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A unified starting procedure for the Houbolt method

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SUMMARY

The method proposed by J.C. Houbolt in 1950 is one of the pioneering methods of time integration. Nevertheless, especially due to its multi-step fashion and not having a well-defined starting procedure, the method has not met considerable acceptance. The conversion of the Houbolt method to a one-step method is reported in the literature. However, the resulting method still lacks an appropriate starting procedure for all practical cases. In this paper, a parameter-less unified starting procedure is proposed for time integration with the Houbolt method. Copyright © 2006 John Wiley & Sons, Ltd.

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KEY WORDS: time integration; Houbolt method; starting procedure; initial conditions; accuracy

1. INTRODUCTION

After discretization in space, the behaviour of many structural systems can be explained by the following mathematical model:

$$[M]\{\ddot{u}(t)\} + \{f_{\text{int}}\} = \{f(t)\}, \quad 0 \leqslant t \leqslant T \tag{1a}$$

$$\{u(t=0)\} = \{u_0\} \tag{1b}$$

$$\{\dot{u}(t=0)\} = \{\dot{u}_0\} \tag{1c}$$

(see [1]). In Equation (1), $[M]$ is the mass matrix; $\{u\}$, $\{\dot{u}\}$, and $\{\ddot{u}\}$, respectively, denote the vectors of displacement, velocity, and acceleration; $\{u_0\}$ and $\{\dot{u}_0\}$ imply the initial conditions;

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$\{f_{\text{int}}\}$ and $\{f(t)\}$, respectively, represent the vectors of the total internal force (due to damping and stiffness) and the external excitation; and, t and T stand for the time as the independent variable and the length of the time interval under consideration. In special nonlinear cases, additional initial conditions should be considered, e.g.

$$\{f_{\text{int}}(t=0)\} = \{f_{\text{int}_0}\} \tag{2}$$

for elastic–plastic dynamic models [2]. It is also worth noting that inequality constraints should be satisfied in some nonlinear problems, e.g. dynamic problems involved in contact or plasticity [3, 4]. To solve the mathematical models defined by Equations (1) and (2), time integration is the most versatile [5, 6], practically accepted [6, 7], and still under-progress approach [8–10]. One of the pioneering time integration methods is introduced by Houbolt [11]. This method investigates solutions to problems defined in Equations (1) and (2), in view of the three equations below

$$\begin{aligned} [M]\{\ddot{u}_{n+1}\} + \{f_{\text{int}_{n+1}}\} &= \{f_{n+1}\} \\ \{\dot{u}_{n+1}\} &= \frac{1}{6\Delta t}[11\{u_{n+1}\} - 18\{u_n\} + 9\{u_{n-1}\} - 2\{u_{n-2}\}] \\ \{\ddot{u}_{n+1}\} &= \frac{1}{\Delta t^2}[2\{u_{n+1}\} - 5\{u_n\} + 4\{u_{n-1}\} - \{u_{n-2}\}] \end{aligned} \tag{3}$$

(see [11, 12]). The variables in Equations (3) are defined in correspondence to the variables in Equations (1), however, for the discretized time interval shown in Figure 1. As apparent in Equations (3), the original Houbolt method is a three-step method [11, 12], i.e. to compute the response at a time station, responses at three prior time stations are to be known in advance. In order to determine the sufficient initial values (see Equations (1b), (1c), and (2)) and start the step-by-step implementation of Equations (3), it is an accepted practice to implement a one-step method in the starting two steps [12, 13]. Three most common one-step integration methods in this regard are the central difference method [14], and the average and linear acceleration methods of Newmark [15]; see [12, 13]. Such an approach implies the fact that the original Houbolt method cannot, by itself, result in the responses of Equations (1); and the computed responses differ depending on the integration method implemented in the starting steps. To put it better, the original Houbolt method does not uniquely define the response; by considering the original Houbolt method as the analysis tool, one cannot talk about specific and unique responses computed by the integration process. Lack of a specific starting procedure also prevents/complicates the implementation of the Houbolt method in problems involved in nonlinearity, adaptive time stepping, etc. [12, 16]. Research on the Houbolt method continued in the past decades [17–19]; and, regarding single-step time integration, yielded the following one-step numerical model

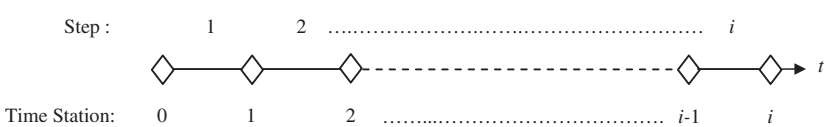

Figure 1. Typical arrangement of steps and stations in time integration analyses.

