15.2 (9, 16)

9. C : line segment from (0,0) to (1, 1)

a) parametrization of the path C

$\vec{r}(t) = (1-t)\langle 0,0\rangle + t\langle 1,1\rangle$

$= \langle t,t\rangle = ti + tj \qquad 0 \leq t \leq 1$

b) $\int_C (x^2+y^2)\,ds$

$\vec{r}'(t) = \langle 1, 1\rangle$

$ds = \sqrt{2}\,dt$

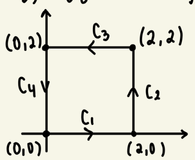
$\int_C 2t^2\sqrt{2}\,dt = 2\sqrt{2}\ \frac{1}{3}t^3\Big|_0^1$

$= \frac{2\sqrt{2}}{3}$

16. C : CCW □ (0,0), (2, 0), (2,2), (0, 2)

a) C

b) $\int_C (2x + 3\sqrt{y})\,ds$



C_1 : $\vec{r}(t) = (1-t)\langle 0,0\rangle + t\langle 2,0\rangle$

$= \langle 2t, 0\rangle$

$\vec{r}'(t) = \langle 2, 0\rangle \qquad 0 \leq t \leq 1$

$ds = 2\,dt$

$\int_0^1 4t\ 2\,dt = \cancel{8}^{4}\ \frac{1}{\cancel{2}}t^2\Big|_0^1 = 4$

C_2 : $\vec{r}(t) = (1-t)\langle 2,0\rangle + t\langle 2,2\rangle$

$= \langle 2-2t, 0\rangle + \langle 2t, 2t\rangle$

$= \langle 2, 2t\rangle \qquad 0 \leq t \leq 1$

$\vec{r}'(t) = \langle 0, 2\rangle$

$ds = 2\,dt$

$\int_0^1 (4 + 3\sqrt{2t})\,2\,dt = 2\left(4t + \cancel{3}\sqrt{2}\cdot\frac{2}{\cancel{3}}t^{3/2}\right)\Big|_0^1$

$= 2(4 + 2\sqrt{2}) = 8 + 4\sqrt{2}$

C_3: $\vec{r}(t) = (1-t)\langle 2,2\rangle + t\langle 0,2\rangle$

$= \langle 2-2t, 2-2t\rangle + \langle 0, 2t\rangle$

$= \langle 2-2t, 2\rangle \qquad 0 \leq t \leq 1$

$\vec{r}'(t) = \langle -2, 0\rangle$

$ds = 2\,dt$

$\int_0^1 (4 - 4t + 3\sqrt{2})\,2\,dt = 2\left(4t - 2t^2 + 3\sqrt{2}t\right)\Big|_0^1 = 2(4-2+3\sqrt{2}) = 4 + 6\sqrt{2}$

C_4 : $\vec{r}(t) = (1-t)\langle 0,2\rangle + t\langle 0,0\rangle$

$= \langle 0, 2-2t\rangle \qquad 0 \leq t \leq 1$

$\vec{r}'(t) = \langle 0, -2\rangle$

$ds = 2\,dt$

$\int_0^1 3\sqrt{2-2t}\;2\,dt = -\cancel{3}\cdot\frac{2}{\cancel{3}}(2-2t)^{3/2}\Big|_0^1$

$d(2-2t) = -2\,dt \qquad = -2\,(0-2\sqrt{2}) = 4\sqrt{2}$

$\int_C (2x+3\sqrt{y})\,ds = 4+8+4\sqrt{2}+4+6\sqrt{2}+4\sqrt{2} = 16+14\sqrt{2}$

15.3 (26)

26. $\int_C (2x-3y+1)\,dx - (3x+y-5)\,dy$

M N

$M_y = -3$ $N_x = -3$

$M_y = N_x \Rightarrow$ conservative

a) & d) $W = 0$

b) $M = f_x = 2x - 3y + 1$

$f(x,y) = \int 2x - 3y + 1 \, dx$

$= x^2 - 3xy + x + h(y)$

$f_y = -3x + h'(y) \longrightarrow$ $f_y = N$

$\cancel{-3x} + h'(y) = \cancel{-3x} - y + 5$

$h'(y) = -y + 5$

$h(y) = -\frac{1}{2}y^2 + 5y$

$f(x,y) = x^2 - 3xy + x - \frac{1}{2}y^2 + 5y$

$W = f(\text{end pt.}) - f(\text{init. pt.})$

$= f(0,1) - f(0,-1)$

$= -\frac{1}{2} + 5 - (-\frac{1}{2} - 5)$

$= 10$

c) $W = f(\text{end pt.}) - f(\text{init. pt.})$

$= f(2, e^2) - f(0,1)$

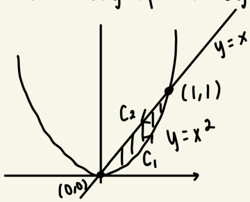
$= 4 - 6e^2 + 2 - \frac{1}{2}e^4 + 5e^2 - (-\frac{1}{2} + 5)$

