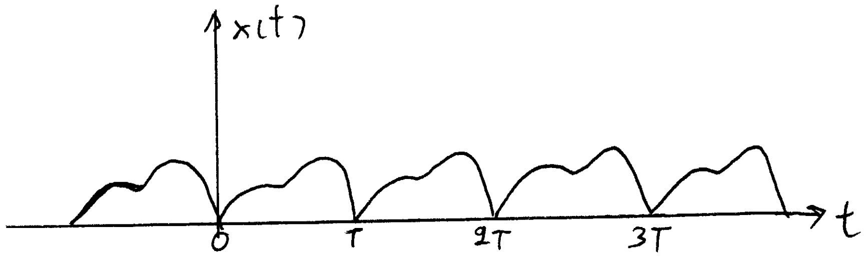
ENE208

# Fourier Series

- Fourier Series is named after a French mathematical physicist by the name "Jean Baptiste Joseph, Baron de Fourier (March 21, 1768 - May 16, 1830)".
- He stated that "no matter how complicated it is, a wave that is periodic, with a pattern that repeats itself, consists of the sum of many simple waves."
- The concept of the Fourier series is based on representing periodic signal as a sum of harmonically related sinusoidal functions. With the aid of Fourier series, complex periodic waveforms can be represented in terms of sinusoidal functions, whose properties are familiar to us.
- Periodic function

$$y = f(x) = f(x+T)$$




This signal maybe represented by the sum of a series of sine and/or cosine functions plus a dc term. The resulting series is called a Fourier series.

The lowest freq of the sinusoidal components is a freq $f$, given by

$$\boxed{f = \frac{1}{T}}$$

This freq is referred to as the fundamental component.

All other freq in the signal will be integer multiples of the fundamental.

These various components are referred to as harmonics, with the order of a given harmonic indicated by the ratio of its freq to the fundamental freq

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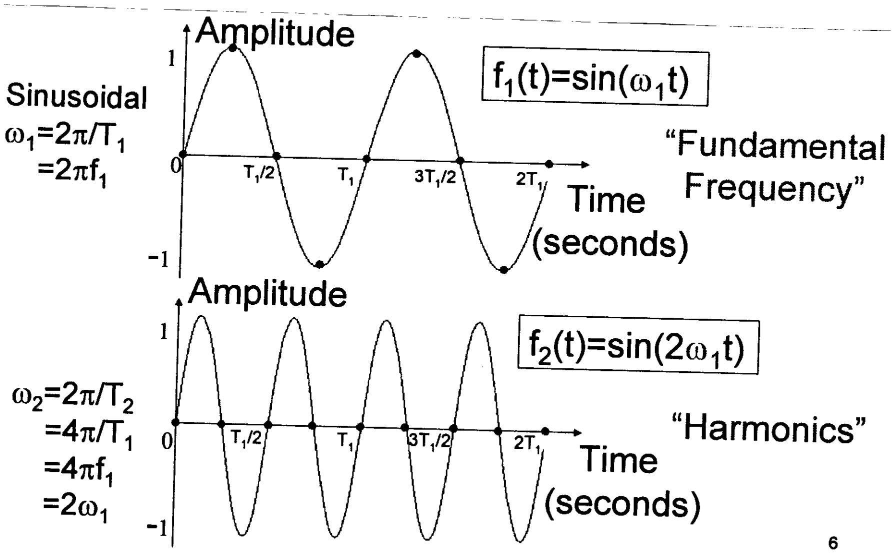
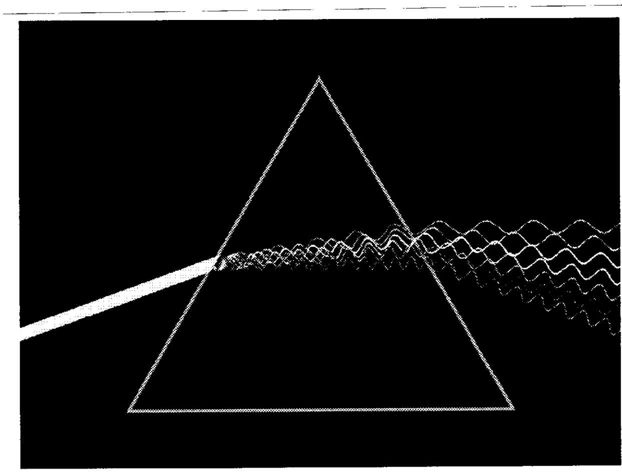
Example




note: angular velocity : $\omega = \frac{\theta}{t}$ [rad/s]

$$\text{rad} = \frac{\pi}{180} \times \text{degree}$$

Example Prism = Fourier transform



Any transmission system through which a given signal passes must have a bandwidth sufficiently large to pass all significant freq of the signal.
In a purely mathematical sense, many common waveforms theoretically contain an infinite number of harmonics. So, must we need an infinite bandwidth to process such signals?

