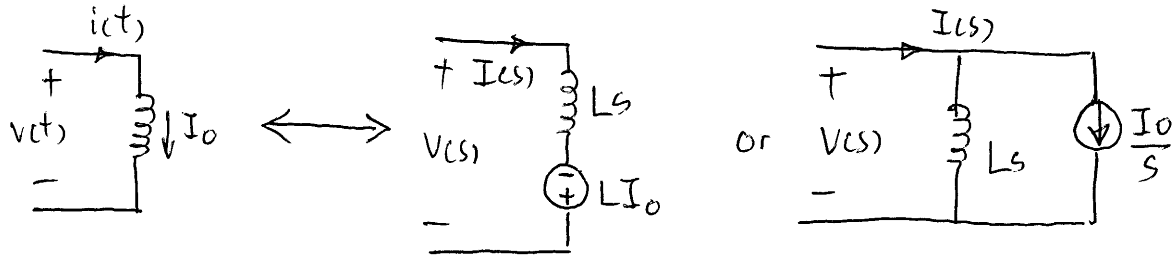
$$V(s) = L[sI(s) - I_0] = LsI(s) - LI_0 = Ls\left[I(s) - \frac{I_0}{s}\right]$$

$$Z(s) = sL$$

$$Y(s) = \frac{1}{sL}$$

A fluxed inductor is equivalent, for $t>0$, to an unfluxed inductor in parallel with a dc or step-current source.
In the dc steady state, the voltage across a pure inductor is zero, implying that the inductor acts as a short circuit under this condition.

Therefore,



③ Transform equivalent of resistance

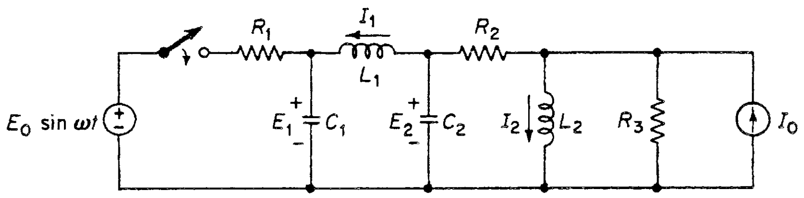
$$v(t) = Ri(t)$$

$$V(s) = RI(s)$$

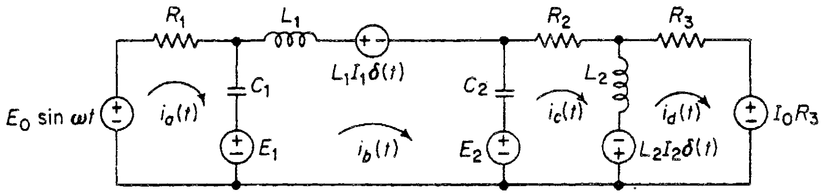
$$Z(s) = R$$

Example

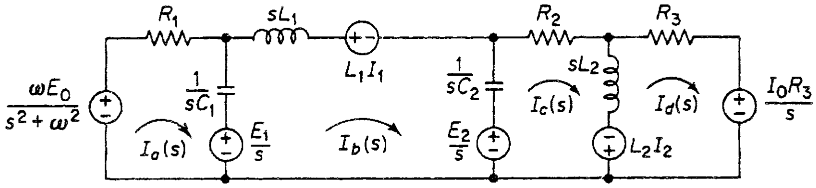
The switch in the circuit of Figure 6–5(a) is closed at $t = 0$. The initial values of inductive currents and capacitive voltages are shown. Draw the transformed circuit in a form most suitable for mesh current analysis.



(a)



(b)



(c)

Solutions of complete circuits in the transform domain

After a ckt is completely transformed according to the procedure discussed above, it may then be manipulated by any standard algebraic or circuit analysis technique. Among the possible methods of solution are mesh current analysis, node voltage analysis, Thévenin's theorem, Norton's theorem, successive reduction technique, and many others.

In general, any dc ckt analysis scheme may be employed as long as we remember that both sources and impedances that appear in the ckt are functions of the variable "s".

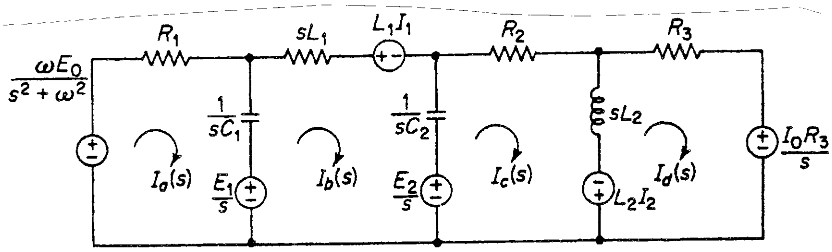
After the desired voltage or current is obtained in the transform domain, its inverse transform can be determined to yield the final time function. In many problems, such as are encountered in network and controlled system design studies the entire analysis and design may be carried out in the transform domain, no inversion may be necessary.