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(scheme) [17]:

$$
\begin{aligned}
[M]\{\ddot{u}_{n+1}\} + \{f_{\text{int}_{n+1}}\} &= \{f_{n+1}\} \\
\{u_{n+1}\} &= \{q_0\} + b_0\Delta\{\dddot{u}_n\} \\
\{\dot{u}_{n+1}\} &= \{q_1\} + b_1\Delta\{\dddot{u}_n\} \\
\{\ddot{u}_{n+1}\} &= \{q_2\} + b_2\Delta\{\dddot{u}_n\} \\
\{\dddot{u}_{n+1}\} &= \{q_3\} + b_3\Delta\{\dddot{u}_n\}
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
\{q_0\} &= \{u_n\} + \{\dot{u}_n\}\Delta t + \{\ddot{u}_n\}\frac{\Delta t^2}{2} + \{\dddot{u}_n\}\frac{\Delta t^3}{6} \\
\{q_1\} &= \{\dot{u}_n\} + \{\ddot{u}_n\}\Delta t + \{\dddot{u}_n\}\frac{\Delta t^2}{2} \\
\{q_2\} &= \{\ddot{u}_n\} + \{\dddot{u}_n\}\Delta t \\
\{q_3\} &= \{\dddot{u}_n\}
\end{aligned} \tag{5}
$$

$$
b_0 = \Delta t^3, \quad b_1 = \tfrac{11}{6}\Delta t^2, \quad b_2 = 2\Delta t, \quad b_3 = 1 \tag{6}
$$

Provided that the initial conditions are set consistently, Equations (4)–(6) lead to responses identical to those obtained from Equations (3). Though different from Equations (3), Equations (4)–(6) define a one-step integration method; the computation of $\{\dddot{u}_0\}$ is still a question to be answered. The existing suggestion in this regard is as noted below [17]

$$
\{\dddot{u}_0\} = [M]^{-1}[\{\dot{f}(0)\} - [K]\{\dot{u}_0\} - [C]\{\ddot{u}_0\}] \tag{7}
$$

Since, Equation (7) is set for linear behaviours ($[K]$ and $[C]$ are considered as constant-valued matrices) and excitations continuous at origin (see $\{\dot{f}_0\}$ in Equation (7)); we cannot consider Equation (7) as an appropriate starting procedure. The objective of this paper is to develop an appropriate starting procedure for the one-step scheme of the Houbolt method (Equations (4)–(6)); with slight changes, also applicable to the original Houbolt method (Equations (3)). (The importance of well-defined initial conditions is apparent in the literature [20].)

After arriving at an appropriate starting procedure for the one- and three-step Houbolt methods in Section 2; the performance of the procedure is studied via simple examples in Section 3; and finally, the paper is concluded in Section 4.

2. THEORY

2.1. Introduction

As, notified in Section 1, the one- and three-step schemes of the Houbolt method are equivalent [16, 17] (and provided the starting procedures are consistent, result in identical

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responses). Therefore, with sufficient and appropriate initial conditions for the one-step scheme, we will be able to implement Equations (4)–(6) twice and arrive at appropriate initial conditions for the original Houbolt method. This approach, i.e. introduction of an appropriate starting procedure for the one-step Houbolt method and extending it to the three-step (original) scheme of the Houbolt method, is followed in the rest of this section.

At the initial time instant, the state of a dynamic system is, in general, definable by Equations (1b), (1c), and (2) [16], and the external excitation is independently given by

$$\{f(t=0)\} = \{f_0\} \tag{8}$$

where, practically, the members of the vectors $\{u_0\}$, $\{\dot{u}_0\}$, $\{f_{\text{int}_0}\}$, and $\{f_0\}$ are all finite.

Equations (1a), (2), and (8) result in the initial values of acceleration

$$\{\ddot{u}_0\} = [M]^{-1}(\{f_0\} - \{f_{\text{int}_0}\}) \tag{9}$$

Hence, in order to start the analysis of the problem defined by Equations (1) and (2) with the one-step Houbolt method (Equations (4)–(6)), $\{\dddot{u}_0\}$ is the only unknown to be determined in advance.

Furthermore, it is essential to note that, since the Houbolt method is dedicated to the analysis of initial value problems originated in structural motion, it is reasonable to consider $\{f(t)\}$, in Equations (1), (and hence, $\{\dot{f}_0\}$) as finite-valued vectors, and arrive at Figure 2 as an illustration of the sufficient cases in this study (an additional very rare case is also briefly discussed later). With these considerations, appropriate formulation for $\{\dddot{u}_0\}$ is developed in Section 2.2 and then the corresponding starting procedure is set in Section 2.3.

