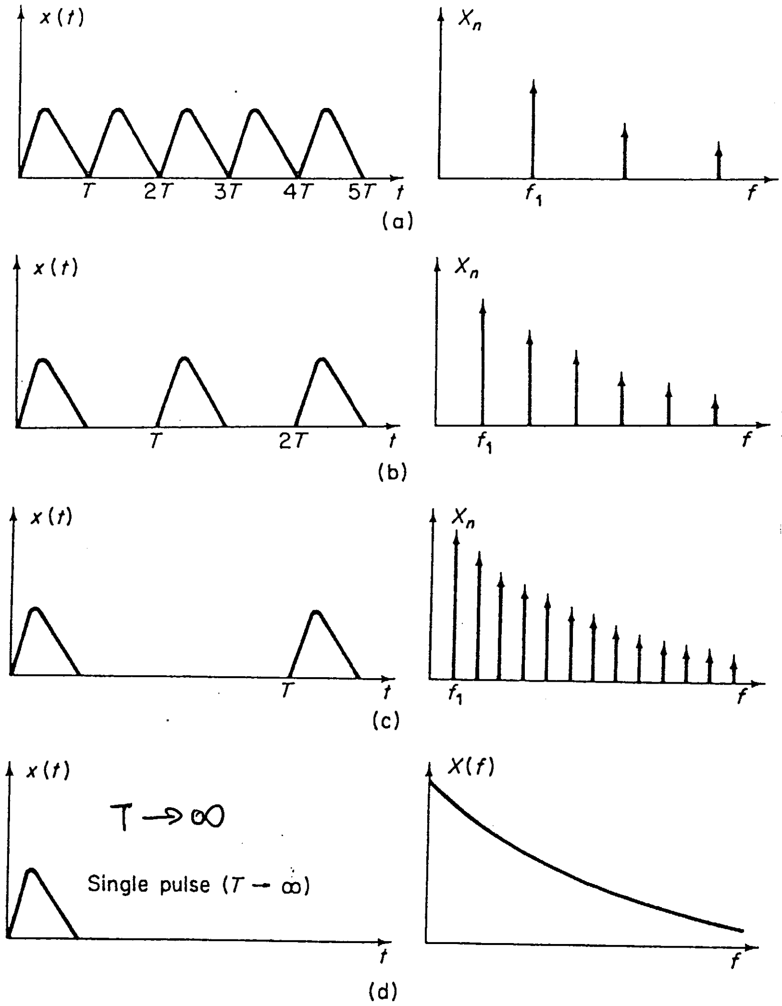
$\boxed{\text{ENE 208}}$ Fourier Transform

The emphasis on spectral analysis so far has focused on periodic functions, whose spectra consist of discrete components at integer multiples of the fundamental repetition freq.

Next we'll consider the process of spectral analysis for $\boxed{\text{nonperiodic signals,}}$ which is accomplished by the Fourier transform. To satisfy certain mathematical restrictions, assume that all nonperiodic signals of interest have finite energy.



For example: some arbitrary, periodic, pulse-type signal $x(t)$ and its assumed amplitude spectrum X_n are shown in (a).
The time function is periodic, with period T, and the spectrum is discrete. The fundamental component is $f_1 = \frac{1}{T}$, and spectral components appear at integer multiples of that frequency.

In Fig. b), the pulse width and shape remain the same, but the period is doubled by inserting a space between successive pulses. An expression for X_n would be the same as before, since the integrand hasn't changed.

What has changed, however, is the fundamental freq f_1. When T is doubled, f_1 is halved, so the spacing between spectra lines is halved as shown.

The effect of this trend is that the relative shape of the envelope of the $\bar{X}_n$ coefficients remain the same, but the number of components in a given freq interval increases as the period increases. In the limit as $T \to \infty$, the freq difference approaches zero.

In this limiting form, the spectral lines all merge together and form a "continuous curve". The spectrum could appear at any frequency.

The Fourier transform is the commonly used name for the mathematical function that provides the freq spectrum of a nonperiodic signal.