Example Referring back to the previous example, write a set of mesh current equations that characterize the network.

Solution

The transformed circuit is shown again in Figure 6–6 with mesh currents assigned. Remember, the transform impedances and transform sources are treated exactly like dc quantities in writing equations. The mesh equations are

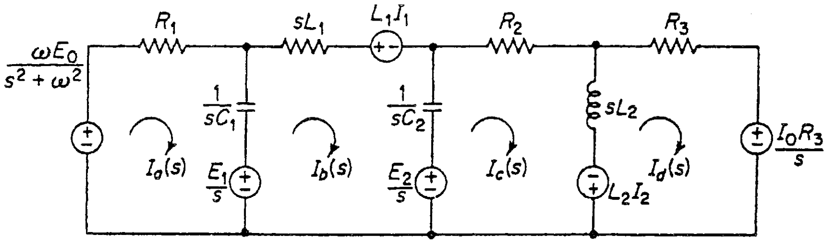
$$\left[R_1 + \frac{1}{sC_1}\right]I_a(s) - \left[\frac{1}{sC_1}\right]I_b(s) = \frac{\omega E_0}{s^2+\omega^2} - \frac{E_1}{s} \qquad \textbf{(6–26)}$$



$$\left[\frac{-1}{sC_1}\right]I_a(s) + \left[\frac{1}{sC_1} + sL_1 + \frac{1}{sC_2}\right]I_b(s) - \left[\frac{1}{sC_2}\right]I_c(s) = \frac{E_1}{s} - L_1 I_1 - \frac{E_2}{s} \quad (6\text{–}27)$$

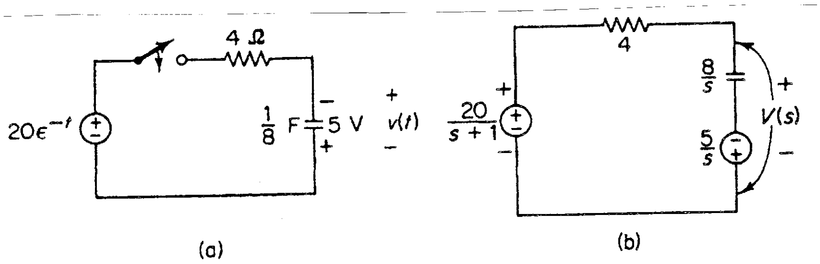
$$-\left[\frac{1}{sC_2}\right]I_b(s) + \left[\frac{1}{sC_2} + R_2 + sL_2\right]I_c(s) - [sL_2]I_d(s) = \frac{E_2}{s} + L_2 I_2 \quad (6\text{–}28)$$

$$-[sL_2]I_c(s) + [sL_2 + R_3]I_d(s) = -L_2 I_2 - \frac{I_0 R_3}{s} \quad (6\text{–}29)$$



$\boxed{\text{First-order circuit}}$ we'll consider problems involving ckts whose time-domain differential eqns are of first order

Example solve for i(t) and v(t) for ckt below



$$\frac{1}{sC} = \frac{1}{s\times\frac{1}{8}} = \frac{8}{s}$$

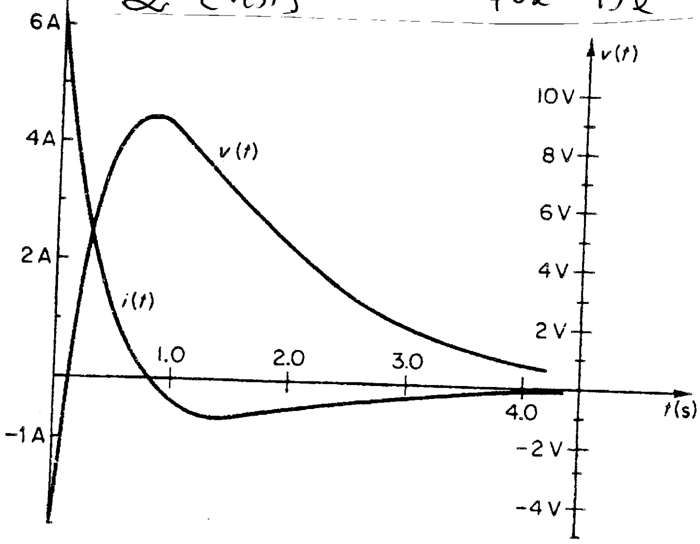
$$I(s) = \frac{V(s)}{Z(s)} = \frac{\left[20/(s+1) + \frac{5}{s}\right]}{4+\frac{8}{s}} = \frac{5s}{(s+1)(s+2)} + \frac{1.25}{(s+2)}$$

$$= \frac{-5}{s+1} + \frac{11.25}{s+2}$$

$$\mathcal{L}^{-1}[I(s)] = i(t) = 11.25e^{-2t} - 5e^{-t}$$

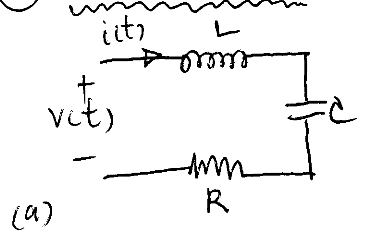
$$V(s) = I(s)\times\frac{8}{s} - \frac{5}{s} = \frac{40}{(s+1)(s+2)} + \frac{10}{s(s+2)} - \frac{5}{s} = \frac{40}{s+1} - \frac{45}{s+2}$$