2.2. Appropriate initial conditions for the one-step scheme

In the case displayed in Figure 2(a), we can differentiate the equation of motion with respect to time and obtain $\{\dddot{u}_0\}$ from

$$\{\dddot{u}_0\} = [M]^{-1}(\{\dot{f}_{0^+}\} - [K_0]\{\dot{u}_0\} - [C_0]\{\ddot{u}_0\}) \tag{10}$$

seemingly, under the assumption of

$$[\dot{C}_0] = [\dot{K}_0] = 0 \tag{11}$$

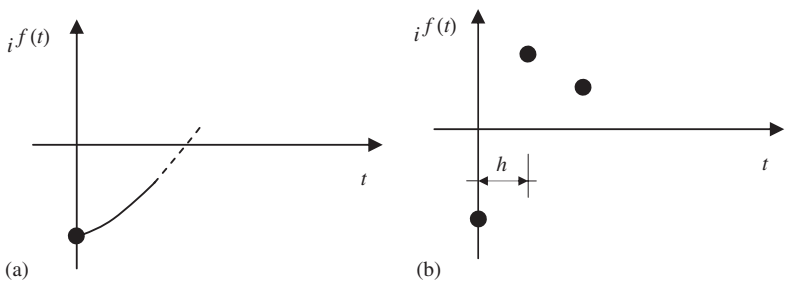

Figure 2. Schematic changes of the ith arbitrarily selected member of $\{f(t)\}$, at the close vicinity of $t = 0$: (a) continuous excitation; and (b) digitized excitation.

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In Equation (10), $[K_0]$ and $[C_0]$, respectively, represent the stiffness and damping matrices at $t=0$, and $\{\dot{f}_{0^+}\}$ is the temporal derivative of $\{f(t)\}$ in the limiting vicinity, at the right of $t=0$ (and, in Figure 2(a), equal to the right derivative of $\{f(t)\}$ at $t=0$). Regarding the validity of Equation (11), since the equation of motion is of the second order, $\{\ddot{u}_0\}$ is not a characteristic of the mathematical model, but an essentiality of the one-step Houbolt time integration analysis. Considering the numerical model, step-by-step integration converts the mathematical model to a piece-wisely linear numerical model [21–23], which, in the limit of zero time steps, should coincide with the mathematical model (convergence). From this point of view, the fact that time steps are never zero [1, 23] results in the existence of a close vicinity of $t=0$, where stiffness and damping are considered constant in time integration analysis. Hence, Equations (10) and (11) hold. It is also worth noting that in the case of Figure 2(a), when Equation (7) is applicable, Equations (7) and (10) are computationally identical; other than that, according to the knowledge of the authors, no starting procedure is addressed in the literature. Therefore, Equation (10) can be considered computationally appropriate.

In the case displayed in Figure 2(b), $\{\dot{f}_{0^+}\}$ is unknown and hence Equation (10) is inapplicable. Instead, the time axis can be considered starting from the -1th time station (Figure 3), with quiescent initial conditions, i.e.

$$\{u(-\Delta t')\}=\{\dot{u}(-\Delta t')\}=\{\ddot{u}(-\Delta t')\}=\{\ddot{u}\ (-\Delta t')\}=\{f(-\Delta t')\}=\bar{\mathbf{O}} \tag{12}$$

Extending the initial characteristics to the time interval prior to $t=0$, and applying the one-step scheme of the Houbolt method (Equations (4)–(6)), leads to

$$2\Delta t'[M]\{\ddot{u}_0\}+\tfrac{11}{6}(\Delta t')^2[C_0]\{\ddot{u}_0\}+(\Delta t')^3[K_0]\{\ddot{u}_0\}=\{f_0\} \tag{13}$$

Consequently,

$$\{\ddot{u}_0\}=[K_{\text{eff}}]^{-1}\{f_0\} \tag{14a}$$

$$[K_{\text{eff}}]=2\Delta t'[M]+\tfrac{11}{6}(\Delta t')^2[C_0]+(\Delta t')^3[K_0] \tag{14b}$$

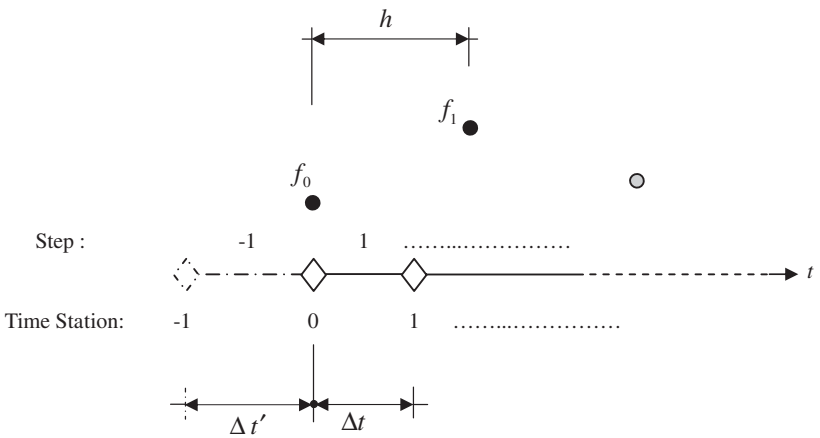

Figure 3. The consideration of an additional time station before t_0 for the case in Figure 2(b).