So, the goal is to write this known signal $f(t)$ (or $x(t)$) in time domain as

$$f(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

$$= a_0 + a_1 \cos(\omega t) + b_1 \sin(\omega t)$$

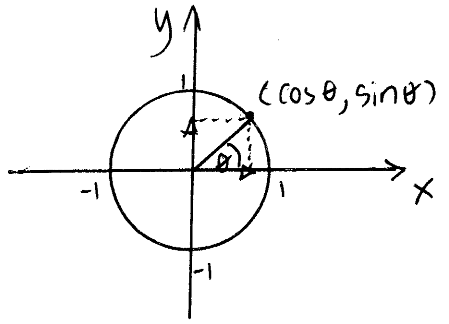
$$+ a_2 \cos(2\omega t) + b_2 \sin(2\omega t) + \dots$$

$$+ \dots + a_n \cos(n\omega t) + b_n \sin(n\omega t)$$

where $a_n, b_n$ are all constants

Fourier coefficients

So why sine and cosine? Because they are "sinusoidal basis functions" in Euclidean space ($\mathbb{R}^2$).




their inner product $\overrightarrow{\cos\theta} \cdot \overrightarrow{\sin\theta} = 0$

They are orthogonal to each other

So, if we know $a_0, a_1, \dots a_n, b_1, \dots b_n$, we should be able to write our $f(t)$ as an infinite sum of sine & cosine functions.

So, how do we find them?

Use these formulas:

$$a_0 = \frac{1}{T}\int_0^T x(t)\,dt = \frac{\text{area under curve in one cycle}}{\text{period } T}$$

$$a_n = \frac{2}{T}\int_0^T x(t)\cos(n\omega t)\,dt$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin(n\omega t)\,dt$$

To prove such relations, we first review some important trigonometric formulas

# Trigonometric Formulas

- $\sin(-\theta) = -\sin\theta$
- $\cos(-\theta) = \cos\theta$
- $\sin(\theta_1 \pm \theta_2) = \sin\theta_1\cos\theta_2 \pm \sin\theta_2\cos\theta_1$
- $\cos(\theta_1 \pm \theta_2) = \cos\theta_1\cos\theta_2 \mp \sin\theta_1\sin\theta_2$
- $\sin^2\theta + \cos^2\theta = 1$
- $\tan^2\theta + 1 = \sec^2\theta$
- $1 + \cot^2\theta = \csc^2\theta$
- $\sin 2\theta = 2\sin\theta\cos\theta$
- $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
- $e^{j\theta} = \cos\theta + j\sin\theta$ where $j = \sqrt{-1}$
- $\sin\theta = 1/2j\,(e^{j\theta} - e^{-j\theta})$
- $\cos\theta = 1/2\,(e^{j\theta} + e^{-j\theta})$
- For sinusoidal signal, $\omega = 2\pi/T = 2\pi f$

$\boxed{\text{Finding } a_0}$ since $f(t) = a_0 + \sum_{n=1}^{\infty}[a_n\cos(n\omega t) + b_n\sin(n\omega t)]$

Integrate both sides

$$\int_0^T f(t)\,dt = \int_0^T a_0\,dt + \sum_{n=1}^{\infty}\left[\int_0^T a_n\cos(n\omega t)\,dt + \int_0^T b_n\sin(n\omega t)\,dt\right]$$

$$= a_0\int_0^T dt + \sum_{n=1}^{\infty}\left[a_n\int_0^T\cos(n\omega t)\,dt + b_n\int_0^T\sin(n\omega t)\,dt\right]$$

$$= a_0 T + \sum_{n=1}^{\infty}[a_n \times 0 + b_n \times 0]$$

$$= a_0 T$$

$$\therefore \quad a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

$\boxed{\text{Finding } a_n}$ $f(t) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$

- multiply both sides by $\cos(\omega t)$

$\therefore f(t)\cos(\omega t) = a_0\cos(\omega t) + a_1\cos^2(\omega t) + b_1\sin(\omega t)\cos(\omega t) + a_2\cos(2\omega t)\cos(\omega t) + b_2\sin(2\omega t)\cos(\omega t) + \ldots$

- integrate both sides

$$\therefore \int_0^T f(t)\cos(\omega t)dt = a_0\int_0^T \cos(\omega t)dt + a_1\int_0^T \cos^2(\omega t)dt + b_1\int_0^T \sin(\omega t)\cos(\omega t)dt + a_2\int_0^T \cos(2\omega t)\cos(\omega t)dt + b_2\int_0^T \sin(2\omega t)\cos(\omega t)dt + \ldots$$

Let's look at each of these terms on the right

① $a_0\int_0^T \cos(\omega t)\,dt = 0$

② $a_1\int_0^T \cos^2(\omega t)dt = a_1\int_0^T \frac{(1+\cos(2\omega t))}{2}dt = \frac{a_1}{2}\left[\int_0^T dt + \int_0^T \cos(2\omega t)dt\right]$
$= \frac{a_1}{2}\left[T + 0\right] = \frac{a_1 T}{2}$