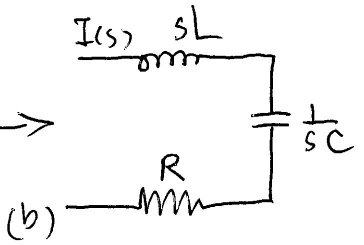
$$\mathcal{L}^{-1}[V(s)] = v(t) = 40e^{-t} - 45e^{-2t} = 5e^{-t}(8-9e^{-t})$$



$\boxed{\text{Second-order circuit}}$ → The ckt for which the describing differential eqn is of second order. A second-order system may still be an RL or an RC form, or it may be an RLC form.

① Series RLC ckt





Consider the ckt shown in Fig (a), with no initial energy storage assumed, and its transform shown in (b). Depending on the desired quantity, we can solve for a transform response by writing a mesh current eqn or by means of the impedance concept. The latter Interpretation results in

$$I(s) = \frac{E(s)}{Z(s)}$$

where
$$Z(s) = sL + R + \frac{1}{sC} = \frac{s^2LC + sRC + 1}{sC}$$
$$= \frac{s^2 + sR/L + 1/LC}{s/L}$$

substitute this result in for the eqn of the $I(s)$:

$$I(s) = \frac{sE(s)/L}{s^2 + sR/L + 1/LC}$$

The poles of $I(s)$, which determine the form of the time response, are determined by the poles of $E(s)$ and the roots of the quadratic $s^2 + sR/L + 1/RC$. The latter roots are the zeros (numerator roots) of the impedance $Z(s)$. Since $E(s)$ may be almost anything, in general let us turn our attention to the form of the transient response of the network due primarily to the poles of the ckt.

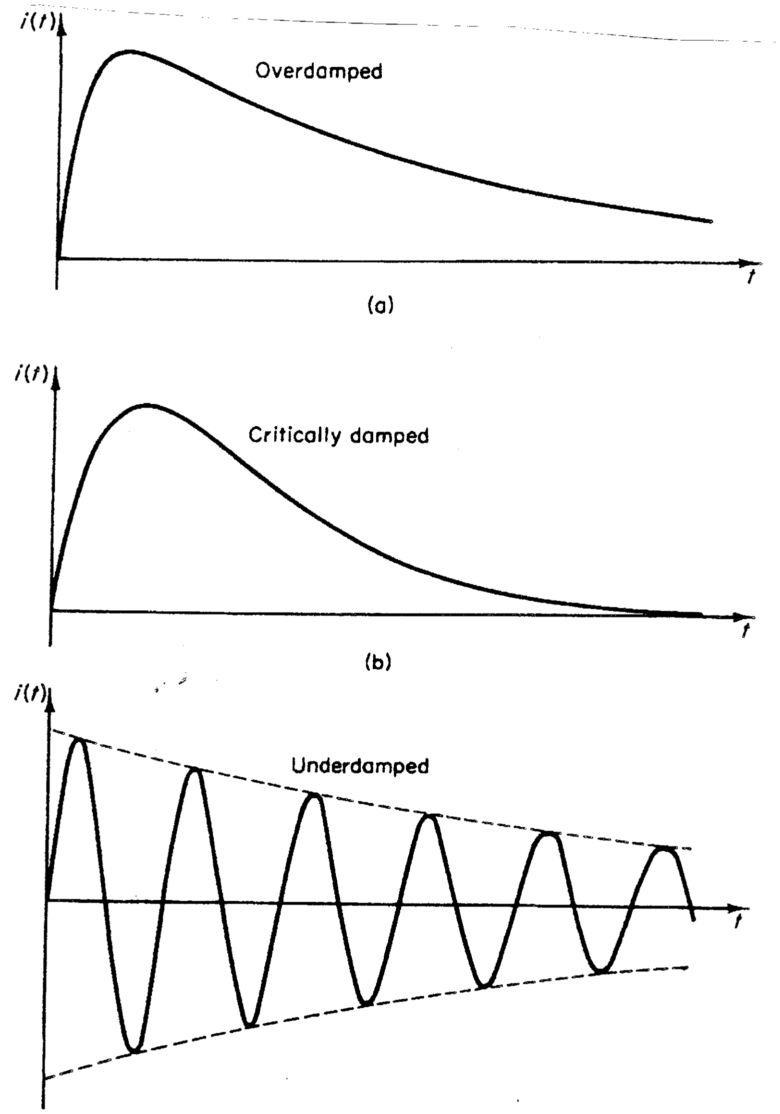
Assume that $e(t)$ is a dc voltage of E volts applied at $t=0$, the transient excitation is $E(s) = E/s$.