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In Equations (14), $[K_{\text{eff}}]$ is, in general, invertible for structural dynamic problems and practical $\Delta t'$s [1, 16, 17]. Hence, by selecting an appropriate value for $\Delta t'$, we can arrive at an appropriate value for $\{\ddot{u}_0\}$. As a general approach in numerical analysis, the most meaningful points of view to study the appropriateness are easy-implementation, computational cost, convergence, and accuracy [5, 24]. Starting from accuracy, in view of the error equations developed for time integration analyses [1], and the spectral radius, $\rho_{\max}$, of the Houbolt method [1, 12, 16], the error of the starting procedure will damp out after sufficient time steps. However, still, Equations (4) and (5) result in the transfer of the errors in $\{\ddot{u}_0\}$ to other steps. This recommends using smaller values for $\Delta t'$. In view of Equations (14a) and (14b) and also the existing literature [23], $\Delta t' = 0$ is not acceptable; and besides, for the sake of responses proper convergence [1, 16, 24], $\Delta t'$ would rather be set independent of the integration step. In view of these considerations, $\Delta t'$ is herein set equal to the time step by which the excitation is digitized at $t = 0^+$, h (see Figures 2 and 3), i.e.

$$\Delta t' = h \tag{15}$$

Due to the simplicity of Equation (15), the computational cost of Equations (14) and (15) is about the same as the one-step Houbolt method [17] at one step. Such a cost is negligible especially considering time integration throughout the integration interval. Therefore, Equation (15) seems an appropriate formulation from all the standpoints noted above (further study is presented later). Since, according to Equations (1b) and (1c), $\{u_0\}$ and $\{\dot{u}_0\}$ are known vectors, time integration in the step prior to $t = 0$ looks in contradiction with Equations (1a)–(1c) and (9). However, we are using Equations (14) and (15) merely for computing $\{\ddot{u}_0\}$. Therefore, the contradiction can be explained by considering the application of constant external acceleration, velocity, and displacement, point-wisely at $t = 0$. This assumption is equivalent to a discontinuous response at $t = 0$, i.e.

$$\{u_{0^-}\} \neq \{u_{0^+}\} = \{u_0\} \tag{16a}$$

$$\{\dot{u}_{0^-}\} \neq \{\dot{u}_{0^+}\} = \{\dot{u}_0\} \tag{16b}$$

$$\{\ddot{u}_{0^-}\} \neq \{\ddot{u}_{0^+}\} = \{\ddot{u}_0\} \tag{16c}$$

which is acceptable, in view of our interest in the response at $t \geqslant 0$.

For the completeness of the discussion in this paper, it is essential to briefly study a very rare case, when the two cases in Figure 2 occur together; see Figure 4. If we return to Equation (10)

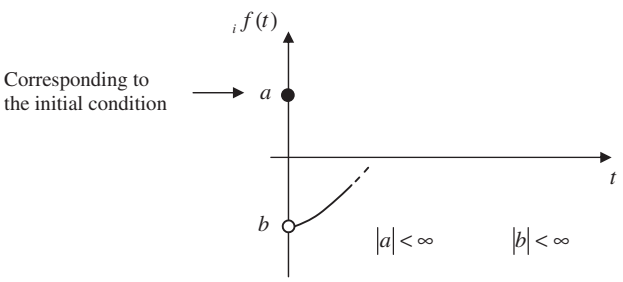

Figure 4. A combination of the cases displayed in Figures 2(a) and (b).

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and the definition of $\{\dot{f}_{0+}\}$, i.e. the temporal derivative of $\{f(t)\}$ in the limiting vicinity at the right of $t=0$, it becomes obvious that we can deal with the cases displayed in Figures 2(a) and 4, in a similar way.

2.3. Practical implementation

According to the foregoing discussions, the initial conditions essential for Houbolt time integration of Equations (1) and (2) are available from the procedure below

1. Compute $\{\ddot{u}_0\}$ from Equation (9),
2. If the RHS (right hand side) of Equation (1a) is defined as displayed schematically in Figure 2(a) or 4, compute $\{\dddot{u}_0\}$ from Equation (10); continue to step 5,
3. The RHS of Equation (1a) is available as a digitized record (Figure 2(b)); compute $\Delta t'$ from Equation (15),
4. Compute $\{\ddddot{u}_0\}$ from Equations (14a) and (14b),
5. If the one-step scheme of the Houbolt method is to be applied, the initial conditions are sufficient for starting the time integration analysis according to Equations (4)–(6); stop,
6. March two time steps according to Equations (4)–(6); the initial conditions are then sufficient to continue the time integration by the original Houbolt method (Equations (3)); stop.