③ $b_1\int_0^T \sin(\omega t)\cos(\omega t)dt = b_1 \times \frac{1}{\omega}\int_0^T \sin(\omega t)\,d(\sin\omega t)$
$= \frac{b_1}{2\omega}\sin^2(\omega t)\Big|_0^T = 0$

④ $a_2\int_0^T \cos(2\omega t)\cos(\omega t)dt = \frac{a_2}{\omega}\int_0^T [1 - 2\sin^2(\omega t)]\,d(\sin\omega t)$
$= \frac{a_2}{\omega}\left[\int_0^T d(\sin(\omega t)) - 2\int_0^T \sin^2(\omega t)d(\sin(\omega t))\right]$
$= \frac{a_2}{\omega}\left[\sin(\omega t)\Big|_0^T - \frac{2}{3}\sin^3(\omega t)\Big|_0^T\right]$
$= \frac{a_2}{\omega}(0) = 0$

⑤ $b_2\int_0^T \sin(2\omega t)\cos(\omega t)dt = b_2 \times 2\int_0^T \sin(\omega t)\cos(\omega t)\cos(\omega t)dt$
$= 2b_2\int_0^T \sin(\omega t)\cos^2(\omega t)dt = -\frac{2b_2}{\omega}\int_0^T \cos^2(\omega t)d(\cos(\omega t))$
$= -\frac{2b_2}{3\omega}\cos^3(\omega t)\Big|_0^T = 0$

$$\therefore \int_0^T \cos(3\omega t)\cos(\omega t)\,dt = 0$$

$$\int_0^T \sin(3\omega t)\cos(\omega t)\,dt = 0$$

$$\dots$$

Therefore,

$$\int_0^T f(t)\cos(\omega t)\,dt = a_0(0) + a_1\left(\frac{T}{2}\right) + b_1(0) + a_2(0) + b_2(0) + \dots$$

$$a_1 = \frac{2}{T}\int_0^T f(t)\cos(\omega t)\,dt$$

From this result, we conclude : $\boxed{a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)\,dt}$

**Finding $b_n$**

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega t) + b_n\sin(n\omega t)\right]$$

multiply both sides with $\sin(\omega t)$ and integrate :

$$f(t)\sin(\omega t) = a_0\sin(\omega t) + a_1\cos(\omega t)\sin(\omega t) + b_1\sin^2(\omega t) + a_2\cos(2\omega t)\sin(\omega t) + b_2\sin(2\omega t)\sin(\omega t) + \dots$$

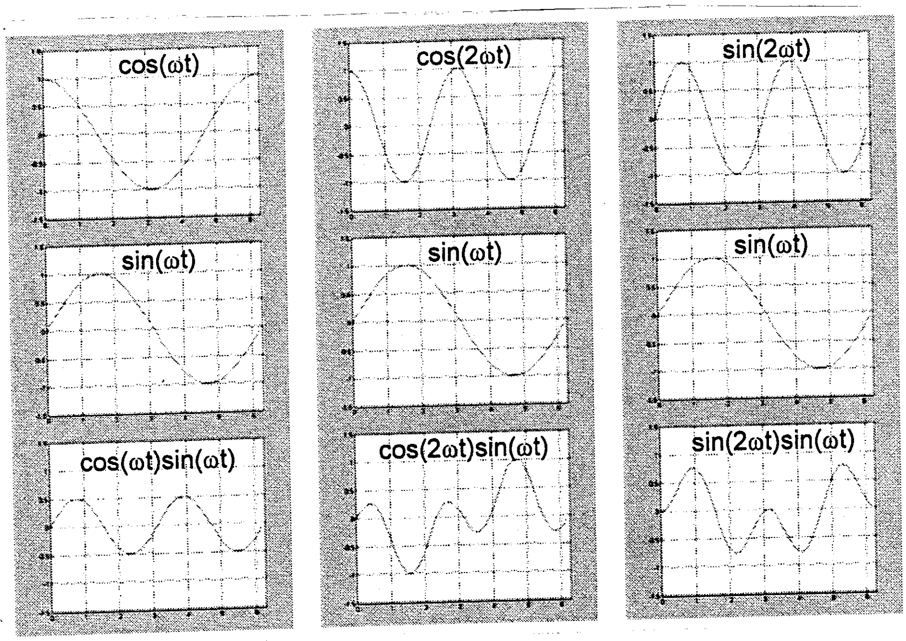
$$\int_0^T f(t)\sin(\omega t)\,dt = a_0\int_0^T \sin(\omega t)\,dt + a_1\int_0^T \cos(\omega t)\sin(\omega t)\,dt + b_1\int_0^T \sin^2(\omega t)\,dt + a_2\int_0^T \cos(2\omega t)\sin(\omega t)\,dt + b_2\int_0^T \sin(2\omega t)\sin(\omega t)\,dt + \dots$$

$$= 0 + 0 + \frac{b_1}{2}\int_0^T 1 - \cos(2\omega t)\,dt + 0 + 0 = \frac{b_1}{2}\left[\int_0^T dt - \int_0^T \cos(2\omega t)\,dt\right]$$

$$= \frac{b_1 T}{2}$$

$$\therefore \quad b_1 = \frac{2}{T}\int_0^T f(t)\sin(\omega t)\,dt$$

we can conclude that $\boxed{b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)\,dt}$