$$\therefore \quad I(s) = \frac{E/L}{s^2 + s(R/L) + 1/LC}$$

The poles due to the network are determined from the equation

$$s^2 + \frac{sR}{L} + \frac{1}{LC} = 0$$

Let s_1, s_2 represent the poles, we have $s_1, s_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{\sqrt{LC}}\right)^2}$



Typical step responses of series *RLC* circuit. (c)

① Overdamped: If $\frac{R}{2L} > \frac{1}{\sqrt{LC}}$, the roots are both real, negative in sign and of simple order

$$I(s) = \frac{E/L}{(s+\alpha_1)(s+\alpha_2)} = \frac{A_1}{s+\alpha_1} + \frac{A_2}{s+\alpha_2}$$

The coefficients A_1, A_2 can be readily determined by partial fraction expansion.

$$i(t) = A e^{-\alpha_1 t} + A_2 e^{-\alpha_2 t}$$

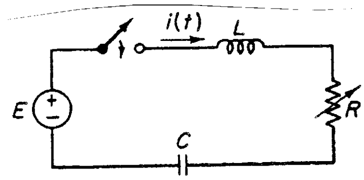
② Critically damped. If $\frac{R}{2L} = \frac{1}{\sqrt{LC}}$, the roots are both real, negative in sign and equal.

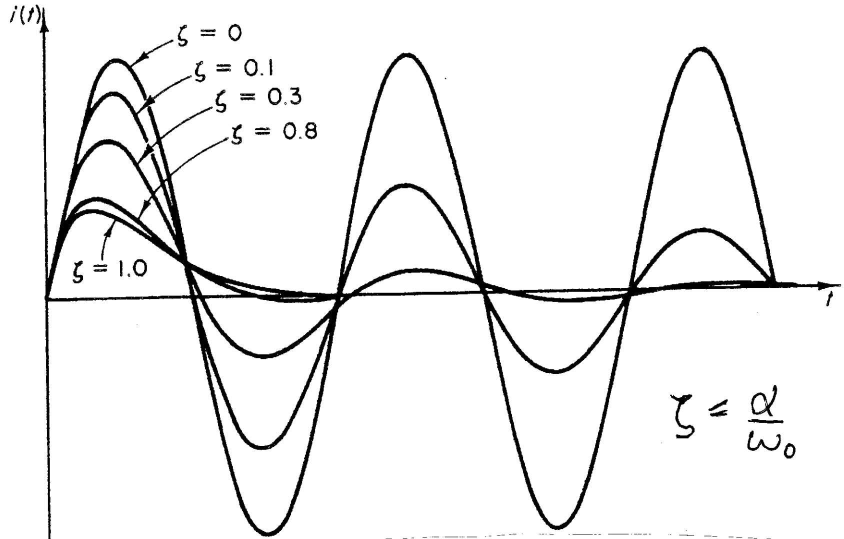
$$I(s) = \frac{E/L}{(s+\alpha)^2} \quad \text{where } \alpha = -\frac{R}{2L}$$

$$i(t) = \frac{Et}{L} e^{-\alpha t}$$

③ Underdamped If $\frac{R}{2L} < \frac{1}{\sqrt{LC}}$, the roots are complex and of first order, with negative real parts.

Circuit in which the damping is to be varied.





Step responses of series *RLC* circuit for different values of damping.

(underdamped continued...) $s_1, s_2 = -\frac{R}{2L} \pm j\sqrt{\left(\frac{1}{\sqrt{LC}}\right)^2 - \left(\frac{R}{2L}\right)^2}$

Since the real part corresponds to a damping constant and the imaginary part corresponds to an oscillatory response, let us define some useful terms. Let

$$\alpha = \frac{R}{2L} = \text{damping const}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \text{Undamped natural resonant freq}$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \text{damped natural resonant freq}$$

The quantity ω_0 is the angular freq of oscillation if there is no resistance in the ckt (i.e. $R = 0$). However, the damped freq ω_d is always less than the undamped freq.

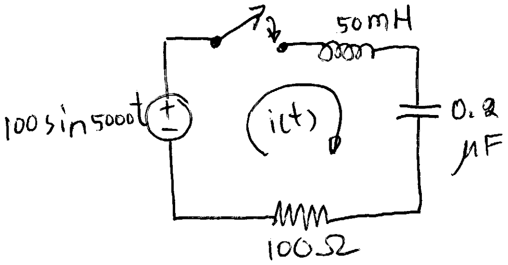
For the underdamped case, we may write $I(s) = \frac{E/L}{(s+\alpha)^2 + \omega_d^2}$