$= \frac{3}{2} - \frac{1}{2}e^4 - e^2$

15.4 (5, 6, 18)

5-6) verify $\int_C y^2 dx + x^2 dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$

5. C: boundary of the region lying between the graphs of $y=x$ and $y=x^2$



$$x = x^2$$
$$x^2 - x = 0$$
$$x(x-1) = 0$$
$$x=0, \quad x=1$$

RHS: $N = x^2 \qquad M = y^2$

$N_x = 2x \qquad M_y = 2y$

$N_x - M_y = 2x - 2y = 2(x-y)$

$$2\int_0^1 \int_{x^2}^{x} x-y \; dy\,dx = 2\int_0^1 xy - \frac{1}{2}y^2 \Big|_{x^2}^{x} dx$$
$$= 2\int_0^1 x^2 - \frac{1}{2}x^2 - \left(x^3 - \frac{1}{2}x^4\right) dx$$
$$= 2\int_0^1 \frac{1}{2}x^2 - x^3 + \frac{1}{2}x^4 \, dx$$
$$= 2\left(\frac{1}{6}x^3 - \frac{1}{4}x^4 + \frac{1}{10}x^5\right)\Big|_0^1$$
$$= 2\left(\frac{1}{6} - \frac{1}{4} + \frac{1}{10}\right) = \frac{1}{3} - \frac{1}{2} + \frac{1}{5} = \frac{10-15+6}{30} = \frac{1}{30}$$

LHS: C_1: $y = x^2$

$dy = 2x\,dx$

$$\int_0^1 x^4 dx + x^3 \, 2dx = \int_0^1 x^4 + 2x^3 \, dx$$
$$= \frac{1}{5}x^5 + \frac{1}{2}x^4 \Big|_0^1 = \frac{1}{5} + \frac{1}{2} = \frac{7}{10}$$

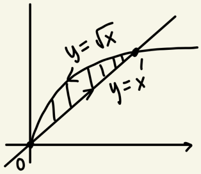
C_2: $y = x$

$dy = dx$

$$\int_1^0 x^2 dx + x^2 dx = \int_1^0 2x^2 dx = \frac{2}{3}x^3\Big|_1^0 = -\frac{2}{3}$$

$$\int_C y^2 dx + x^2 dy = \frac{7}{10} - \frac{2}{3} = \frac{21-20}{30} = \frac{1}{30}$$

6. C: boundary of the region lying between the graphs of $y=x$ & $y=\sqrt{x}$



$N = x^2 \qquad M = y^2$

$N_x = 2x \qquad M_y = 2y$

$N_x - M_y = 2(x-y)$

RHS:
$$2\int_0^1 \int_x^{\sqrt{x}} x-y \, dy\, dx = 2\int_0^1 xy - \frac{1}{2}y^2 \Big|_x^{\sqrt{x}} dx$$
$$= 2\int_0^1 x\sqrt{x} - \frac{1}{2}x - \left(x^2 - \frac{1}{2}x^2\right) dx$$
$$= 2\int_0^1 x^{3/2} - \frac{1}{2}x - \frac{1}{2}x^2 \, dx$$
$$= 2\left(\frac{2}{5}x^{5/2} - \frac{1}{4}x^2 - \frac{1}{6}x^3\right)\Big|_0^1$$
$$= 2\left(\frac{2}{5} - \frac{1}{4} - \frac{1}{6}\right) = \frac{4}{5} - \frac{1}{2} - \frac{1}{3} = \frac{24-15-10}{30} = -\frac{1}{30}$$

LHS: C_1 : $y = x$

$dy = dx$
$$\int_0^1 x^2 dx + x^2 dx = 2\int_0^1 x^2 dx$$
$$= 2 \cdot \frac{1}{3}x^3 \Big|_0^1 = \frac{2}{3}$$

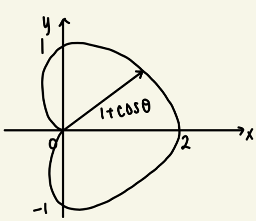
C_2 : $y = \sqrt{x}$

$dy = \frac{1}{2\sqrt{x}} dx$
$$\int_1^0 x\,dx + x^2 \frac{1}{2\sqrt{x}} dx = \int_1^0 x + \frac{1}{2}x^{3/2} \, dx$$
$$= \frac{1}{2}x^2 + \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{5}x^{5/2} \Big|_1^0$$
$$= -\frac{1}{2} - \frac{1}{5} = -\frac{7}{10}$$
$$\int_C y^2 dx + x^2 dy = \frac{2}{3} - \frac{7}{10} = \frac{20-21}{30} = -\frac{1}{30}$$