The process of Fourier transformation of a time function is designated symbolically as

$$\bar{X}(f) = \mathcal{F}[x(t)]$$

The inverse operation is designated symbolically as

$$x(t) = \mathcal{F}^{-1}[\bar{X}(f)]$$

The actual mathematical processes involved in these operations are:

$$\boxed{\begin{aligned} \bar{X}(f) &= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \\ x(t) &= \int_{-\infty}^{\infty} \bar{X}(f)\, e^{+j\omega t}\, df \end{aligned}}$$

The Fourier transform $\bar{X}(f)$ is, in general, a complex function and has both magnitude and an angle. Thus, $\bar{X}(f)$ can be expressed as

$$\bar{X}(f) = X(f)\, e^{j\phi(f)} = X(f) \angle \phi(f)$$

where $X(f)$ represents amplitude spectrum and $\phi(f)$ is the phase spectrum.

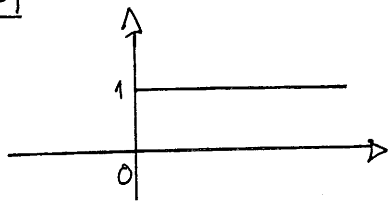
Fourier transform symmetry conditions

Condition	$\bar{X}(f)$	Comment
General	$\int_{-\infty}^{\infty} x(t)\varepsilon^{-j\omega t}\, dt$	
Even function $x(-t) = x(t)$	$2\int_0^{\infty} x(t)\cos \omega t\, dt$	$\bar{X}(f)$ is an even, real function of f
Odd function $x(-t) = -x(t)$	$-2j\int_0^{\infty} x(t)\sin \omega t\, dt$	$\bar{X}(f)$ is an odd, imaginary function of f

Some Important nonperiodic waveforms

① **A unit step function $u(t)$**

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

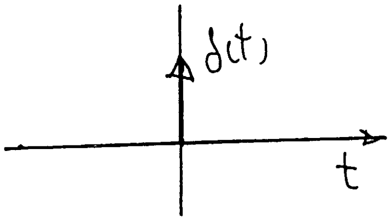


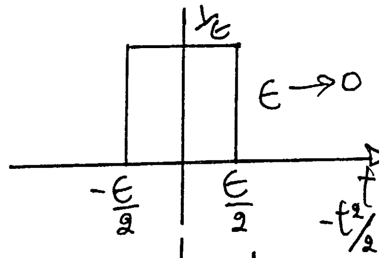
$$\frac{du(t)}{dt} = \delta(t)$$

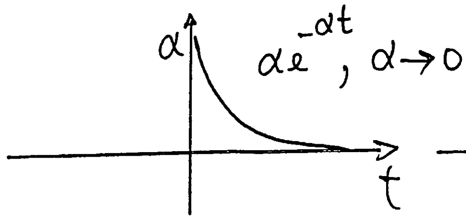
② **A unit impulse function $\delta(t)$**

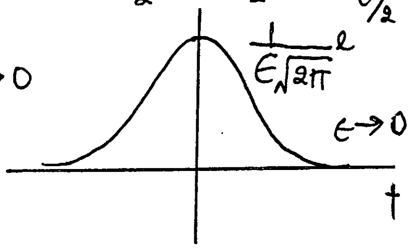
$$\delta(t) = 0, \ t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$









multiplication:

$$\phi(t)\,\delta(t) = \phi(0)\,\delta(t)$$

$$\phi(t)\,\delta(t-T) = \phi(T)\,\delta(t-T)$$

$$\int_{-\infty}^{t} \delta(\tau)\,d\tau = u(t)$$

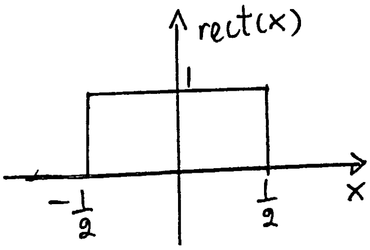
Sampling property of unit impulse function:

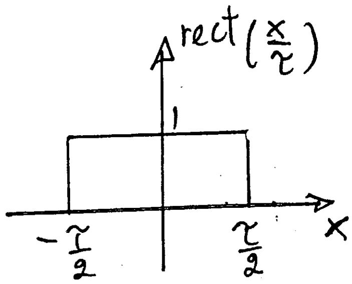
$$\int_{-\infty}^{\infty} \phi(t)\,\delta(t)\,dt = \phi(0)\int_{-\infty}^{\infty} \delta(t)\,dt = \phi(0), \text{ provided } \phi(t) \text{ is continuous at } t=0$$

$$\int_{-\infty}^{\infty} \phi(t)\,\delta(t-T)\,dt = \phi(T)$$

③ **A unit gate function (rect(x))**

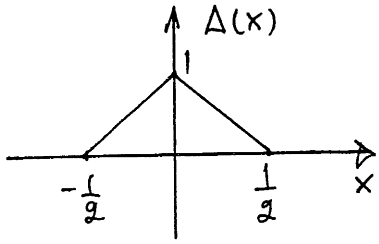
$$\mathrm{rect}(x) = \begin{cases} 0 & |x| > \frac{1}{2} \\ \frac{1}{2} & |x| = \frac{1}{2} \\ 1 & |x| < \frac{1}{2} \end{cases}$$