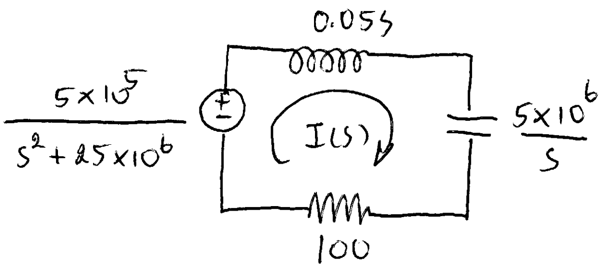
$$\boxed{i(t) = \frac{E e^{-\alpha t}}{\omega_d L} \sin \omega_d t}$$

It's an interesting problem to investigate how the response changes as the damping factor is increased. Consider the ckt at the top of this page, in which L and C are fixed but R is adjustable. Thus, ω_0 is fixed, but α varies directly with R and ω_d decreases with an increase of R. The ckt becomes overdamped for $\alpha/\omega_0 = \zeta = 1$

When an RLC ckt is excited by a more general excitation, the response will consist of 2 parts. The ~natural~ part will be due to the ckt itself and will always be similar to one of the forms discussed, depending on whether the ckt is underdamped, critically damped, or overdamped. As long as there is any resistance at all in the ckt, this response will be transient in nature and will disappear after a sufficiently long time. The forced part of the response will be due to the nature of the source, and if the source is such as to maintain a response after the transient disappears, such response is, of course, the steady-state response.

Example The relaxed series RLC ckt is excited at $t=0$ by the sinusoidal source shown. Solve for the current $i(t)$ for $t>0$





$$\frac{R}{2L} = \frac{100}{2\times0.05} = 10^3$$

$$\frac{1}{\sqrt{LC}} = -\frac{1}{\sqrt{0.05\times0.2\times10^{-6}}} = 10^4$$

since $\frac{R}{2L} < \frac{1}{\sqrt{LC}}$, the ckt is underdamped and oscillatory. We have

$$\alpha = 10^3 \text{ nepers}$$
$$\omega_0 = 10^4 \text{ rad/s}$$

$$\omega_d \sqrt{\omega_0^2-\alpha^2} = 9.95\times10^3 \text{ rad/s}$$

Using the impedance concept, we have $Z(s) = 0.05s + 100 + \frac{5\times10^6}{s} = \frac{s^2+2000s+10^8}{20s}$

The current $I(s) = \frac{E(s)}{Z(s)} = \frac{10^7 s}{(s^2+25\times10^6)(s^2+2000s+10^8)}$

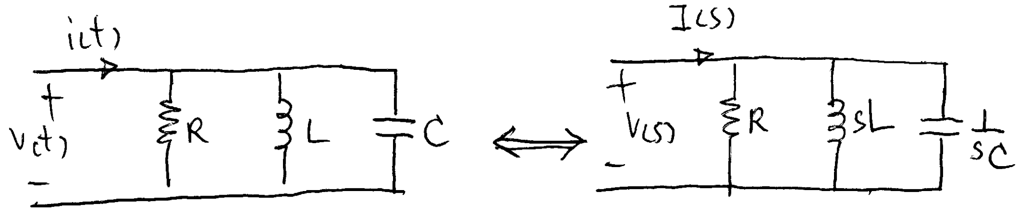
The poles due to the quadratic with 3 terms are $s_1, s_2 = -10^3 \pm j9.95\times10^3$

$$i(t) = \mathcal{L}^{-1}[I(s)] = 0.1332e^{-1000t}\sin(9.95\times10^3 t - 99.51^\circ)$$
$$+0.1329\sin(5000t + 82.41^\circ)$$

The response consists of a damped sinusoidal term whose freq is the natural damped resonant freq of the ckt, and an undamped sinusoidal whose freq is that of the excitation. The former is transient in nature, whereas the latter is the steady-state response. After the transient disappears, the steady-state or forced response is

$$i_{ss}(t) = 0.1329\sin(5000t + 82.41^\circ)$$

② Parallel RLC ckt



The admittance of the network is given by $Y(s) = sC + \frac{1}{R} + \frac{1}{sL}$

$$= \frac{s^2LC + sL/R + 1}{sL}$$

$$= \frac{s^2 + s/RC + 1/LC}{s/C}$$

The impedance $Z(s) = \frac{1}{Y(s)} = \frac{s/C}{s^2 + s/RC + 1/LC}$

If a current $I(s)$ excites the network, the resulting voltage is

$$V(s) = Z(s)I(s) = \frac{sI(s)/C}{s^2 + s/RC + 1/LC}$$

Let $I(s) = I/s$, $V(s) = \frac{I/C}{s^2 + s/RC + 1/LC}$

poles are $s_1, s_2 = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \left(\frac{1}{\sqrt{LC}}\right)^2}$

we can have 3 possibilities

① Overdamped : $\frac{1}{2RC} > \frac{1}{\sqrt{LC}}$

② critically damped : $\frac{1}{2RC} = \frac{1}{\sqrt{LC}}$

③ Underdamped $\frac{1}{2RC} < \frac{1}{\sqrt{LC}}$

The undamped resonant freq $\omega_0 = \frac{1}{\sqrt{LC}}$

The damping factor $\alpha = \frac{1}{2RC}$

The damped resonant freq $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

The relative shapes and forms of the 3 possible types of response in this case are analogous to the current responses of the series RLC ckt.