Some good points to note:
For m and n are integers:

$$\int_0^T \sin(m\omega t)\sin(n\omega t)\,dt = \begin{cases} 0 & \text{for } m \neq n \\ T/2 & \text{for } m = n \end{cases}$$

$$\int_0^T \cos(m\omega t)\cos(n\omega t)\,dt = \begin{cases} 0 & \text{for } m \neq n \\ \frac{T}{2} & \text{for } m = n \end{cases}$$

$$\int_0^T \sin(m\omega t)\cos(n\omega t)\,dt = 0$$

$\boxed{\text{proof}}$

$$\int_0^T \sin(m\omega t)\sin(n\omega t)\,dt = \frac{1}{2}\int_0^T \cos((m-n)\omega t) - \cos((m+n)\omega t)\,dt$$

( ∵ $\cos(\theta_1 - \theta_2) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2$
$\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$ )

$$\frac{1}{2}\int_0^T \cos((m-n)\omega t) - \cos((m+n)\omega t)\,dt = \frac{1}{2\omega(m-n)} \times \sin(m-n)\omega t\Big|_0^T - \frac{1}{2\omega(m+n)} \times \cos(m+n)\omega t\Big|_0^T$$

$$= \frac{1}{2\omega}\left[\frac{\sin(m-n)\omega T}{(m-n)} - \frac{\sin(m+n)\omega T}{(m+n}\right]$$

$$= \frac{1}{2\omega}\left[\frac{\sin(2\pi(m-n))}{(m-n)} - \frac{\sin(2\pi(m+n))}{(m+n)}\right]$$

① For $m \neq n$; the whole thing is 0 ∵ $\sin 2\pi k = 0$ ($k$ = integer)

② For $m = n$; the term $\frac{\sin 2\pi(m-n)}{(m-n)} = \frac{0}{0}$ → we use L'Hôpital's rule

Let $m-n = x$; $\frac{\sin 2\pi x}{x} \rightarrow \frac{d(\sin 2\pi x)/dx}{dx/dx} = \frac{(\cos 2\pi x) \times 2\pi}{1} = 2\pi$

$\therefore \frac{1}{2\omega} \times [2\pi - 0] = \frac{\pi}{\omega} = \frac{\pi}{2\pi f} = \frac{1}{2f} = \frac{T}{2}$

## Conclusions

→ Decomposition : we know $f(t)$ and need to find Fourier coefficients $a_n, b_n$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(2\pi nt/T)\,dt = \frac{2}{T}\sum_{t=0}^{T} f(t)\cos(2\pi nt/T)\,\Delta t$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(2\pi nt/T)\,dt = \frac{2}{T}\sum_{t=0}^{T} f(t)\sin(2\pi nt/T)\,\Delta t$$

→ Reconstruction: we know the Fourier Coefficients $a_n$, and $b_n$ and find $f(t)$

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(2\pi nt/T) + b_n\sin(2\pi nt/T)\right]$$

Let's do some examples:

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There are 3 forms of Fourier Series:

- sine cosine form
- amplitude phase form
- complex exponential form

① Sine-Cosine form: → we just studied a few minutes ago

② amplitude phase form: $$x(t) = C_0 + \sum_{n=1}^{\infty} C_n\cos(n\omega t + \phi_n)$$

or $$x(t) = C_0 + \sum_{n=1}^{\infty} C_n\sin(n\omega t + \theta_n)$$

where $C_n = \sqrt{a_n^2 + b_n^2}$ , $C_0 = a_0$

$\phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right)$ $\theta_n = \phi_n + 90^\circ$

③ Complex Exponential form:

$$x(t) = \sum_{n=-\infty}^{\infty} \overline{X}_n\, e^{jn\omega t}$$

$\overline{X}_0 = a_0 = C_0$ ; $\overline{X}_n = \frac{a_n - jB_n}{2}$ for $n \neq 0$

$\overline{X}_n = X_n e^{j\phi_n}$

**Example** $x(t) = 18 + 40\cos 2000\pi t - 30\sin 2000\pi t$
$-24\cos 4000\pi t + 10\sin 4000\pi t$

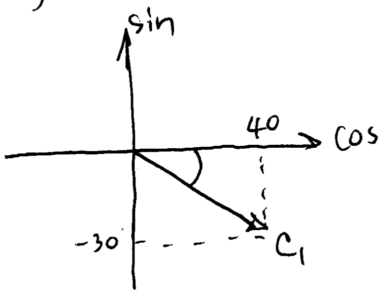
Express the signal in (a) amplitude-phase form
(b) complex exponential form