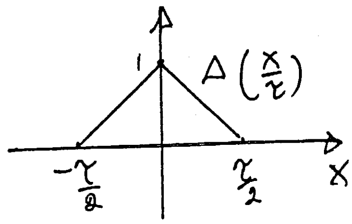




④ **A unit triangle function ($\Delta(x)$)**

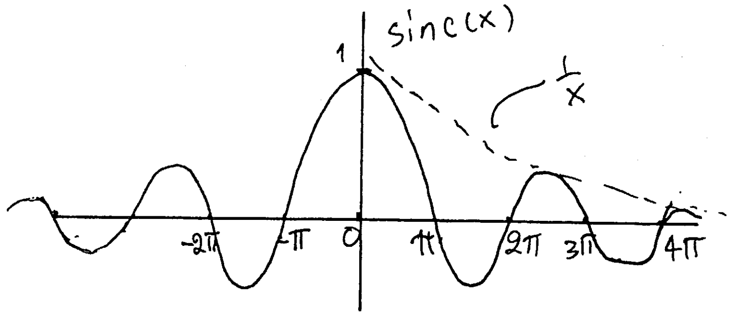
$$\Delta(x) = \begin{cases} 0 & |x| \geq \frac{1}{2} \\ 1-2|x| & |x| < \frac{1}{2} \end{cases}$$





(5) An interpolation function, or filtering function

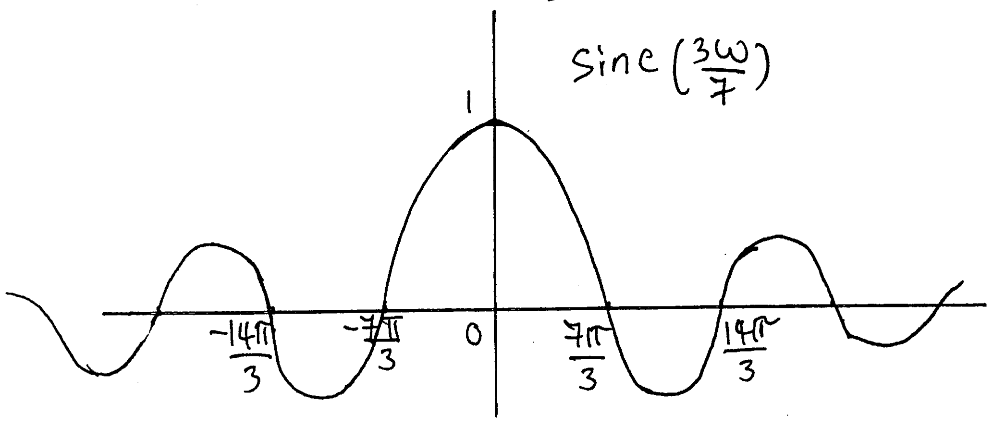
$$\text{sinc}(x) = \frac{\sin(x)}{x}$$



1. even function of x
2. $\text{sinc}(x) = 0$ when $\sin(x) = 0$ except at $x = 0$
   $\sin x = 0$ for $x = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$
3. Using L'Hopital's rule, we find $\text{sinc}(0) = 1$

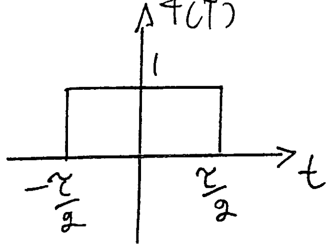
$$\lim_{x \to 0} \frac{\sin x}{x} = \frac{\cos x}{1} = \frac{\cos 0}{1} = \frac{1}{1} = 1$$

4. sinc(x) is the product of an oscillating signal $\sin x$ (of period 2π) and a monotonically decreasing function $\frac{1}{x}$. Therefore sinc(x) exhibits sinusoidal oscillations of period 2π, with amplitude decreasing continuously as $\frac{1}{x}$