EXAMPLE 6–11 The relaxed circuit of Figure 6–25(a) is excited at $t = 0$ by a pulse that approximates an impulse of area 100 V · s. Determine the voltage across the tuned circuit, $v(t)$, for $t > 0$.

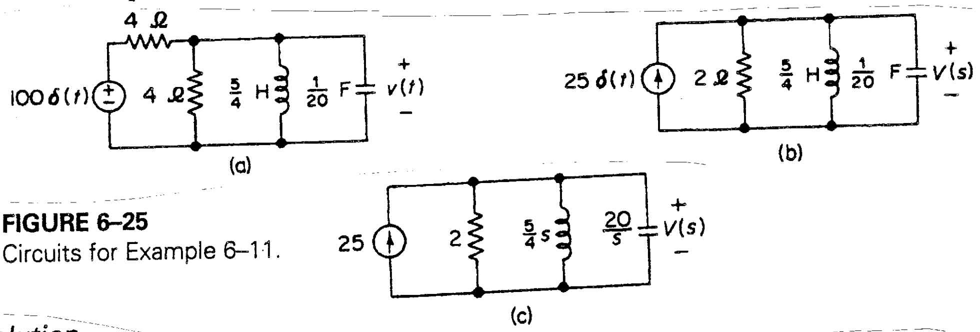


FIGURE 6–25
Circuits for Example 6–11.

Solution

The first step in solving the problem is to rearrange the circuit in the simplest form for analysis. We do this by converting the impulse-voltage source to an impulse-current source and combining the two resistors. The result is shown in Figure 6–25(b) and its transform is shown in (c). The admittance is

$$Y(s) = \frac{1}{20}s + \frac{1}{2} + \frac{4}{5s} = \frac{s^2 + 10s + 16}{20s} \tag{6–145}$$

$$Z(s) = \frac{20s}{s^2 + 10s + 16} \tag{6–146}$$

The voltage $V(s)$ is

$$V(s) = Z(s)I(s) = \frac{500s}{s^2 + 10s + 16} \tag{6–147}$$

The poles are

$$\left.\begin{cases} s_1 \\ s_2 \end{cases}\right. = -5 \pm \sqrt{25 - 16} = -5 \pm 3 = -2 \quad \text{and} \quad -8 \tag{6–148}$$

Thus, the response in this case is overdamped. We may write $V(s)$ as

$$V(s) = \frac{500s}{(s + 2)(s + 8)} = \frac{-500/3}{s + 2} + \frac{2000/3}{s + 8} \tag{6–149}$$

The time response is thus

$$v(t) = \frac{500}{3}[4\varepsilon^{-8t} - \varepsilon^{-2t}] \tag{6–150}$$

A sketch of the response is shown in Figure 6–26.

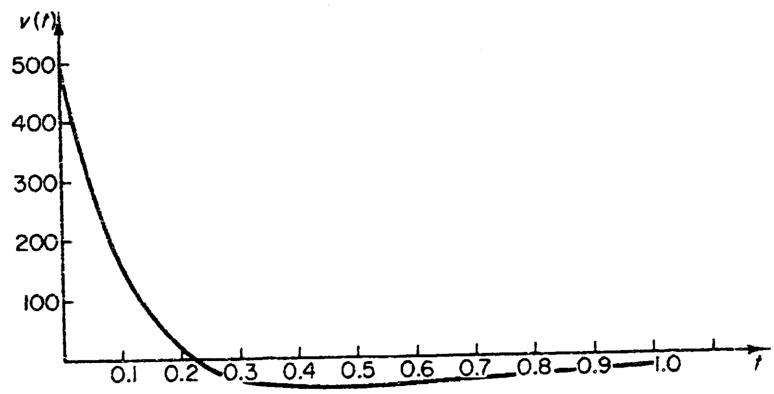


FIGURE 6–26
Response of the circuit for Example 6–11.

Redundancy and Higher Order A ckt containing 2 or more ckt components of the same type is said to be redundant if these components can be reduced to a single equivalent components under all conditions external to their terminals. Hence, the order of a ckt is the number of energy storage element after redundancy is removed. Furthermore, the order of the ckt is the order of the denominator polynomial of a given response when the only excitations in the ckt are impulse functions. In effect, the latter statement says that poles due to excitations must not be counted in determining the order of a ckt from a transform function.