18. use Green's Thm to evaluate

$\int_C (x^2 - y^2)\,dx + 2xy\,dy$

C: $r = 1 + \cos\theta$



$M = x^2 - y^2 \qquad N = 2xy$

$M_y = -2y \qquad N_x = 2y$

$N_x - M_y = 2y + 2y = 4y$

$$\int_0^{2\pi}\int_0^{1+\cos\theta} 4y \, r\,dr\,d\theta = 4\cancel{\int_0^{2\pi}}\int_0^{1+\cos\theta} r\underbrace{\cancel{\sin\theta}}_{0} \, r\,dr\,d\theta = 0$$

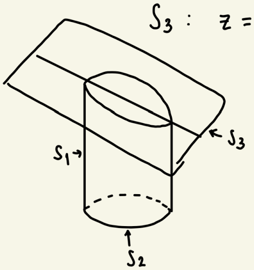
15.6 (class example)

Evaluate $\iint_S z \, dS$

S: S_1 : cylinder $x^2+y^2=1$

S_2 : $z=0$

S_3 : $z = 1 + x$



S_1: $r(z,\theta) = \langle \cos\theta, \sin\theta, z \rangle$

$r_z = \langle 0,0,1 \rangle$

$r_\theta = \langle -\sin\theta, \cos\theta, 0 \rangle$

$$r_z \times r_\theta = \begin{vmatrix} i & j & z \\ 0 & 0 & 1 \\ -\sin\theta & \cos\theta & 0 \end{vmatrix} = -\cos\theta \, i - \sin\theta \, j = \langle -\cos\theta, -\sin\theta, 0 \rangle$$

$$\|r_z \times r_\theta\| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

$$\int_0^{2\pi}\int_0^{1+\cos\theta} z \, dz \, d\theta = \int_0^{2\pi} \frac{1}{2}(1+\cancel{2\cos\theta}^{\,0}+\cos^2\theta)\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} + \frac{1}{2}\left(\frac{1+\cancel{\cos 2\theta}^{\,0}}{2}\right) d\theta$$

$$= \frac{1}{2} + \frac{1}{4} \Big|_0^{2\pi}$$

$$= \frac{3}{4}(2\pi) = \frac{3\pi}{2}$$

S_2 : $z=0$

$$\iint_S z \, ds = 0$$

S_3 : $z = 1+x$

$g_x = 1 \quad g_y = 0$

$$dS = \sqrt{1+g_x^2+g_y^2}\, dA = \sqrt{2}\, dA$$

$$\int_0^{2\pi}\int_0^1 (1+x)\, dS = \int_0^{2\pi}\int_0^1 (1+r\cos\theta)\sqrt{2}\, r\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 r + r^2\cos\theta \, dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi} \frac{1}{2}r^2 + \frac{1}{3}r^3\cos\theta \Big|_0^1 \, d\theta$$

$$= \sqrt{2}\int_0^{2\pi} \frac{1}{2} + \frac{1}{3}\cancel{\cos\theta}^{\,0} \, d\theta$$

$$= \frac{\sqrt{2}}{2}(2\pi) = \sqrt{2}\,\pi$$

$$V = \int S_1 + S_2 + S_3 = \frac{3\pi}{2} + \pi\sqrt{2} = \pi\left(\frac{3}{2}+\sqrt{2}\right)$$

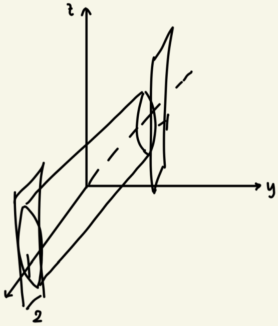
15.7 (class example 1, 18)

CE 1 $\vec{F} = \langle 3xy^2, xe^z, z^3 \rangle$

$S_1: \ y^2 + z^2 = 1$

$S_2: \ x = -1$

$S_3: \ x = 2$



$\text{div}F = 3y^2 + 0 + 3z^2 = 3(y^2+z^2)$

$= 3r^2$

$$\text{Total flux} = \int_0^{2\pi}\int_0^1\int_{-1}^{2} 3r^2 \, dx\,r\,dr\,d\theta$$

$$= 2\pi \int_0^1 9r^3 \, dr$$

$$= 18\pi \cdot \frac{1}{4} = \frac{9\pi}{2}$$

18. $F(x,y,z) = \langle xy, 4y, xz \rangle$

$S: x^2+y^2+z^2 = 16 \longrightarrow$ sphere

Evaluate: $\iint_S F \cdot N \, dS = \iiint_Q \text{div}F \, dV$

$\text{div}F = y + 4 + x = x + y + 4$

$$\text{total flux} = \int_0^{2\pi}\int_0^{\pi}\int_0^{4} (\cancelto{0}{4\sin\phi\cos\theta} + \cancelto{0}{4\sin\phi\sin\theta} + 4)\,\rho^2\sin\phi \, d\rho\, d\phi\, d\theta$$