Examples

① Find Fourier transform of $f(t) = \mathrm{rect}\left(\frac{t}{\tau}\right)$



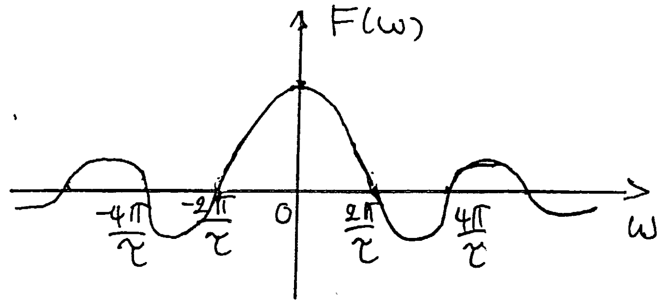
$$F(\omega) = \int_{-\infty}^{\infty} \mathrm{rect}\left(\frac{t}{\tau}\right) e^{-j\omega t}\,dt$$

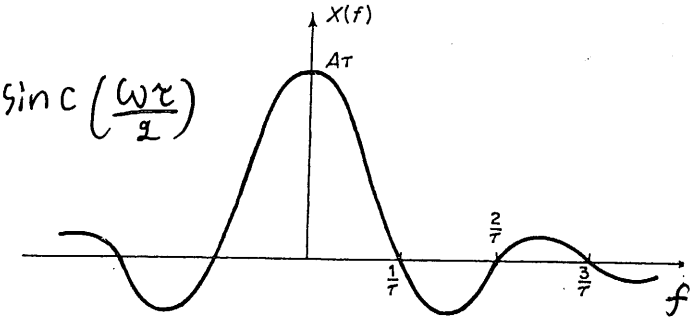
$$= \int_{-\frac{\tau}{2}}^{\tau/2} e^{-j\omega t}\,dt$$

$$= -\frac{1}{j\omega}\left(e^{-j\omega\tau/2} - e^{j\omega\tau/2}\right)$$

$$= \frac{2\sin(\omega\tau/2)}{\omega}$$

$$= \frac{\tau\sin(\omega\tau/2)}{(\omega\tau/2)} = \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2}\right)$$





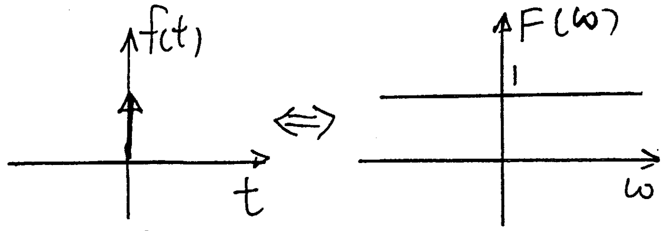
$$\mathrm{rect}\left(\frac{t}{\tau}\right) \longleftrightarrow \tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2}\right)$$

$$\mathrm{sinc}\left(\frac{\omega\tau}{2}\right) = 0 \text{ when } \left(\frac{\omega\tau}{2}\right) = \pm n\pi$$

② $$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\,dt$$

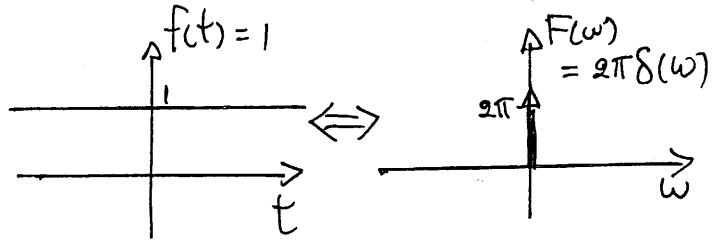
$$= 1$$

$$\therefore \quad \delta(t) \longleftrightarrow 1$$



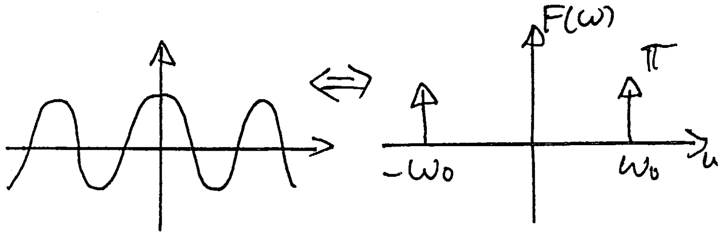
$$\mathcal{F}^{-1}[\delta(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\, e^{j\omega t}\,d\omega$$

$$= \frac{1}{2\pi}$$



$$\therefore \quad \frac{1}{2\pi} \longleftrightarrow \delta(\omega)$$

$$1 \longleftrightarrow 2\pi\,\delta(\omega)$$



③ $$\cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

$$\Leftrightarrow \pi\left[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right]$$