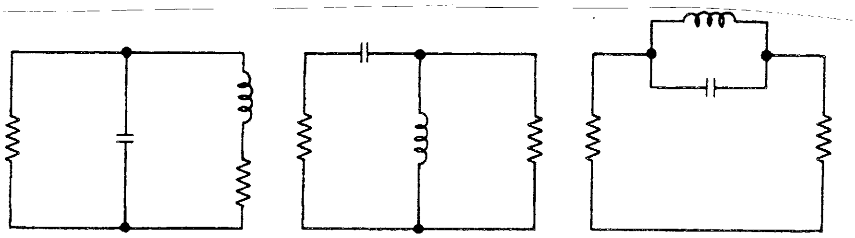


FIGURE 6–27
Some possible forms for second-order circuits.

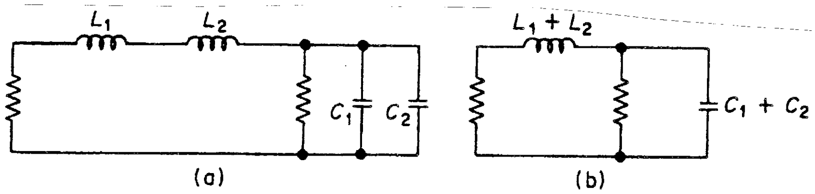


FIGURE 6–28
All illustration of redundancy in a circuit.

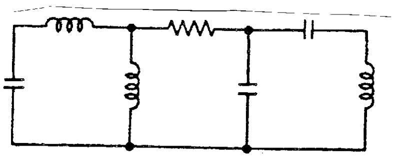


FIGURE 6–29
A circuit without redundancy.

solution from the differential equation

In many cases, the ckt analyst will have occasion to solve a given differential eqn by transform methods. The Laplace transform approach is best suited to ordinary differential eqn of the constant-coefficient type. Such an eqn of order "m" appears in the form

$$b_m \frac{d^m y}{dt^m} + b_{m-1}\frac{d^{m-1}}{dt^{m-1}}y + \dots + b_0 y = f(t)$$

The transform of the 1st derivative is $\mathcal{L}\left[\frac{dy}{dt}\right] = sY(s) - y(0)$

" 2nd " $\mathcal{L}\left[\frac{d^2y}{dt^2}\right] = s\mathcal{L}\left[\frac{dy}{dt}\right] - y'(0)$

$$= s^2Y(s) - sy(0) - y'(0)$$

" 3rd " $\mathcal{L}\left[\frac{d^3y}{dt^3}\right] = s\mathcal{L}\left[\frac{d^2y}{dt^2}\right] - y''(0)$

$$= s^3Y(s) - s^2y(0) - sy'(0) - y''(0)$$

In general, $\mathcal{L}\left[\frac{d^ky}{dt^k}\right] = s^kY(s) - s^{k-1}y(0) - s^{k-2}y'(0) - \dots - y^{k-1}(0)$

Example : The response of a given physical system is described for $t > 0$ by the differential eqn : $4\frac{d^2y}{dt^2} + 24\frac{dy}{dt} + 32y = 100$. The initial values of y and $\frac{dy}{dt}$ are $y(0) = 10$ and $y'(0) = -20$. Solve for $y(t)$ for $t > 0$ using Laplace transform

soln Use the Laplace transform to transform the above eqn:

$$4\left[s^2Y(s) - s(10) - (-20)\right] + 24\left[sY(s) - (10)\right] + 32Y(s) = \frac{100}{s}$$

$$Y(s)\left[s^2+6s+8\right] = \frac{25}{s} + 10s + 40$$

or $$Y(s) = \frac{25}{s(s^2+6s+8)} + \frac{10s+40}{s^2+6s+8}$$

$$\mathcal{L}^{-1}[Y(s)] = y(t) = \frac{5}{8}\left[5 + 6e^{-2t} + 5e^{-4t}\right)$$

z - transform

The z-transform is an operational function that may be applied to discrete-time systems in the same manner as the Laplace transform is applied to continuous-time systems.

We'll develop this concept through the use of the one-sided z-transform, which is most conveniently related to the concepts of continuous-time systems as discussed earlier.

The z-transform of a discrete-time signal $x(n)$ is denoted by $X(z)$

$$\text{i.e.} \quad X(z) = Z[x(n)]$$

$$\text{and} \quad x(n) = Z^{-1}[X(z)]$$

The actual definition of one-sided z-transform is

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

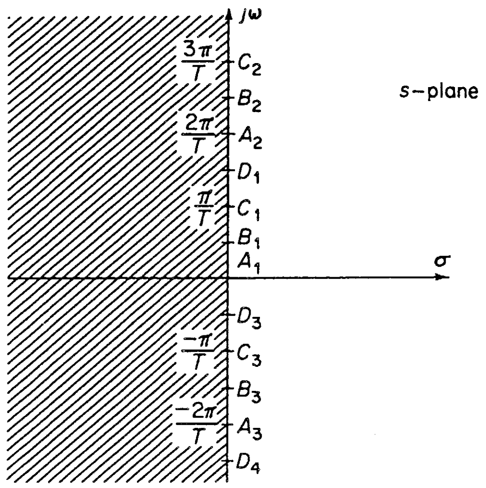
The function $X(z)$ is an infinite series, but it can often be expressed in closed form.

