Brain-Inspired Computing
Class 6
Important types of Network structures

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Contents

Winner-take-all networks

The WTA network structure

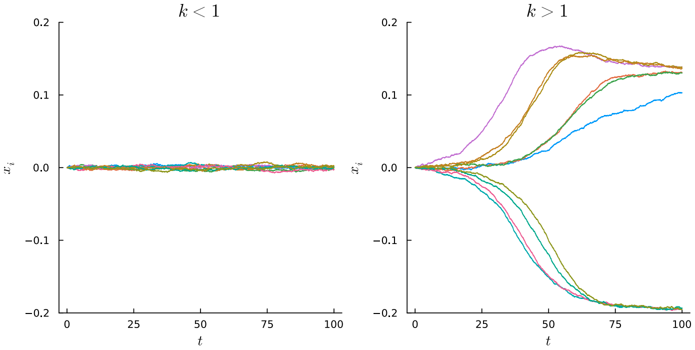
A WTA network is defined by the network model

$$\tau\dot{x}_i(t) = -x_i(t) + \sum_{j=1}^{N} A_{ij}\tanh\left(k_j\sum_{k=1}^{N} B_{jk}x_k(t) + \sum_{k=1}^{N} C_{jk}I_k(t)\right),\quad i = 1,\ldots,N$$

with $A = C = \boldsymbol{I}_N$ (the identity matrix in dimension N), $B = -\mathbf{1}_N\mathbf{1}_N^\top + \boldsymbol{I}_N$ and $k_1 = \ldots = k_N = k$.

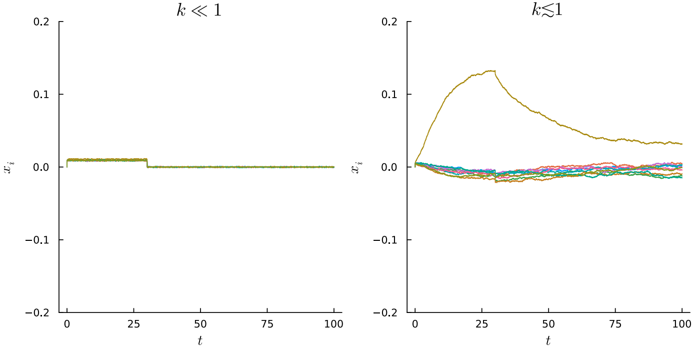
The WTA bifurcation

Observe that recurrent interconnection matrix AB has eigenvalues $\lambda_N = \lambda_{min} = -(N-1)$ and $\lambda_1 = \ldots = \lambda_{N-1} = \lambda_{max} = 1$. Hence, the model undergo a bifurcation for $k = \frac{1}{\lambda_{max}} = 1$. The bifurcation has an $N-1$ dimensional kernel $\mathbf{1}_N^\perp$. This implies that all the equilibria $\boldsymbol{x}_{eq}$ that are born at the bifurcation satisfy $\langle \mathbf{1}_N, \boldsymbol{x}_{eq} \rangle \approx 0$. There are many such equilibria! Which ones are stable depends on the details of the vector field and also on the number of agents.



The WTA input-output behavior

In the presence of inputs, WTA networks can be used as highly selective input-difference amplifiers. In this simulations we are injecting a pulse of current inside each neuron with one neuron receiving a slightly larger input with respect to the others. Close to bifurcations, the network **selects the node with the largest input**. This happens because of the ≈ -0 eigenvalue and because also in the presence of inputs $\langle \mathbf{1}_N, \boldsymbol{x}_{eq} \rangle \approx 0$. Selectivity and amplification decrease as the coupling gain k is decreased, which makes the network tuneable.



Generalized WTA networks

In the bifurcation associated to a WTA network, the resulting equilibria satisfy $\langle \mathbf{1}_N, \boldsymbol{x}_{eq} \rangle \approx 0$, i.e., $\boldsymbol{x}_{eq} \in \mathbf{1}_N^\perp$. It can be shown that only the projection of the inputs on $\mathbf{1}_N^\perp$ get amplified by the network: when all inputs are equal, the network does not respond or responds only linearly.