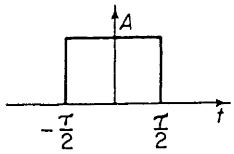
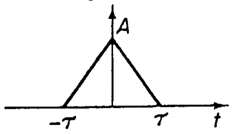
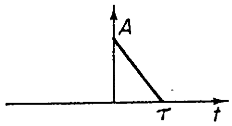
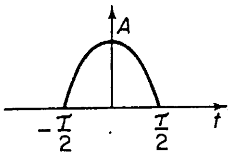
④ $$\mathcal{F}^{-1}[\delta(\omega-\omega_0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_0)\, e^{j\omega t}\,d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$$

$\frac{1}{2\pi} e^{j\omega_0 t} \Leftrightarrow \delta(\omega-\omega_0)$ and so $\boxed{e^{j\omega_0 t} \Leftrightarrow 2\pi\,\delta(\omega-\omega_0)}$

and $\boxed{e^{-j\omega_0 t} \Leftrightarrow 2\pi\,\delta(\omega+\omega_0)}$

⑤ $$\mathcal{F}[\sin\omega_0 t] \rightarrow \mathcal{F}\left[\frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)\right] = j\pi\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right]$$

Some common signals and their Fourier transforms

Signal $x(t)$	Spectrum $\overline{X}(f)$
Rectangular pulse 	$A\tau \dfrac{\sin \pi f\tau}{\pi f\tau}$
Triangular pulse 	$A\tau \left(\dfrac{\sin \pi f\tau}{\pi f\tau}\right)^2$
Sawtooth pulse 	$\dfrac{jA}{2\pi f}\left[\dfrac{\sin \pi f\tau}{\pi f\tau}\,\epsilon^{-j\pi f\tau} - 1\right]$
Cosine pulse 	$\dfrac{2A\tau}{\pi}\,\dfrac{\cos \pi f\tau}{1 - 4f^2\tau^2}$

Time & Frequency Domain

	Time	Frequency
Fourier Series	• Continuous • Periodic	• Discrete • Non-periodic
Continuous Fourier Transform	• Continuous • Non-periodic	• Continuous • Non-Periodic

Fourier Transform Properties

Fourier transform operation pairs

$x(t)$	$\overline{X}(f) = \mathcal{F}[x(t)]$	
$ax_1(t) + bx_2(t)$	$a\overline{X}_1(f) + b\overline{X}_2(f)$	(O–1)
$\dfrac{dx(t)}{dt}$	$j2\pi f\overline{X}(f)$	(O–2)
$\int_{-\infty}^{t} x(t)\,dt$	$\dfrac{\overline{X}(f)}{j2\pi f}$	(O–3)
$x(t-\tau)$	$\varepsilon^{-j2\pi f\tau}\overline{X}(f)$	(O–4)
$\varepsilon^{j2\pi f_0 t}x(t)$	$\overline{X}(f-f_0)$	(O–5)
$x(at)$	$\dfrac{1}{a}\overline{X}\left(\dfrac{f}{a}\right)$	(O–6)

The following notational form will be used here and in certain subsequent sections:

$$x(t) \longleftrightarrow \overline{X}(f)$$

This notation indicates that $x(t)$ and $\overline{X}(f)$ are corresponding transform pair; that is $\overline{X}(f) = \mathcal{F}[x(t)]$.

① Superposition Principle (or Linearity)

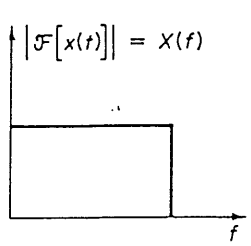
$$a x_1(t) + b x_2(t) \longleftrightarrow a\overline{X}_1(f) + b\overline{X}_2(f)$$

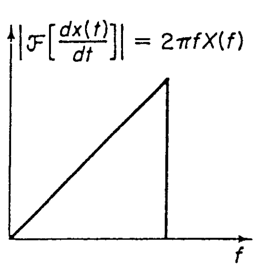
The Fourier transform integral is a linear operation and thus obeys the principle of superposition.