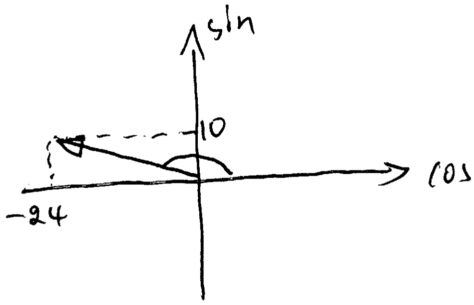
a) $\therefore$ The amplitude phase form should be

$$x(t) = 18 + C_1\cos(2000\pi t + \phi_1) + C_2\cos(4000\pi t + \phi_2)$$

or $$x(t) = 18 + C_1\sin(2000\pi t + \theta_1) + C_2\sin(4000\pi t + \theta_2)$$

using the cosine as basis function



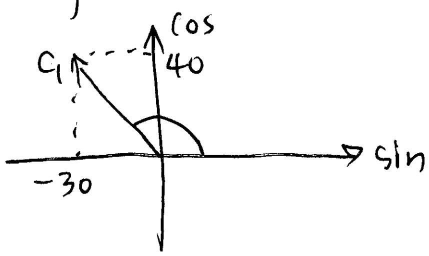



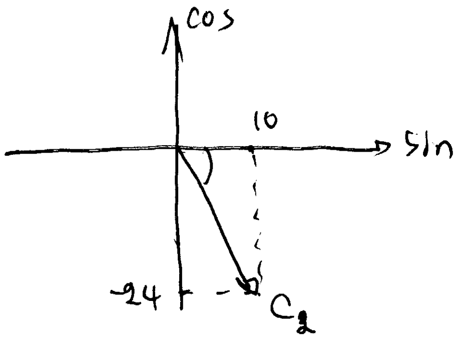

$C_1 = 40 - 30j = 50\angle -36.87^\circ$

$C_2 = -24 + 10j = 26\angle 157.38^\circ$

$\therefore x(t) = 18 + 50\cos(2000\pi t - 36.87^\circ) + 26\cos(4000\pi t + 157.38^\circ)$

Using the sine as basis function




$\bar{C}_1 = -30 + 40j = 50\angle 126.87^\circ$

$\bar{C}_2 = 10 - j24 = 26\angle -67.38^\circ$

$\therefore x(t) = 18 + 50\sin(2000\pi t + 126.87^\circ) + 26\sin(4000\pi t - 67.38^\circ)$

b) Complex Exponential form

$$\bar{X}_1 = \frac{40 - j(-30)}{2} = 20 + j15 = 25\angle 36.87^\circ$$

$$\bar{X}_{-1} = \text{complex conjugate of } \bar{X}_1 = 25\angle -36.87^\circ = 25 - j15$$

$$\bar{X}_2 = \frac{-24 - j10}{2} = -12 - j5 = 13\angle -157.38^\circ$$

$$\bar{X}_{-2} = \text{complex conjugate of } \bar{X}_2 = -12 + j5 = 13\angle 157.38^\circ$$

$$\therefore \; x(t) = 18 + 25e^{j(2000\pi t + 36.87^\circ)} + 25e^{-j(2000\pi t + 36.87^\circ)}$$

$$+ 13e^{j(4000\pi t - 157.38^\circ)} + 13e^{-j(4000\pi t - 157.38^\circ)}$$

## Frequency Spectrum plots:

One of the most useful form for displaying the Fourier Series of a signal is by means of a graphical plot showing the relative strengths of the components as a function of freq. Such a plot is called "the freq spectrum" of a given signal.