We note that $X(z) = \left[X^*(s) \right]_{z=e^{sT}}$

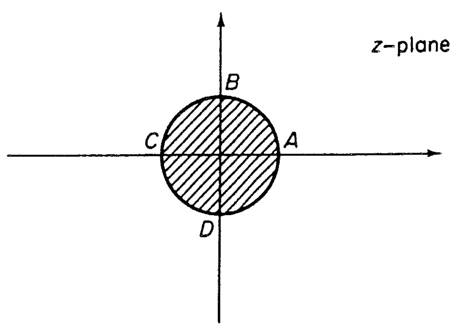
when $X^*(s) = \sum_{n=0}^{\infty} x(nT) e^{-nTs}$ $\quad$ [and $x^*(t) = \sum_{n=0}^{\infty} x(nT)\,\delta(t-nT)$]

The s and z variables are related by $z = e^{sT}$

and $s = \frac{1}{T} \ln z$



$\therefore \quad |z| = |e^{sT}| = 1 \rightsquigarrow$ unit circle

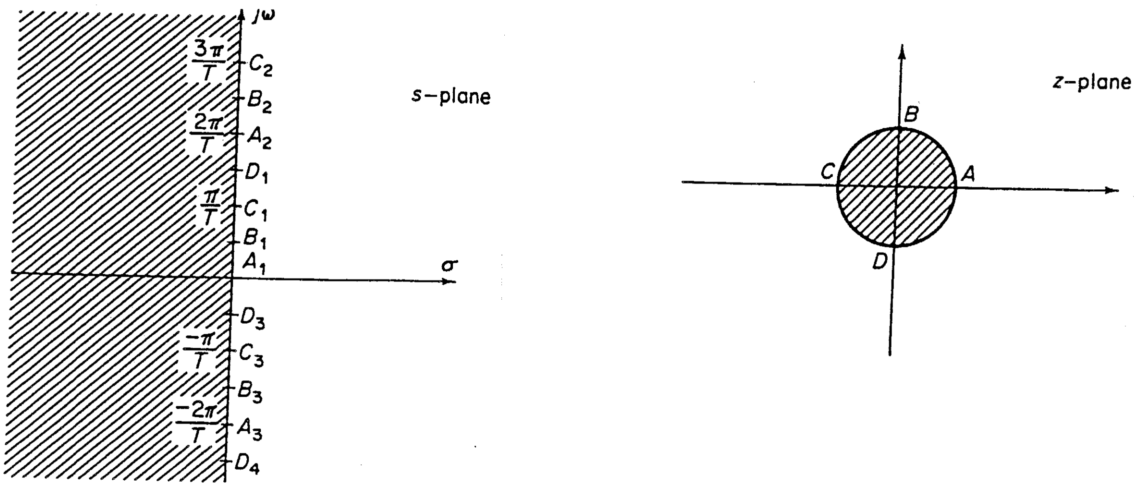


The left half of the s-plane maps to the interior of the unit circle in the z-plane, and the right half of the s-plane maps to the exterior of the unit circle in the z-plane

The $j\omega$ axis in the s-plane maps to the boundary of the unit circle in the z-plane.

$$\text{Let} \quad z = e^{j\omega T}$$

As the cyclic freq varies over the range $-\frac{1}{2T} \leq f \leq \frac{1}{2T}$, the argument of the above eqn varies from $-\pi$ to π. This is equivalent to a complete rotation around the unit circle in the z-plane.



<u>Example</u> pg 442 Derive the z-transform of the discrete unit step func

$$u[n] = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$\text{Definition: } \quad X(z) = \sum_{0}^{\infty} x(n)\, z^{-n} = \sum_{0}^{\infty} (1)\, z^{-n}$$

$$= \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

Inverse z-transform

Partial Fraction Expansion

$$Y(z) = \frac{c_0 z^{l} + c_1 z^{l-1} + \dots + c_l}{z^{l} + d_1 z^{l-1} + \dots + d_l}$$

In case where all the poles are of simple order,

$$\frac{Y(z)}{z} = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \dots + \frac{A_l}{z-p_l}$$

where

$$A_m = (z-p_m)\frac{Y(z)}{z}\Bigg|_{z=p_m}$$

Ex By partial Fraction Expansion, obtain the inverse z-transform of

$$Y(z) = \frac{1}{(1-z^{-1})(1-0.5z^{-1})}$$ pg 447

rewrite $$Y(z) = \frac{z^2}{(z-1)(z-0.5)}$$

$$\frac{Y(z)}{z} = \frac{z}{(z-1)(z-0.5)} = \frac{A_1}{z-1} + \frac{A_2}{z-0.5}$$

$$= \frac{2}{z-1} + \frac{-1}{z-0.5}$$

multiply both sides by z, then inverse

$$y(n) = 2-(0.5)^n$$