② Differentiation

$$\frac{dx(t)}{dt} \longleftrightarrow j2\pi f\,\overline{X}(f)$$

each time a signal is differentiated, the spectrum is multiply by $j2\pi f$. Multiplication by $j2\pi f$ has the effect of decreasing the relative level of the spectrum at low frequencies and increasing the relative level at higher frequencies. Note also that a pure dc component is eliminated

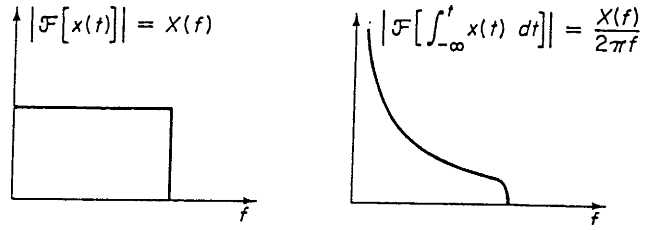




③ integration $\displaystyle\int_{-\infty}^{t} x(t)\,dt \longleftrightarrow \frac{X(f)}{j2\pi f}$

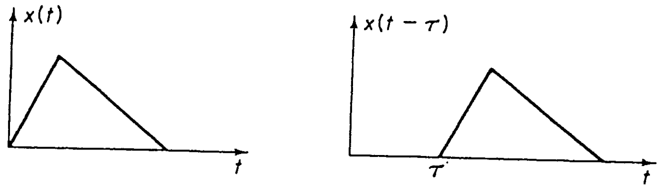
This theorem which is the reverse of ②, indicates that when a signal is integrated, the amplitude spectrum is divided by $2\pi f$.

Division by $2\pi f$ has the effect of increasing the relative level of spectrum at low frequencies and decreasing the relative level at higher freq.



④ Time Delay $x(t-\tau) \longleftrightarrow e^{-j2\pi f\tau}\, X(f)$

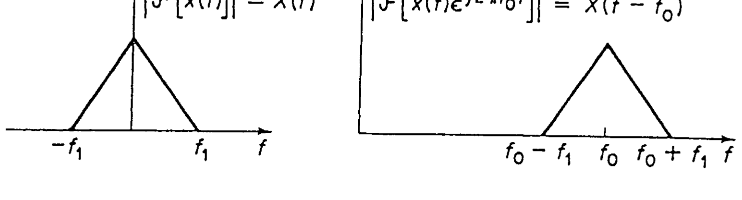
The amplitude spectrum is not changed by the shifting operation, but the phase spectrum is shifted by $-2\pi f t$ radians.



⑤ Modulation $e^{j2\pi f_0 t}\, x(t) \longleftrightarrow X(f-f_0)$

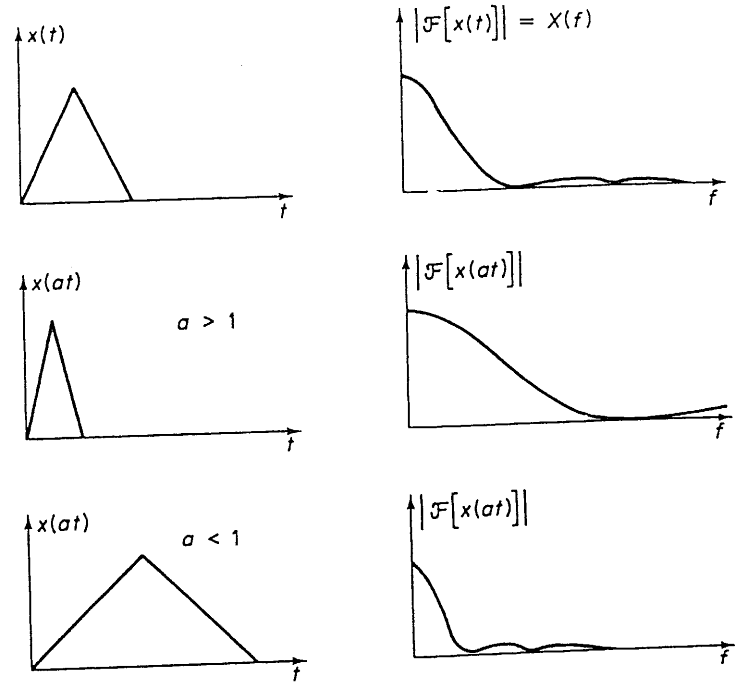
If a time signal is multiplied by a complex exponential, the spectrum is translated to the right by the frequency of the exponential.