As implied in Section 2.2, it is worth emphasizing that, being based on Equations (9), (10), (14), and (15); the proposed procedure, besides, being independent of other integration methods and also applicable to practical problems, is comparable with the existing procedures [12, 13, 17] from the standpoint of computational appropriateness.

Meanwhile, with regard to the probable presence of different types of nonlinearities, it is worth noting that, in the procedure above, the variables are not being affected by nonlinearity (specifically, as explained for Equations (10), $[K_0]$ and $[C_0]$, respectively, stand for the stiffness and damping at the initial instant and are independent of the stiffness and damping within the initial step of the numerical model). Therefore, the procedure presented above is applicable to both linear and nonlinear problems. Nevertheless, due to the different behaviours of the linear and nonlinear problems, the procedure may affect differently the accuracies of the two time histories. This question is also numerically studied in the next section.

In the end of this section, it is essential to note that, as apparent, when implementing the proposed procedure, we can arrive at specific determinate responses from Houbolt time integration, by (1) selecting the integration step size (or the criterion of adaptive integration), and (2) if needed, setting some nonlinearity considerations. This is common among practically accepted time integration methods [1, 12–14]. Therefore, as addressed in the title of this paper, it seems reasonable to consider the starting procedure proposed in this paper as a unified starting procedure for the Houbolt method.

3. NUMERICAL EXAMPLES

3.1. Introduction

As explained in Section 2, easy implementation, trivial computational cost, and having no effect on convergence are the obvious characteristics of the proposed starting procedure. To explain better, the equations implemented in the proposed procedure, i.e. Equations (9), (10), (14), and (15),

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are not more complicated than those existing (for some cases) in the literature [12, 13, 17]; the corresponding computational cost is about the cost associated with Houbolt integration in a single step and hence negligible in view of the considerable number of steps in the integration intervals; and since, the procedure is independent of the integration step size, it does not affect convergence. However, for numerical verification and specifically, in order to study the effects of the proposed procedure on accuracy, the procedure is herein compared to other starting procedures suggested in the literature [12, 13, 17]. Three problems, corresponding to the cases displayed in Figure 2, are analysed by the six methods below.

(1) The one-step Houbolt method with the starting procedure proposed in this paper.
(2) The three-step Houbolt method with the starting procedure proposed in this paper.
(3) The one-step Houbolt method with the one-step starting procedure existing in the literature [17] (if applicable).
(4) The three-step Houbolt method with the two starting steps computed according to the average acceleration method of Newmark (the starting procedure suggested in [12]).
(5) The three-step Houbolt method with the two starting steps computed according to the linear acceleration method of Newmark (the starting procedure suggested in [12]).
(6) The three-step Houbolt method with the two starting steps computed according to the central difference method (the starting procedure suggested in [13]).

3.2. Linear and nonlinear two-DOF systems subjected to continuous excitations (Figure 2(a))

Consider the two-DOF structural system below [25, 26]

$$\begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} \begin{Bmatrix} {}_1u \\ {}_2u \end{Bmatrix} + \begin{bmatrix} 0.2 & -0.1 \\ -0.1 & 0.1 \end{bmatrix} \begin{Bmatrix} {}_1\dot{u} \\ {}_2\dot{u} \end{Bmatrix} + \begin{bmatrix} 20+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} {}_1u \\ {}_2u \end{Bmatrix} = \begin{Bmatrix} {}_1f(t) \\ {}_2f(t) \end{Bmatrix}$$
$$\begin{Bmatrix} {}_1u(t=0) \\ {}_2u(t=0) \end{Bmatrix} = \begin{Bmatrix} {}_1\dot{u}(t=0) \\ {}_2\dot{u}(t=0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad 0 \leqslant t \leqslant 10 \tag{17}$$

where

$$ {}_1f(t) = 0$$
$$ {}_2f(t) = \begin{cases} 0 & \text{for } t<0 \\ 4 & \text{for } t \geqslant 0 \end{cases} \tag{18}$$

and, in the linear case,

$$k_2 = 1 \tag{19}$$

and, in the nonlinear case,

$$k_2 = 1 + 0.5({}_2u - {}_1u)^2 \tag{20}$$

In Houbolt time integration of both of the problems stated above, the starting procedures existing in the literature [12, 13, 17] can be implemented. Considering S.I. units; time integrating the