Project 1: Design networks for which bifurcating equilibria lies on an arbitrary $N-1$-dimensional subspace $\boldsymbol{v}^\perp$, $\boldsymbol{v} \in \mathbb{R}^N$, and for which only the input component on (a possibly different) $N-1$-dimensional subspace $\boldsymbol{w}^\perp$, $\boldsymbol{w} \in \mathbb{R}^N$ is amplified by the network.
Application: "generalized difference" amplification, i.e., if $\mathrm{sign}(\boldsymbol{w}_i) = \mathrm{sign}(\boldsymbol{w}_j)$, then I_i and I_j should have the opposite sign (in such way that $\boldsymbol{w}_i I_i + \boldsymbol{w}_j I_j \approx 0$) to promote amplification, and viceversa when $\mathrm{sign}(\boldsymbol{w}_i) \neq \mathrm{sign}(\boldsymbol{w}_j)$.

Project 2: same as Project 1, but with "amplification subspaces" of dimension $1 < m < N-1$. Of course, for $m = 1$ we recover the pitchfork case.

Neural fields

Rate models

The network model we considered in last class

$$\tau \dot{x}_i(t) = -x_i(t) + \sum_{j=1}^{N} A_{ij} \tanh\left(k_j \sum_{k=1}^{N} B_{jk} x_k(t) + \sum_{k=1}^{N} C_{jk} I_k(t) \right), \quad i = 1, \dots, N$$

is an instance of a **rate model** that aims at representing the collective activity of large group of neurons [4, 3].

In the rate model context, x_i is the "activity rate", usually, the **firing rate**, of a neuron or a neuronal population. The sigmoid $\tanh(\cdot)$ captures the **nonlinearity of synaptic transmission**. Matrices A, B encode the **recurrent synaptic interconnection topology** of the underlying neural network, while matrices A, C encode the **feed-forward synaptic interconnection topology**. The exogenous inputs I_k can either represent sensory stimuli or the activity of presynaptic neurons/neuronal populations.

From rate models to neural fields

It sometimes makes sense to consider **spatially extended** neuronal populations, in which the neuronal index is not discrete but continuous. In this case, the rate model is called a **neural field** [2, 1].

Consider for instance a population of neurons representing information about a continuous variables, for instance, the direction $\theta \in [0, 2\pi)$ of a perceived motion. Let $a(t, \theta)$ be the activity of neurons encoding for the presence of an edge oriented at angle θ at time t. Then, we can let

$$\tau a_t(t, \theta) = -a(t, \theta) + \tanh\left(\int_0^{2\pi} \left(W^{rec}(\theta, \theta')a(t, \theta') + W^{ff}(\theta, \theta')I(\theta')\right) d\theta'\right)$$

This is an **infinite-dimensional** dynamical system. The coupling functions $W^{rec}(\theta, \theta')$ and $W^{ff}(\theta, \theta')$ are called the **recurrent interconnection kernel** and the **feedforward interconnection kernel**. For $\theta \in [0, 2\pi)$, $W^{rec}(\theta, \theta') = W^{rec}(\theta - \theta')$ and $W^{ff}(\theta, \theta') = W^{ff}(\theta - \theta')$ are periodic functions.

In practice, in order to simulate it, we partition the angle $[0, 2\pi)$ into N element and use the Riemann approximation of the integral in order to obtain a rate model as above. We can also apply directly network bifurcation theory to the resulting finite-dimensional system system.

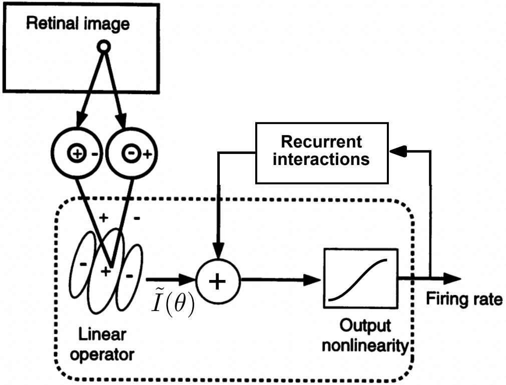
A neural fields for orientation perception

Example: Continuous orientation perception.