There are two different kinds of spectrum plots: one-sided and two-sided

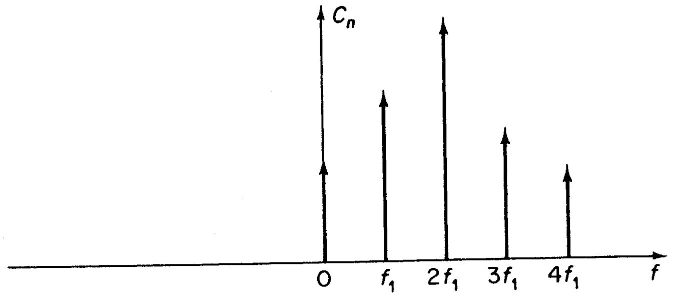



**FIGURE 9–4**
Typical one-sided amplitude-frequency spectrum.

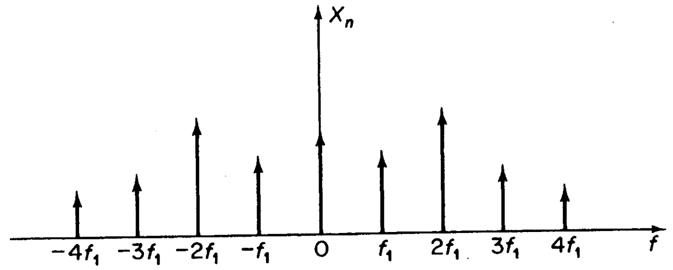
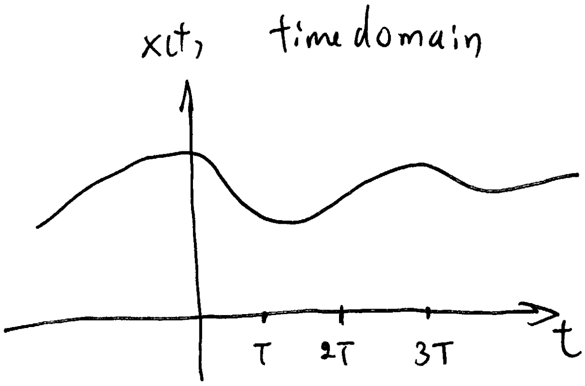
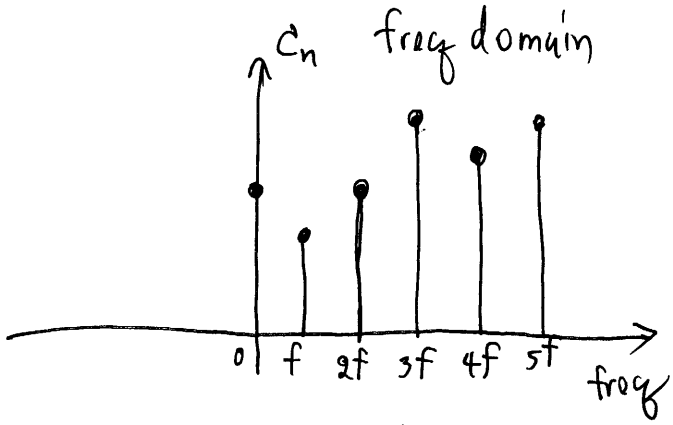



**FIGURE 9–5**
Two-sided amplitude spectrum corresponding to Figure 9–4.




continuous




discrete

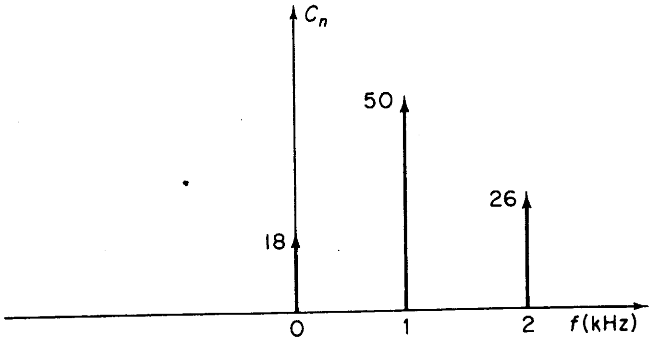
Example From the previous example, plot the spectrum plots

a) single-sided plot:

$$x(t) = 18 + 50\cos(2000\pi t - 36.87^\circ) + 26\cos(4000\pi t + 157.38^\circ)$$

| Frequency (Hz) | 0 | 1000 | 2000 |
|---|---|---|---|
| Amplitude | 18 | 50 | 26 |

The one-sided plot is shown in Figure 9–6(a).

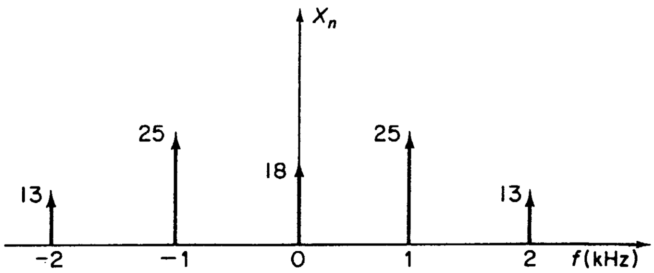



(a)

b) double-sided plot:

$$x(t) = 18 + 25e^{+j(\quad)} + 25e^{-j(\quad)}$$
$$+13e^{+j(\quad)} + 13e^{-j(\quad)}$$

| Frequency (Hz) | 0 | ±1000 | ±2000 |
|---|---|---|---|
| Amplitude | 18 | 25 | 13 |

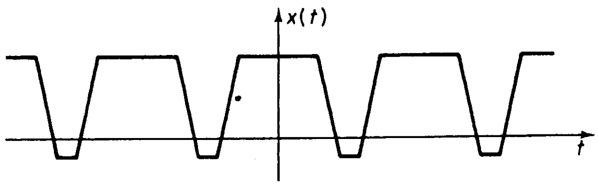



(b)

## Fourier Series Symmetry Conditions

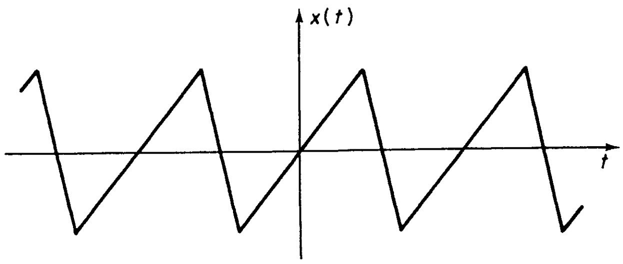
In many cases, there are certain properties that may be used to simplify the computation of spectrum. The types of criteria to be studied in this section are the "symmetry" conditions.