In practical cases, complex exponentials occur in pairs with a term above along with its conjugate.



⑥ Time Scaling $\quad x(at) \longleftrightarrow \frac{1}{a} \bar{X}\left(\frac{f}{a}\right)$

If $a>1$, $x(at)$ represents a "faster" version of the orignal signal, whereas if $a<1$, $x(at)$ represents a "slower" version.
In the former case, the spectrum is broadened, whereas in the latter case, the spectrum is narrowed.



Effect on the spectrum of the time-scaling operation.

⑦ Convolution:

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot y(t-\tau)\, d\tau$$

$$x(t) * y(t) \longleftrightarrow \bar{X}(f)\, \bar{Y}(f)$$

$$x(t) \cdot y(t) \longleftrightarrow \bar{X}(f) * \bar{Y}(f)$$

⑧ Symmetry Property:

If $x(t) \longleftrightarrow \bar{X}(\omega)$, then $\bar{X}(t) \longleftrightarrow 2\pi\, x(-\omega)$

⑨ Parseval's Theorem: "Signal Energy"

$$E(f) = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\bar{X}(\omega)|^2 d\omega$$

$$x(t) \longrightarrow \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \longrightarrow \bar{X}(\omega)$$

$$\bar{X}(\omega) \longrightarrow \int_{-\infty}^{\infty} \bar{X}(\omega) e^{+j\omega t} dt \longrightarrow x(t)$$

Proof Fourier Transform

From Euler's formula : $e^{j\theta} = \cos\theta + j\sin\theta$ where $j = \sqrt{-1}$

$$\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$

$$\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$

we can write complex Fourier series

$$x(t) = a_0 + \sum_{n=1}^{\infty}[a_n\cos(n\omega t) + b_n(\sin n\omega t)]$$

$$= a_0 + \sum_{n=1}^{\infty}\left[a_n\frac{(e^{jn\omega t} + e^{-jn\omega t})}{2} + b_n\frac{(e^{jn\omega t} - e^{-jn\omega t})}{2j}\right]$$

$$= a_0 + \sum_{n=1}^{\infty}\left[e^{jn\omega t}\frac{(a_n - jb_n)}{2} + e^{-jn\omega t}\frac{(a_n + jb_n)}{2}\right]$$

From Fourier Series coefficients,

$$a_n = \frac{2}{T}\int_0^T x(t)\cos(n\omega t)dt = \frac{2}{T}\int_0^T x(t)\frac{(e^{jn\omega t} + e^{-jn\omega t})}{2}dt$$

$$= \frac{1}{T}\int_0^T x(t)e^{jn\omega t}dt + \frac{1}{T}\int_0^T x(t)e^{-jn\omega t}dt$$

$$b_n = \frac{2}{T}\int_0^T x(t)\sin(n\omega t)dt = \frac{2}{T}\int_0^T x(t)\frac{(e^{jn\omega t} - e^{-jn\omega t})}{2j}dt$$

$$jb_n = \frac{1}{T}\int_0^T x(t)e^{jn\omega t}dt - \frac{1}{T}\int_0^T x(t)e^{-jn\omega t}dt$$

Therefore,

$$\frac{(a_n - jb_n)}{2} = \frac{1}{T}\int_0^T x(t)e^{-jn\omega t}dt = A_n$$

$$\frac{(a_n + jb_n)}{2} = \frac{1}{T}\int_0^T x(t)e^{jn\omega t}dt = B_n$$

Then,

$$x(t) = a_0 + \sum_{n=1}^{\infty}[A_n e^{jn\omega t} + B_n e^{-jn\omega t}]$$

$$= \sum_{n=0}^{\infty}A_n e^{jn\omega t} + \sum_{n=1}^{\infty}B_n e^{-jn\omega t}$$

$$= \sum_{n=0}^{\infty}A_n e^{jn\omega t} + \sum_{n=-1}^{-\infty}B_{-n}e^{jn\omega t}$$

$$= \sum_{n=-\infty}^{\infty}A_n e^{jn\omega t}$$

since $A_n = B_{-n}$

Finally, we get complex Fourier series,

$\boxed{\text{Decomposition}}$: know periodic $x(t)$, to find A_n

$$A_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega t}dt = \Delta f\int_0^T x(t)e^{-jn\omega t}dt$$