In the neural field model, define $W^{rec}(\theta,\theta')=0$ and

$$W^{ff}(\theta,\theta')=A^{ff}e^{-\frac{\left|\sin\left(\frac{\theta-\theta'}{2}\right)\right|}{\sigma_\theta}}$$

and let $I(\theta)$ be the convolution of a small region of a retinal image with a Gabor filter at orientation θ. The parameter $\sigma_\theta>0$ determines the **selectivity** of the neuronal responses. For $\sigma_\theta\to 0$ and $A^{ff}\propto\frac{1}{\sigma_\theta}$ the function $A^{ff}e^{-\frac{\left|\sin\left(\frac{\theta-\theta'}{2}\right)\right|}{\sigma_\theta}}$ becomes a Dirac delta and selectivity is maximal.



$$\tilde{I}(\theta)=\int_0^{2\pi}W^{ff}(\theta,\theta')I(\theta')d\theta$$

More general neural fields

The continuous variable parameterizing the activity of a neural field can live in a general space Ω. In this case,

$$\tau a_t(t,\theta) = -a(t,\theta) + \tanh\left(\int_\Omega \left(W^{rec}(\theta,\theta')a(t,\theta') + W^{ff}(\theta,\theta')I(\theta')\right)d\theta'\right)$$

In this case, the kernels must satisfy functional forms compatible with the **topology** of the space Ω.

[1] Paul C Bressloff. "Waves in neural media". In: *Lecture notes on mathematical modelling in the life sciences* (2014), pp. 18–19.

[2] Stephen Coombes et al. *Neural fields: theory and applications*. Springer, 2014.

[3] X-J. Wang. *Theoretical Neuroscience of Cognition*. 2023.

[4] Hugh R Wilson. "Spikes, decisions, and actions: the dynamical foundations of neurosciences". In: (1999).

Normalization networks

Input normalization

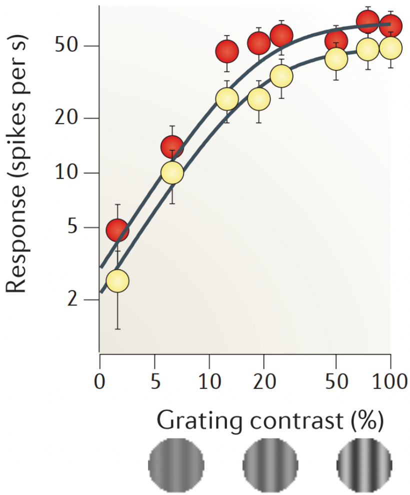
Normalization is a simple way of detecting **contrast** in incoming inputs. Here *contrast* is in a **generalized sense**, for instance, contrast between perceived orientations.

In the example above, including the effects of input normalization leads to

$$\tilde{I}(\theta) = \int_0^{2\pi} A^{ff} \frac{I(\theta)^n}{I_0^n + I_{sum}^n} e^{-\frac{\left|\sin\left(\frac{\theta-\theta'}{2}\right)\right|}{\sigma_\theta}} I(\theta')d\theta',$$

where $I_{sum} = \frac{1}{2\pi}\int_0^{2\pi} I(\theta')d\theta'$ that is, the input at orientation θ is normalized by the average input intensity at all orientations. The discrete version can be defined similarly.

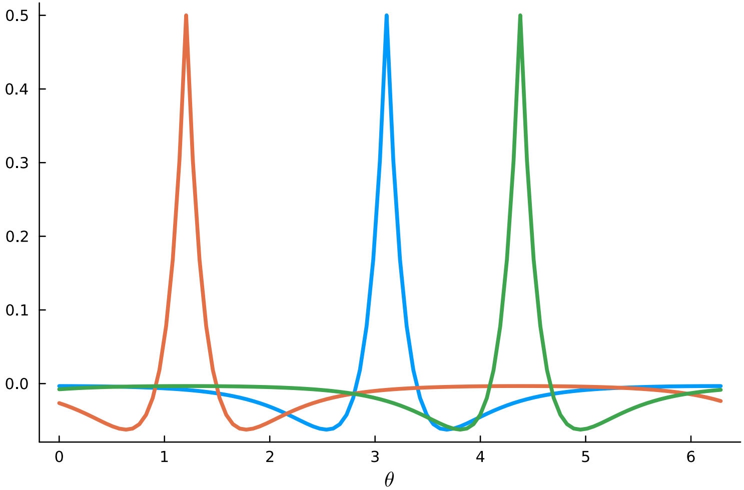
Normalization is a fundamental computational primitive of the brain [1].