① Even function : $x(-t) = x(t)$



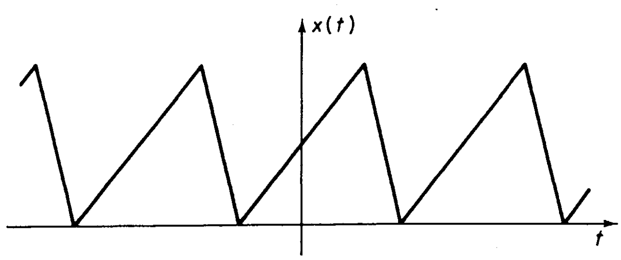

all $b_n = 0$ (sine functions disappear!)

② odd function : $x(-t) = -x(t)$



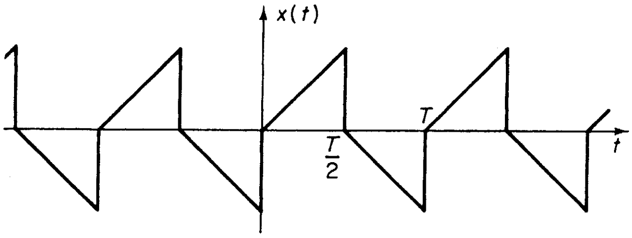

all $a_n = 0$

or it can be disguised by the presence of a dc component.




$a_0 \neq 0$
$a_n = 0$
$b_n \neq 0$

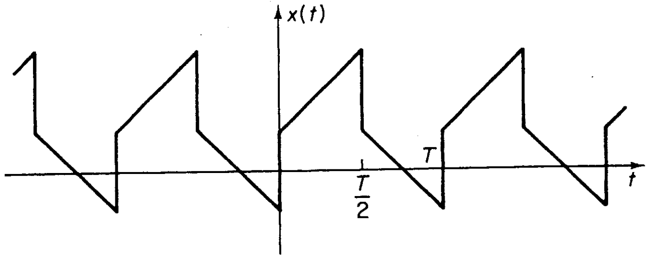
③ Half wave Symmetry: $x(t+\frac{T}{2}) = -x(t)$



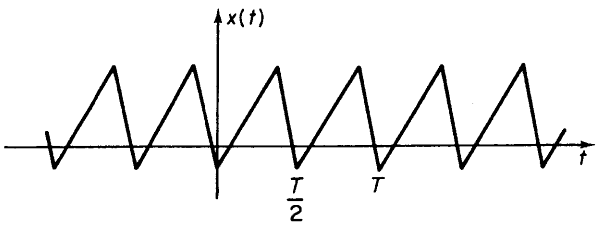

There will only be odd-numbered harmonics present in the series.

$(n = 1, 3, 5, 7, \dots)$

or with a presence of a dc component:




④ Full wave symmetry: $x(t+\frac{T}{2}) = x(t)$




There will only be even-numbered harmonics present in the series

$(n = 0, 2, 4, 6, \dots)$

Table of summary on the symmetry

**TABLE 9–1**
Fourier series symmetry conditions

Sine–cosine form: $x(t) = A_0 + \sum_{n=1}^{\infty} (A_n \cos n\omega_1 t + B_n \sin n\omega_1 t)$, $\quad \omega_1 = 2\pi f_1 = \frac{2\pi}{T}$

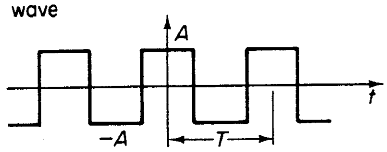
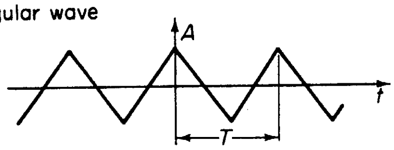
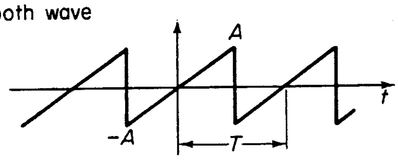
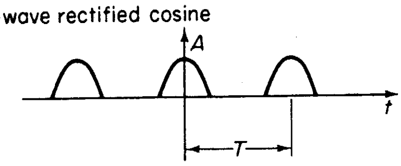
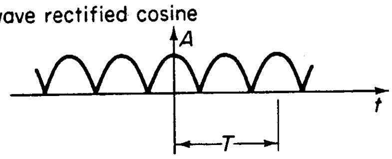
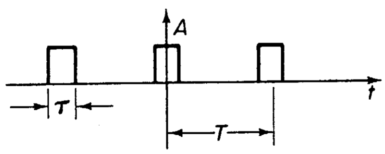
Amplitude-phase form: $x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_1 t + \phi_n) = C_0 + \sum_{n=1}^{\infty} C_n \sin(n\omega_1 t + \theta_n)$, $\quad C_n = \sqrt{A_n^2 + B_n^2}$