$\boxed{\text{Reconstruction}}$: know A_n, to find $x(t)$

$$x(t) = \sum_{n=-\infty}^{\infty} A_n e^{jn\omega t}$$

more on the complex Fourier Series coefficients,

$$A_n = \lim_{T\to\infty} \Delta f \int_0^T x(t)e^{-jn\omega t}dt$$

and
$$x(t) = \sum_{n=-\infty}^{\infty} A_n e^{jn\omega t} = \sum_{n=-\infty}^{\infty}\left[\lim_{T\to\infty}\Delta f\int_0^T x(t)e^{-jn\omega t}dt\right]e^{jn\omega t}$$

$$= \lim_{T\to\infty}\sum_{n=-\infty}^{\infty}\left[\int_0^T x(t)e^{-jn\omega t}dt\right]e^{jn\omega t}\Delta f$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jn\omega t}dt\right]e^{jn\omega t}df$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(t)e^{-jn\omega t}dt\right]e^{jn\omega t}dF$$

$$= \underbrace{\int_{-\infty}^{\infty}\Bigg[\underbrace{\int_{-\infty}^{\infty} x(t)e^{-j\omega_n t}dt}_{\text{Fourier Transform}}\Bigg]e^{j\omega_n t}df}_{\text{Inverse Fourier Transform}}$$

$\boxed{\text{Continuous Fourier Transform}}$

Forward Transform to find coefficient $\bar{X}(f)$

$$\mathcal{F}\{x(t)\} = \bar{X}(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

$$\bar{X}(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

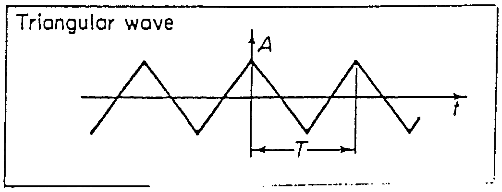
Inverse Transform to find reconstruct $x(t)$

$$\mathcal{F}^{-1}\{\bar{X}(f)\} = x(t) = \int_{-\infty}^{\infty} \bar{X}(f)e^{j2\pi ft}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{X}(\omega)e^{j\omega t}d\omega$$

ENE 208 homework 7

① Given a periodic signal as shown:



a) show that its Fourier Series is given by

$$\frac{8A}{\pi^2}\left(\cos\omega_1 t + \frac{1}{9}\cos 3\omega_1 t + \frac{1}{25}\cos 5\omega_1 t + \dots\right)$$

b) If we were to this in a complex exponential form of the Fourier Series $x(t) = \sum_{n=-\infty}^{\infty} A_n e^{jn\omega t}$

find A_n

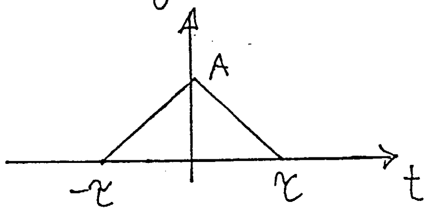
c) Sketch the corresponding frequency spectrum plots (Both one-sided and two-sided)

② Derive the Fourier transform of the exponential function given by

$$x(t) = \begin{cases} Ae^{-\alpha t} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

where $\alpha > 0$

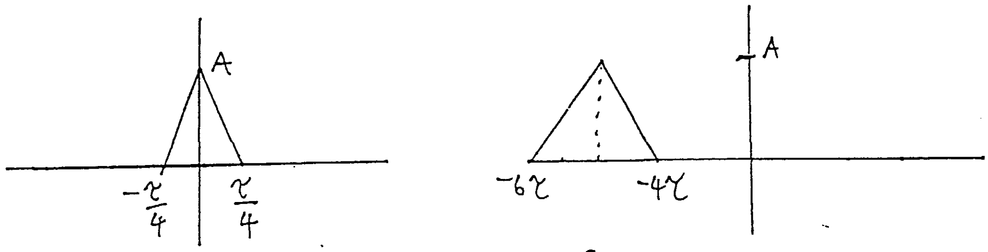
③ Given a triangular pulse as shown:



a) what is the expression of this Function $x(t)$?

b) show that its corresponding Fourier transform is $A\tau\left(\frac{\sin\pi f\tau}{\pi f\tau}\right)^2$

c) Find the Fourier transform of the signals below:



d) Sketch the function $A\tau\left(\frac{\sin\pi f\tau}{\pi f\tau}\right)^2 = F(f)$