[1] Matteo Carandini and David J Heeger. “Normalization as a canonical neural computation”. In: *Nature Reviews Neuroscience* 13.1 (2012), pp. 51–62.

Periodic local-excitation global-Inhibition network: non-linear spatial band-pass filtering

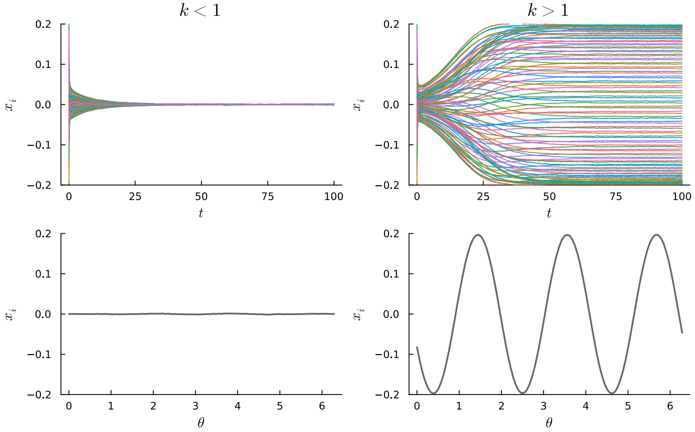
The local excitation, global inhibition recurrent interconnection kernel



Eigenvalues, eigenvectors, and network bifurcation

We can understand the
Because the matrix B (the discrete N-spatial point realization of the kernel) is **circulant**, its eigenvectors are all the **discrete fourier modes** in dimension N.
The Fourier modes corresponding to the largest eigenvalue of B determines the type of equilibria emerging at a bifurcation in the resulting neural field. Using the network theory, the bifurcation will happen for $k = \frac{1}{\lambda_{max}}$ and it has a 2-dimensional kernel spanned by the dominant Fourier mode.

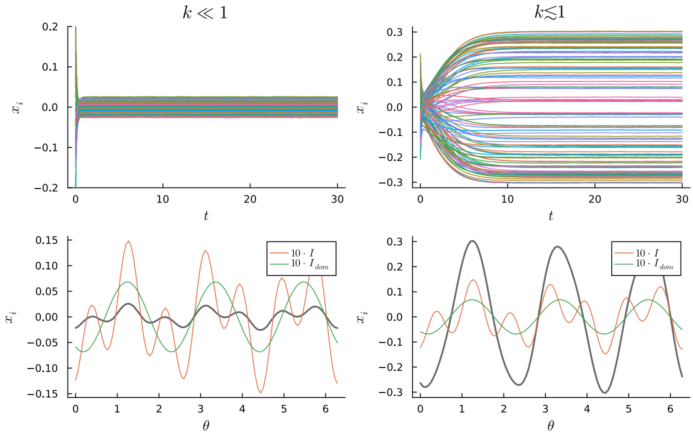
Eigenvalues, eigenvectors, and network bifurcation



Nonlinear, tunable spatial mode amplification

We now study the effect of inputs on the local-excitation, global inhibition model. Using the network theory, it is easy to see that inputs that are resonant with the dominant Fourier mode are strongly amplified close to bifurcation. In the next example with add two small inputs, one that is resonant with the dominant Fourier mode and one that is not resonant.

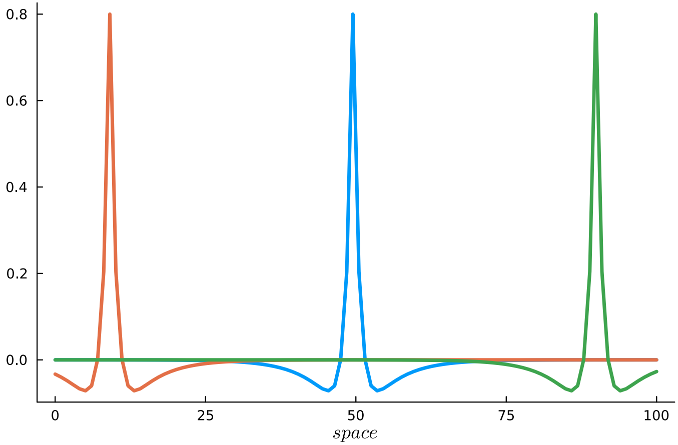
Nonlinear, tunable spatial mode amplification