Complex exponential form: $x(t) = \sum_{n=-\infty}^{\infty} \overline{X}_n \varepsilon^{jn\omega_1 t}$, $\quad \overline{X}_n = \frac{A_n - jB_n}{2}$, for $n \neq 0$ $\quad \overline{X}_0 = A_0$

| Condition | $A_n$(except $n = 0$) | $B_n$ | $\overline{X}_n$ | Comments |
|---|---|---|---|---|
| General | $\frac{2}{T}\int_0^T x(t) \cos n\omega_1 t\, dt$ | $\frac{2}{T}\int_0^T x(t) \sin n\omega_1 t\, dt$ | $\frac{1}{T}\int_0^T x(t)\varepsilon^{-jn\omega_1 t}\, dt$ | |
| Even function $x(-t) = x(t)$ | $\frac{4}{T}\int_0^{T/2} x(t) \cos n\omega_1 t\, dt$ | 0 | $\frac{2}{T}\int_0^{T/2} x(t) \cos n\omega_1 t\, dt$ | One-sided forms have only cosine terms<br>$\overline{X}_n$ terms are real |
| Odd function $x(-t) = -x(t)$ | 0 | $\frac{4}{T}\int_0^{T/2} x(t) \sin n\omega_1 t\, dt$ | $\frac{-2j}{T}\int_0^{T/2} x(t) \sin n\omega_1 t\, dt$ | One-sided forms have only sine terms<br>$\overline{X}_n$ terms are imaginary |
| Half-wave symmetry $x\left(t + \frac{T}{2}\right) = -x(t)$ | $\frac{4}{T}\int_0^{T/2} x(t) \cos n\omega_1 t\, dt$ | $\frac{4}{T}\int_0^{T/2} x(t) \sin n\omega_1 t\, dt$ | $\frac{2}{T}\int_0^{T/2} x(t)\varepsilon^{-jn\omega_1 t}\, dt$ | Odd-numbered harmonics only |
| Full-wave symmetry $x\left(t + \frac{T}{2}\right) = x(t)$ | $\frac{4}{T}\int_0^{T/2} x(t) \cos n\omega_1 t\, dt$ | $\frac{4}{T}\int_0^{T/2} x(t) \sin n\omega_1 t\, dt$ | $\frac{2}{T}\int_0^{T/2} x(t)\varepsilon^{-jn\omega_1 t}\, dt$ | Even-numbered harmonics only |

Some common periodic wave forms and their Fourier Series

**TABLE 9–2**
Some common periodic signals and their Fourier series

| Signal $x(t)$ | Fourier series |
|---|---|
| Square wave   | $\frac{4A}{\pi}(\cos\omega_1 t - \frac{1}{3}\cos 3\omega_1 t + \frac{1}{5}\cos 5\omega_1 t - \frac{1}{7}\cos 7\omega_1 t + \cdots)$ |
| Triangular wave   | $\frac{8A}{\pi^2}(\cos\omega_1 t + \frac{1}{9}\cos 3\omega_1 t + \frac{1}{25}\cos 5\omega_1 t + \cdots)$ |
| Sawtooth wave   | $\frac{2A}{\pi}(\sin\omega_1 t - \frac{1}{2}\sin 2\omega_1 t + \frac{1}{3}\sin 3\omega_1 t - \frac{1}{4}\sin 4\omega_1 t + \cdots)$ |
| Half–wave rectified cosine   | $\frac{A}{\pi}(1 + \frac{\pi}{2}\cos\omega_1 t + \frac{2}{3}\cos 2\omega_1 t - \frac{2}{15}\cos 4\omega_1 t + \frac{2}{35}\cos 6\omega_1 t - \cdots (-1)^{n/2+1}\frac{2}{n^2-1}\cos n\omega t + \cdots)$ $n$ even |
| Full–wave rectified cosine   | $\frac{2A}{\pi}(1 + \frac{2}{3}\cos 2\omega_1 t - \frac{2}{15}\cos 4\omega_1 t + \frac{2}{35}\cos 6\omega_1 t - \cdots (-1)^{n/2+1}\frac{2}{n^2-1}\cos n\omega_1 t + \cdots)$ $n$ even |
|   | $Ad\left[1 + 2\left(\frac{\sin\pi d}{\pi d}\cos\omega_1 t + \frac{\sin 2\pi d}{2\pi d}\cos 2\omega_1 t + \frac{\sin 3\pi d}{3\pi d}\cos 3\omega_1 t + \cdots\right)\right]$ $d = \tau/T$ |