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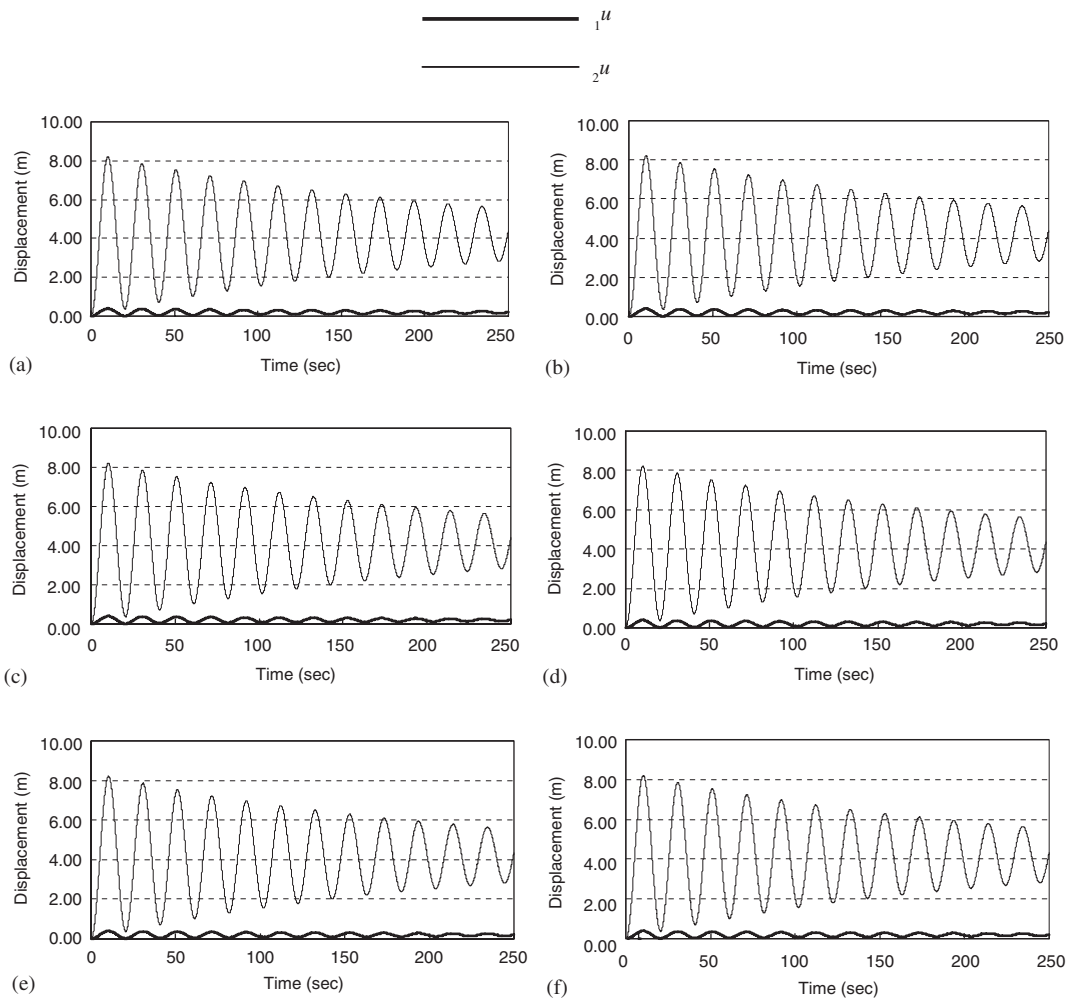

Figure 5. Displacement history computed for the linear case of the first example ($\Delta t = 0.05$ s): (a) implementing the proposed starting procedure and the one-step Houbolt method; (b) implementing the proposed starting procedure and the three-step Houbolt method; (c) implementing the existing starting procedure [17] and then the one-step Houbolt method; (d) implementing the average acceleration method at the first two steps [12] and then the three-step Houbolt method; (e) implementing the linear acceleration method at the first two steps [12] and then the three-step Houbolt method; and (f) implementing the central difference method at the first two steps [13] and then the three-step Houbolt method.

linear and nonlinear cases of Equation (17) with the time steps all equal to 0.05 s, the six methods noted in Section 3.1, and the procedure proposed in Section 2.3, leads to Figures 5 and 6. Figures 5(a)–(f), being in very close resemblance (they are depicted separately because of this similarity), evidence the good performance of the proposed starting procedure. The same is apparent in Figures 6(a)–(f). It is meanwhile worth noting that as can be concluded from Section 2, Figures 5(a)–(b) (and similarly Figures 6(a)–(b)) are identical; see Table I. This implies the