Non-periodic local-excitation global-Inhibition network

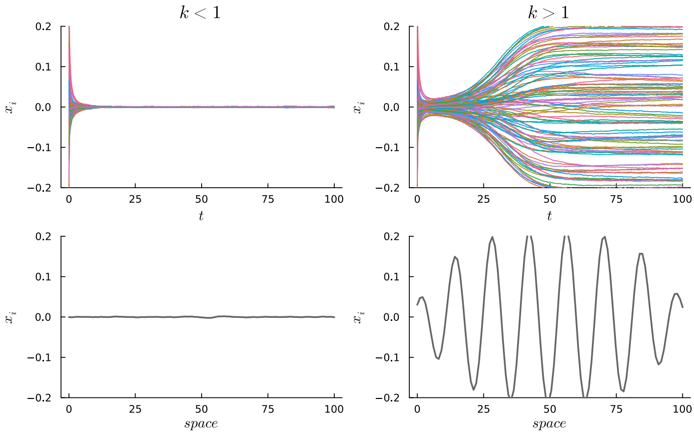
Non-periodic LEGI network

The network is similar to the periodic case, but the recurrent interconnection kernel now is not periodic.



Non-periodic LEGI network: bifurcation behavior

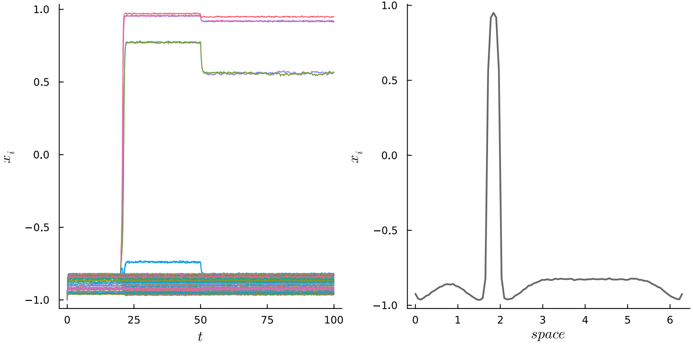
The discrete coupling matrix B is not circulant, but its modes are still quite close to those of a circulant matrix. Using the network theory, the bifurcation will happen for $k = \frac{1}{\lambda_{max}}$ and it has a 2-dimensional kernel spanned by the dominant modes.



Non-periodic LEGI network: bump attractors

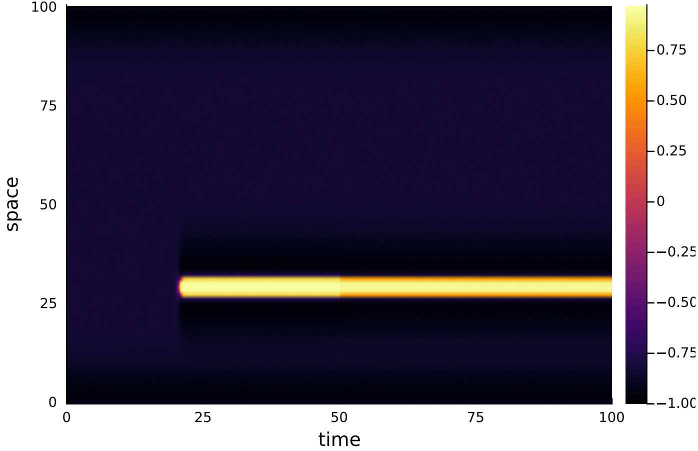
We now study the behavior of the non-periodic LEGI network **past bifurcation** but with a global inhibitory input that stabilizes a low spatially-homogeneous state.

Furthermore we add small spatially-localized inputs that locally excite the network. The resulting behavior is a **bump attractor** that keeps a **dynamical working memory** of the excitatory input.



