Linear Algebra

C.1 Systems of linear equations

Ex. Linear equation in one variable x

unknown / variable

$(a x = b)$

numbers

1) $2x = 4 \rightarrow x = \frac{4}{2} = 2$ one solution

2) $0 \cdot x = 0 \rightarrow$ every num. is a solution : infinitely many solutions

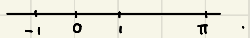
3) $0 \cdot x = 1 \rightarrow$ no solution : 0 solution

Show

in general $ax = b$ has either 0, 1, or infinitely many solutions

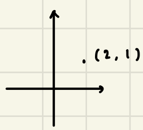
Notion : $\mathbb{R}$: set of real numbers

e.g. $-1, 0, \pi, \sqrt{2}, \ldots$

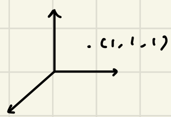


- $\mathbb{R}^n = \{(a_1, a_2, \ldots, a_n) : a_1 \cdots a_n \in \mathbb{R}\}$

R^2



R^3



- We denote numbers by a, b, c $a_1, a_2,$ etc.

  variables by x, y, z $x_1, x_2, x_3, \ldots$

Def.

- A linear equation in n-variables $x_1, x_2, \ldots, x_n$ is an equation of the form

  $a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$

- A system of linear equations is a list of linear equations

  (a linear system)

Ex. $x_1 - 2x_2 = 5$: linear eq. in 2 variables x_1, x_2

$$\begin{cases} x_1 - 2x_2 = 5 \\ x_1 + x_2 = 2 \end{cases}$$: linear system

Def.

- A solution of a linear system in n-variables is an n-tople $(t_1, t_2, \cdots, t_n) \in \mathbb{R}^n$ that makes all the equations true.
- solution set : collection of all solutions

Ex. $x_1 - 2x_2 = 5$ → $2x_2 = x_1 - 5$ → $x_2 = \frac{x_1 - 5}{2}$

$(1, -2)$ is a solution : $1 - (-2) = 5$

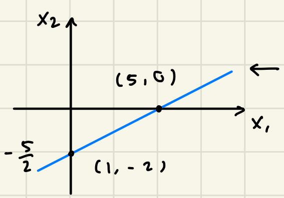
$(5, 0)$ is a solution : $5 - 2 \cdot 0 = 5$

given any number x_2, set $x_1 = 2x_2 + 5$

then $(x_1, x_2) = (2x_2 + 5, x_2)$ is a solution for any x_2

solution set $= \{(2x_2 + 5, x_2) : x_2 \in \mathbb{R}\}$

$= \left\{\left(x_1, \frac{x_1 - 5}{2}\right) ; x \in \mathbb{R}\right\}$



$x_1 - 2x_2 = 5$

Ex. $\begin{cases} x_1 - 2x_2 = 5 & \cdots ① \\ x_1 + x_2 = 2 & \cdots ② \end{cases}$

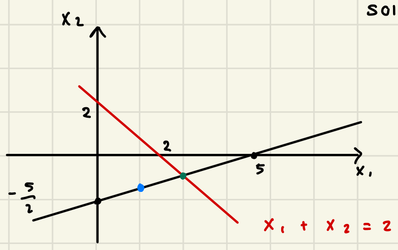
$① - ②$: $(x_1 - 2x_2) - (x_1 + x_2) = 5 - 2$

$-3x_2 = 3$

$x_2 = -1 \Rightarrow x_1 = 2x_2 + 5$

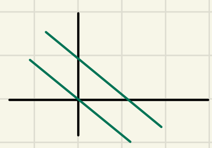
$(3, -1)$ is only solution $= -2 + 5 = 3$

solution set $= \{(3, -1)\}$



Ex. $\begin{cases} x_1 + x_2 = 2 \\ x_1 + x_2 = 0 \end{cases}$ no solution!

solution set $\neq 0$



Def. A linear system without any solution is inconsistent. Otherwise, it is consistem

Def. Two linear systems of n-variables are equivalent if they have the same solution set

EX. Consider linear systems in 3 variables x_1, x_2, x_3:

$$① \begin{cases} x_1 - cx_2 = 0 \\ x_1 + x_3 = 0 \end{cases} \qquad ② \begin{cases} 2x_1 - x_2 + x_3 = 0 \\ x_2 + x_3 = 0 \end{cases}$$

Q. Find $c \in \mathbb{R}$ s.t. they are equivalent

such that

solution set for ② : • solve second eqn. for x_2:

• $x_2 = -x_3$

write x_1 in terms of x_3:

$2x_1 = x_2 - x_3 = -x_3 - x_3 = -2x_3$

$\Rightarrow x_1 = -x_3$

For any $x_3 \in \mathbb{R}$, $(x_1, x_2, x_3) = (-x_3, -x_3, x_3)$ is a solution