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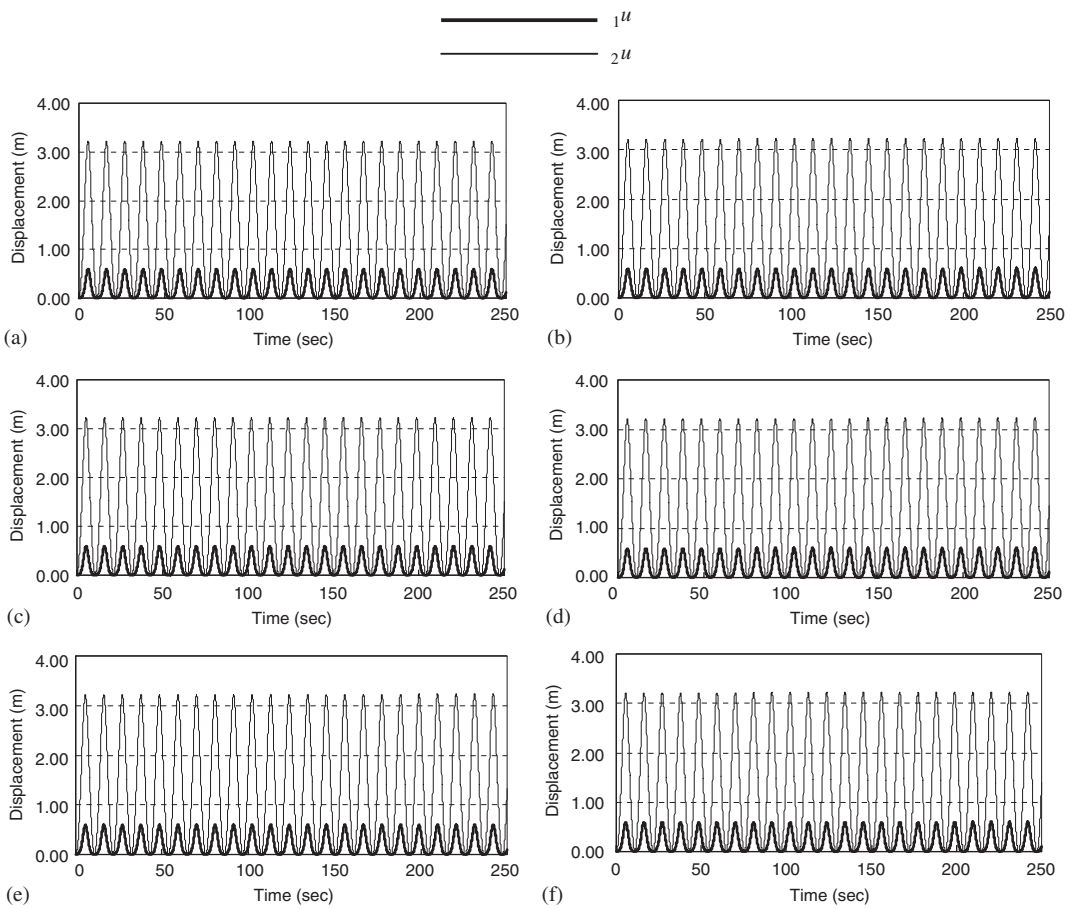

Figure 6. Displacement history computed for the nonlinear case of the first example ($\Delta t = 0.05$ s): (a) implementing the proposed starting procedure and the one-step Houbolt method; (b) implementing the proposed starting procedure and the three-step Houbolt method; (c) implementing the existing starting procedure [17] and then the one-step Houbolt method; (d) implementing the average acceleration method at the first two steps [12] and then the three-step Houbolt method; (e) implementing the linear acceleration method at the first two steps [12] and then the three-step Houbolt method; and (f) implementing the central difference method at the first two steps [13] and then the three-step Houbolt method.

fact that the starting procedure proposed in Section 2.3 causes the Houbolt method to produce unique numerical results for the problems defined by Equations (17)–(20). Regarding computational cost; no meaningful difference is obtained between the computational cost associated with Figures 5(a) and (c); similarly, between Figures 5(b), (d), (e), and (f); similarly between Figures 6(a) and (c); and finally, similarly, between Figures 6(b), (d), (e), and (f). A similar study is carried out with some larger time steps and similar consequences are obtained.

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Table I. Identical effect of the proposed procedure on different schemes of the Houbolt method.

		Figure 5(a)	Figure 5(b)	Figure 6(a)	Figure 6(b)
Maximum	${}_1u$	0.391955	0.391955	0.599955	0.599955
	${}_2u$	8.210101	8.210101	3.228676	3.228676
Average	${}_1u$	0.199189	0.199189	0.198039	0.198039
	${}_2u$	4.181458	4.181458	1.459340	1.459340
Final	${}_1u$	0.211043	0.211043	0.128143	0.128143
	${}_2u$	4.393488	4.393488	1.5136204	1.5136204