Non-periodic LEGI network: bump attractors

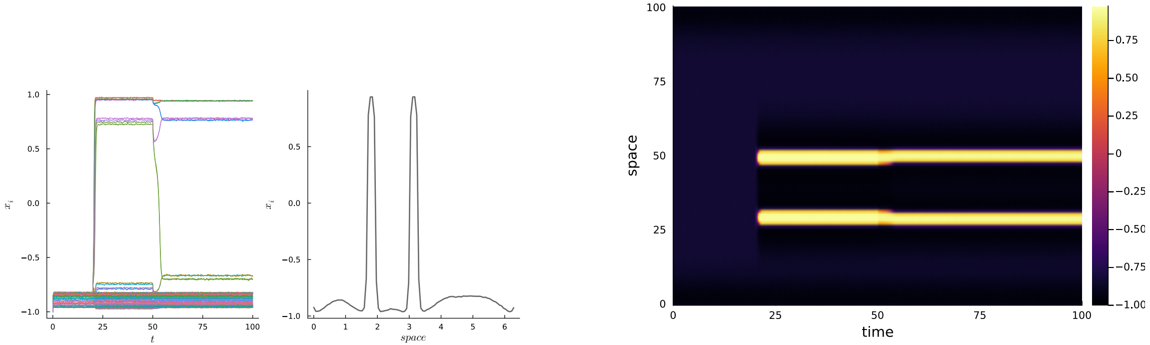
It is useful to visualize this response using a spatio-temporal plot (heatmap)



Bump attractors as "spatial spikes"

Non-periodic LEGI network: bump attractors

When multiple inputs excite the network, both can be remembered, provided they are sufficiently far apart.



Non-periodic LEGI network: bump attractors

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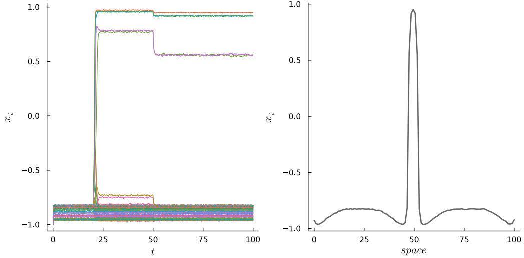
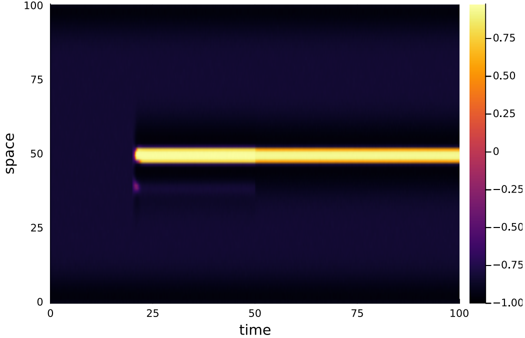
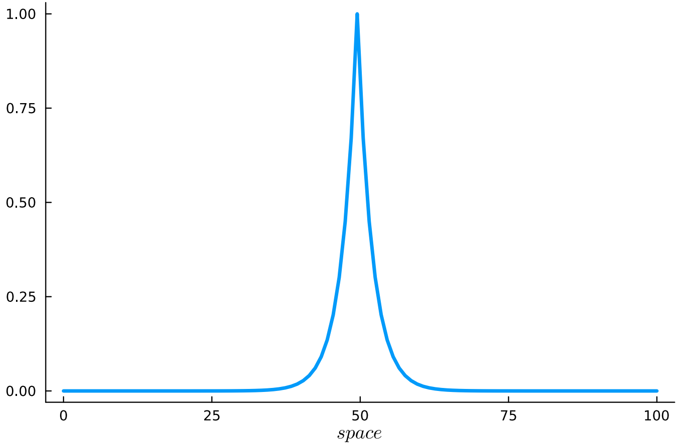




Traveling waves

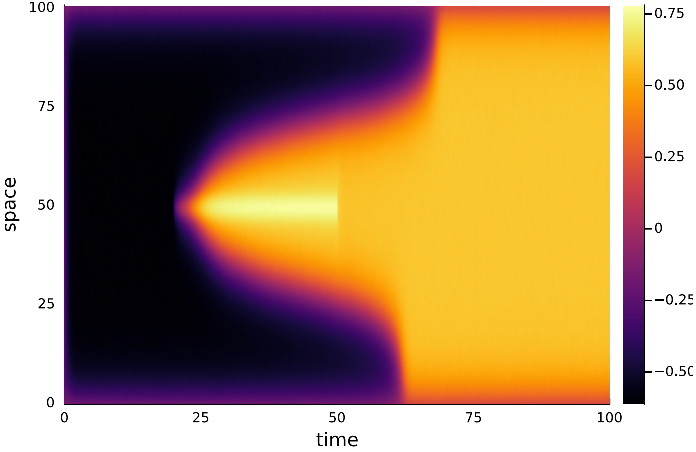
Traveling front

Consider a network with recurrent interconnection kernel.