Solution set for ② $= \{(-x_3, -x_3, x_3) : x_3 \in \mathbb{R}\}$

Exercise: solve system ① & verify that it has the same solution set

$x_1 - cx_2 = 0 \rightarrow x_1 = cx_2 \rightarrow x_2 = \frac{x_1}{c}$

$x_1 + x_3 = 0 \rightarrow x_1 = -x_3 \rightarrow x_3 = -x_1$

$(x_1, x_2, x_3) = \left(x_1, \frac{x_1}{c}, -x_1\right)$

$\left\{\left(x_1, \frac{x_1}{c}, -x_1\right) : x_1 \in \mathbb{R}\right\}$

1.2 Row Reduction

goal: give an algorithm to solve any linear system

matrices: rectangular array of numbers.

ex. $[1]$, $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $[2 \quad 3 \quad 0]$, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{pmatrix} 2 & 3 \\ 5 & 1 \\ 0 & -2 \end{pmatrix}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$$

m rows

n columns

$m \times n$ matrix

$a_{ij} \in \mathbb{R}$ is called the (i, j) entry of A

- consider a linear system

$$\begin{cases} 1 \cdot x_1 - 2x_2 + x_3 = 0 \\ \qquad 2x_2 - 8x_3 = 8 \\ 5x_1 \quad - \quad 5x_3 = 10 \end{cases} \longrightarrow \begin{pmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{pmatrix}$$

coefficient matrix

(i, j) - entry of coefficient matrix is the coefficient of x_j in i-th equation

- Augmented matrix: combine $\begin{pmatrix} 0 \\ 8 \\ 10 \end{pmatrix}$ to coeff. matrix

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right)$$

- How to solve them?

① Elimination method: combine equations & remove variables to get a simpler yet equivalent system

R stands for row

Ex.
$$\begin{cases} 1\cdot x_1 - 2x_2 + x_3 = 0 & : R_1 \\ 2x_2 - 8x_3 = 8 & : R_2 \\ 5x_1 - 5x_3 = 10 & : R_3 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right)$$

step 1: Replace R_3 by $R_3 - 5\cdot R_1$

$$\begin{array}{r} 5x_1 \qquad\qquad - 5x_3 = 10 \\ 5x_1 - 10x_2 + 5x_3 = 0 \quad - \\ \hline 10x_2 - 10x_3 = 10 \end{array} \quad : \text{New } R_3$$

For augmented matrix, this amounts to replace row 3 to row 2 - 5 · row 1

step 2: multiply $\frac{1}{2}$ to R_2 & $\frac{1}{10}$ to R_3

then get

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 & R_1 \\ x_2 - 4x_3 = 4 & R_2 \cdot \frac{1}{2} \quad \text{new} \\ x_2 - x_3 = 1 & R_3 \cdot \frac{1}{10} \quad \text{new} \end{cases}$$

this amounts to replace Row 2 → $\frac{1}{2}$ · Row 2

Row 3 → $\frac{1}{10}$ · Row 3

ori. R_2 ←
$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

step 3: Remove x_2 in R_3 : $R_3 \mapsto R_3 - R_2$

$$\begin{array}{r} \cancel{x_2} - x_3 = 1 \quad \cdots R_3 \\ \cancel{x_2} - 4x_3 = 4 \quad - \cdot R_2 \\ \hline 3x_3 = -3 \\ x_3 = -1 \end{array}$$

mult. $\frac{1}{3}$ to R_3

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 1 & -1 & 1 \end{array}\right) \xrightarrow[\text{Row } 3-2]{\text{Row } 3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 3 & -3 \end{array}\right) \quad \text{in REF}$$

Row 3 · $\frac{1}{3}$
$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

Step 4 : solve from the last to the first equation

$x_3 = -1$

$x_2 - 4x_3 = 4$

$x_2 = 4 + 4x_3$

$= 4 - 4 = 0$

$x_1 - 2x_2 + x_3 = 0$

$x_1 = -x_3 = -(-1)$

$= 1$

Thus $(1, 0, -1)$ is the only solution.

This amounts to

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow[R_1 = R_1 - R_3]{R_2 = R_2 + 4R_3} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{R_1 = R_1 + 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

(not RREF)

(The RREF of the augmented matrix)

$\Updownarrow$

$x_1 = 1$

$x_2 = 0$

$x_3 = -1$

$(1, 0, -1)$ is the only solution

RMK. We have transformed an augmented matrix to a matrix in a special form which is easier to work with.

Def. Row Echelon Form, REF

An $m \times n$ matrix is in REF if

E1) All zero rows are at the bottom

E2) The first non-zero entry of a row is to the right of the first non-zero entry of the row above

E3) Every entry below the first nonzero entry is 0.