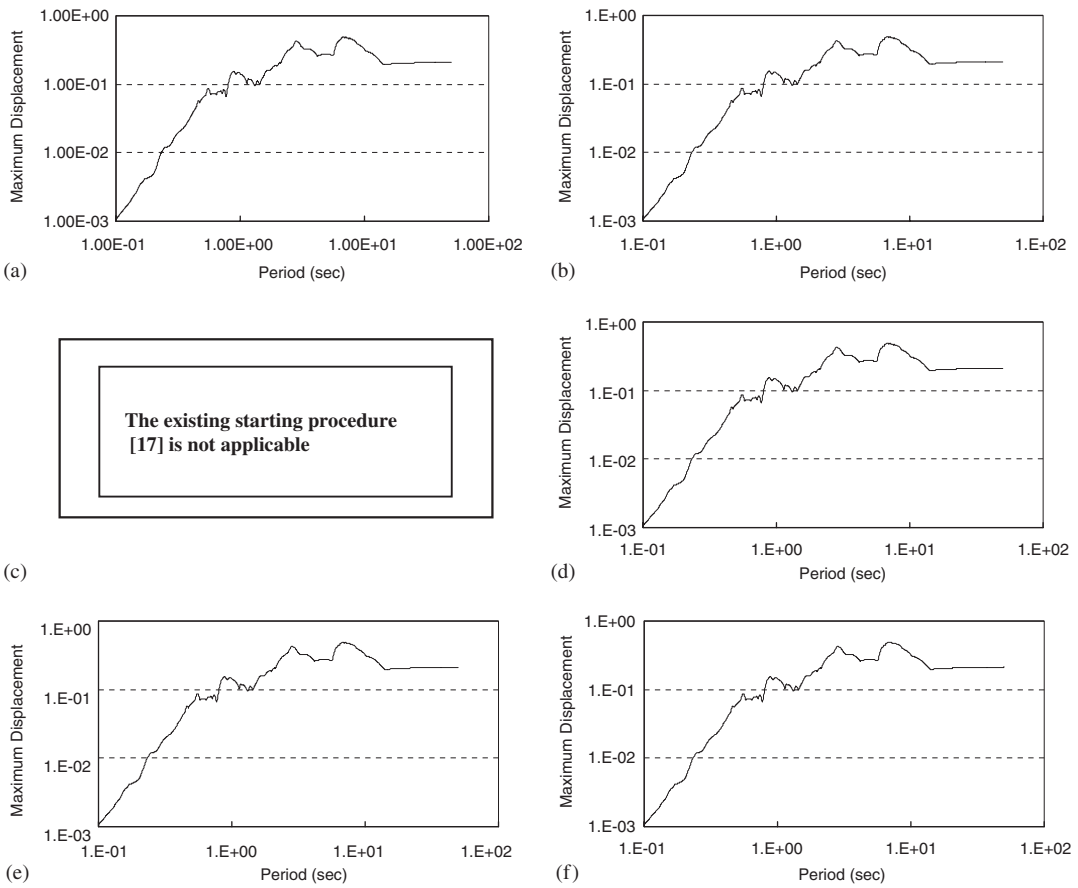

Figure 7. Displacement spectra computed for the north–south component of the El Centro strong motion ($\Delta t = 0.05$ s): (a) implementing the proposed starting procedure and the one-step Houbolt method; (b) implementing the proposed starting procedure and the three-step Houbolt method; (c) implementing the existing starting procedure [17] and then the one-step Houbolt method; (d) implementing the average acceleration method at the first two steps [12] and then the three-step Houbolt method; (e) implementing the linear acceleration method at the first two steps [12] and then the three-step Houbolt method; and (f) implementing the central difference method at the first two steps [13] and then the three-step Houbolt method.

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3.3. Systems subjected to digitized excitation (Figure 2(b))

In order to examine the performance of the proposed starting procedure also in a more practical case; the 2% damped displacement spectra [6, 7] of the 30 s north–south component of the El Centro strong motion [7] is computed in Figure 7 (different from the two problems studied in Section 3.2, the starting procedure existing for the one-step Houbolt method is not applicable here, see Figure 7(c)). Figure 7 not only reveals the good performance of the proposed procedure for digitized excitations, but also, as a secondary result, is an evidence for the independence of this performance from characteristics of dynamic systems, e.g. natural period. (This numerical study is repeated by considering the second half of the El Centro strong motion [7] as the excitation; the obtained numerical results reveal the same points.)

4. CONCLUSION

A starting procedure is, in this paper, proposed for the Houbolt time integration method.

(1) The proposed starting procedure can be applied to both the one- and three-step schemes of the Houbolt method, resulting in identical responses.
(2) Whereas, the existing starting procedures for the three-step Houbolt method [12, 13] depend on other integration methods, and hence, cannot lead to specific determinate responses; the new starting procedure is not in need of other integration methods.
(3) Whereas, the existing starting procedure for the one-step Houbolt method [17] cannot be applied to all problems, the proposed procedure seems applicable to all types of practical equations of motion (equations of motion with finite excitations at close vicinity of $t = 0$).
(4) The performance of the new starting procedure seems sufficiently appropriate from the points of view of simple implementation, accuracy, and computational cost, both in linear and nonlinear problems.

And, finally, with regard to the points noted above, it seems reasonable to refer to the new starting procedure as a practical unified starting procedure for the Houbolt time integration method.

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