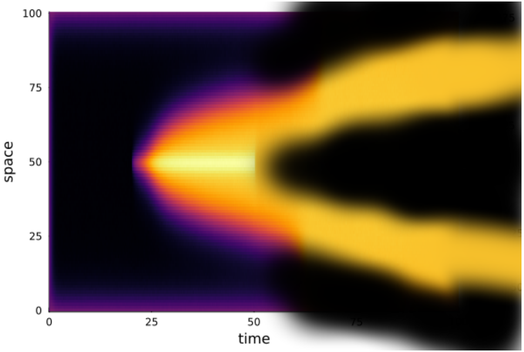
Traveling front

When the network is slightly above the bifurcation point, it is **bistable** and a short pulse can switch it from the low to the up state through a **traveling front**.



From traveling fronts to traveling spikes

If each node in the network is provided with **slow negative feedback**, the front becomes a traveling pulse.

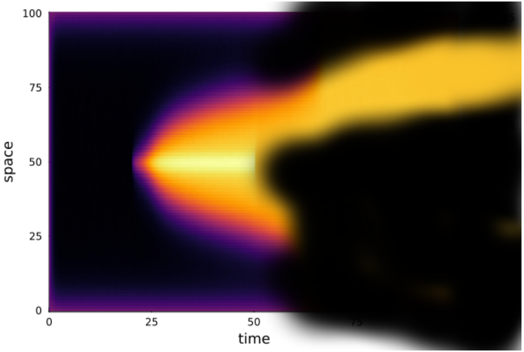


Exercise: encode this model and actually simulate it.

Anisotropic traveling spikes networks

Using an anisotropic excitatory kernel,

we can only allow traveling pulses in one direction.



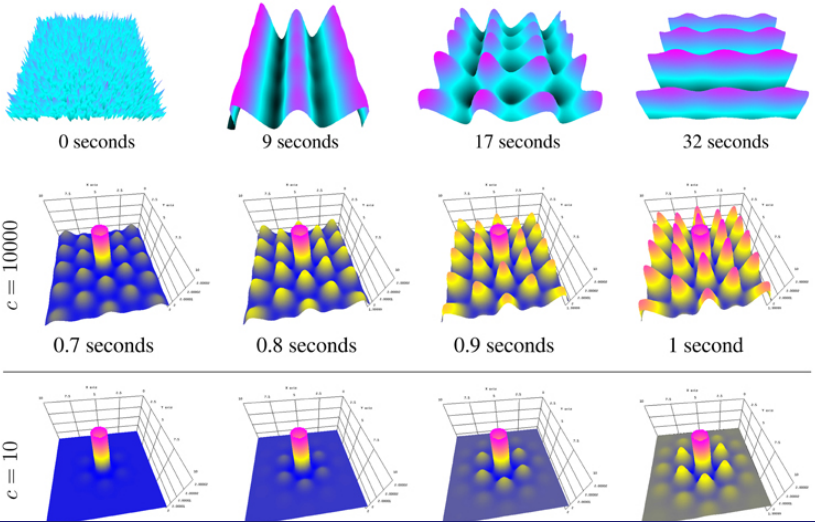
Project: Apply this idea to directed movement detection.

Project: Another application: spatiotemporal movement generation.

2D-spatially extended networks

From one to two spatial dimension

Everything generalize and becomes richer (figures from [3] - see [2, 1] for theory and examples).



[1] Paul C Bressloff. "Waves in neural media". In: *Lecture notes on mathematical modelling in the life sciences* (2014), pp. 18–19.

[2] Stephen Coombes et al. *Neural fields: theory and applications*. Springer, 2014.

[3] Eric J Nichols and Axel Hutt. "Neural field simulator: two-dimensional spatio-temporal dynamics involving finite transmission speed". In: *Frontiers in neuroinformatics* 9 (2015), p. 25.

Visual hallucinations

A spatio-temporal model of visual hallucinations

THE ROYAL SOCIETY

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Geometric visual hallucinations, Euclidean symmetry and the functional architecture of striate cortex

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(*a*)

(*b*)