$$2\pi \int_0^{\pi}\int_0^{4} 4\rho^2 \sin\phi \, d\rho\, d\phi$$

$$\frac{8\pi}{3}\int_0^{\pi} 64\sin\phi \, d\phi$$

$$\frac{512\pi}{3}(-\cos\pi + \cos 0) = \frac{1024\pi}{3}$$

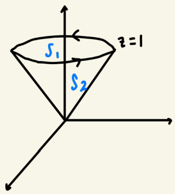
15.8 (class example #1, CE #2, 10)

CE #1 $\vec{F} = \langle \sin x - \frac{y^3}{3},\ \cos y + \frac{x^3}{3},\ xyz \rangle$

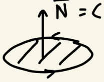
S : S_1, disk $z=1$

S_2, $z = \sqrt{x^2+y^2}$ → cone

orientation : upward



$\vec{N} = \langle 0,0,1 \rangle = \nabla G$

S_1 :  $z=1$

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin x - \frac{y^3}{3} & \cos y + \frac{x^3}{3} & xyz \end{vmatrix}$$

$$= \langle xz,\ -yz,\ x^2+y^2 \rangle$$

$$\text{curl } F \cdot N = \langle xz, -yz, x^2+y^2 \rangle \cdot \langle 0,0,1 \rangle$$

$$= x^2+y^2 = r^2$$

$$\text{flux} = \int_0^{2\pi}\int_0^1 r^2 r\, dr\, d\theta = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}$$

S_2 : $z = \sqrt{x^2+y^2}$

$G = z - \sqrt{x^2+y^2}$

$\nabla G = \langle -\frac{x}{\sqrt{x^2+y^2}},\ -\frac{y}{\sqrt{x^2+y^2}},\ 1 \rangle$

$\text{curl } F \cdot \nabla G$ = sub z

$$= -\frac{x^2 z}{\sqrt{x^2+y^2}} + \frac{y^2 z}{\sqrt{x^2+y^2}} + x^2+y^2$$

$$= -\frac{\cancel{x^2 z}}{\cancel{z}} + \frac{y^2 \cancel{z}}{\cancel{z}} + \cancel{x^2} + y^2$$

$$= 2y^2$$

$$\int_0^{2\pi}\int_0^1 2r^2\sin^2\theta\, r\, dr\, d\theta = 2\int_0^{2\pi} \frac{1}{4} r^4 \sin^2\theta \Big|_0^1 d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(\frac{1-\cancel{\cos 2\theta}^{\,0}}{2}\right) d\theta$$

$$= \frac{1}{4}(2\pi) = \frac{\pi}{2}$$

Total flux: $\frac{\pi}{2} + \frac{\pi}{2} = \pi$

CE #2 $\quad F = \langle x^2y,\ 2xz - y^2,\ xyz \rangle \qquad 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ 0 \leq z \leq 1$

S: union of all faces of a cube except the bottom ← $z=0$

orientation: upward

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & 2xz - y^2 & xyz \end{vmatrix} = \langle xz - 2x,\ -yz,\ 2z - x^2 \rangle$$

$N = \langle 0, 0, 1 \rangle$

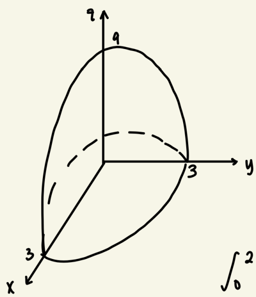
$\text{curl } F \cdot N = 2z - x^2$ (since $z=0$) $= -x^2$

$$\text{flux} = \int_0^1 \cancel{\int_0^1} -x^2 \, \cancel{dy}\,dx = -\frac{1}{3}$$

#10 $\quad F(x,y,z) = \langle 4xz,\ y,\ 4xy \rangle$

$S: z = 9 - x^2 - y^2,\ z \geq 0$

upside down bowl



$$\int_C F \cdot dr = \int_R \int \text{curl } F \cdot \overset{\vec{r}_u \times \vec{r}_v}{DG}\, dA$$

$C: x^2 + y^2 = 9$

$\vec{r}(t) = \langle 3\cos t,\ 3\sin t,\ 0 \rangle$

$\vec{r}'(t) = \langle -3\sin t,\ 3\cos t,\ 0 \rangle$

$F(t) = \langle 0,\ 3\sin t,\ 36\sin t\cos t \rangle$

$F(t) \cdot \vec{r}'(t) = 9\cos t\sin t$

$$\int_0^{2\pi} 9\cos t\sin t \, dt = 0$$