Ex)

$$\begin{pmatrix} \boxed{*} & * & * & * \\ 0 & \boxed{*} & * & * \\ 0 & 0 & 0 & \boxed{*} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\boxed{*}$: non-zero entry → pivots

$*$: any number

Ex. $\begin{bmatrix} 0 & 0 & 3 \\ 0 & 6 & 2 \end{bmatrix}$ does not satisfy

$\begin{bmatrix} 0 & \boxed{6} & 2 \\ 0 & \cancel{3} & 0 \end{bmatrix}$ violated

both not in REF

Def.

- A pivot is the first non-zero entry of a row of a matrix in REF.
- A pivot column is a column containing a pivot.

Def. Reduced Row Echelon Form (RREF)

(even simpler than REF)

A matrix in REF is in RREF if

1) pivot = 1

2) entries above each pivot = 0

shape: $\begin{bmatrix} 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

We can transform any matrix to unique reduced row echelon form by using 3 row operations.

Def. Elementary row operations

- Replacement: add a multiple of a row to another row

  ex/: $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$

  $$\begin{array}{r|rr} R_2 & 2 & 1 \\ 2R_1 & 2 & 4 & - \\ \hline & 0 & -3 \end{array}$$

- Interchange: swap two rows

  ex/: $\begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

- Scaling: scale a row by a non-zero number

  ex/: $\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{2} \cdot R_1} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

RMK. When augmented matrix is transformed by elementary row operations, the solution set doesn't change

(RREF)

Def. Denote the unique reduced row echelon form of a matrix A obtained from elementary row operations by RREF(A)

EX. Find RREF of following matrices

1. $\begin{array}{c} \begin{pmatrix} 2 & 3 \\ 1 & 6 \end{pmatrix} \end{array} \begin{array}{l} R_2 \\ R_1 \end{array}$

$R_2 \leftrightarrow R_1 \begin{pmatrix} 1 & 6 \\ 2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 6 \\ 0 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$R_2 = 2R_1 - R_2$

$(2 \quad 12) - (2 \quad 3)$

$(0 \quad 9)$

$R_2 \cdot \frac{1}{9}$

$R_1 = R_1 - 6R_2$

$(1 \quad 6) - (0 \quad 6)$

$(1 \quad 0)$

2. $\begin{pmatrix} 0 & 2 & -3 & 5 \\ 1 & -1 & 0 & 6 \\ 2 & 1 & 3 & 0 \end{pmatrix} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$

$R_2 \leftrightarrow R_1 \begin{pmatrix} 1 & -1 & 0 & 6 \\ 0 & 2 & -3 & 5 \\ 2 & 1 & 3 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 & 6 \\ 0 & 2 & -3 & 5 \\ 0 & 3 & 3 & -12 \end{pmatrix}$

$R_3 = R_3 - 2R_1$

$$\begin{array}{cccc} 2 & 1 & 3 & 0 \\ 2 & -2 & 0 & 12 \\ \hline 0 & 3 & 3 & -12 \end{array} \; -$$

$R_2 \cdot \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 & 6 \\ 0 & 1 & -\frac{3}{2} & \frac{5}{2} \\ 0 & 3 & 3 & -12 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -\frac{3}{2} & \frac{17}{2} \\ 0 & 1 & -\frac{3}{2} & \frac{5}{2} \\ 0 & & & \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -\frac{3}{2} & \frac{17}{2} \\ 0 & 1 & -\frac{3}{2} & \frac{5}{2} \\ 0 & 0 & \frac{15}{2} & -\frac{39}{2} \end{pmatrix}$

$R_1 = R_1 + R_2$

$R_3 = R_3 - 3R_2$

$$\begin{array}{cccc} 0 & 3 & 3 & -12 \\ 0 & 3 & -\frac{9}{2} & \frac{15}{2} \end{array}$$

$R_3 \cdot \frac{2}{15} \begin{pmatrix} 1 & 0 & -\frac{3}{2} & \frac{17}{2} \\ 0 & 1 & -\frac{3}{2} & \frac{5}{2} \\ 0 & 0 & 1 & -\frac{39}{15} \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & -\frac{3}{2} & \frac{5}{2} \\ 0 & 0 & 1 & -\frac{13}{5} \end{pmatrix}$

$-\frac{13}{5}$

$R_1 = R_1 + \frac{3}{2} R_3$

$$\begin{array}{cccc} 1 & 0 & -\frac{3}{2} & \frac{17}{2} \\ 0 & 0 & \frac{3}{2} & -\frac{39}{10} \end{array}$$

$R_2 = R_2 + \frac{3}{2} R_3$

$$\begin{matrix} 0 & 1 & -\frac{3}{2} & \frac{5}{2} \\ 0 & 0 & \frac{3}{2} & -\frac{39}{10} \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{5}{2} \\ 0 & 1 & 0 & -\frac{5}{2} \\ 0 & 0 & 1 & -\frac{13}{5} \end{pmatrix}$$