Brain-Inspired Computing
Class 3
Multi-option event-based representations

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March 3, 2023

Contents

How do we generalize excitability to multiple options?

The simplest idea and its issues

Let's start with the two-option plus neutral category case (through localized, state-dependent positive feedback)

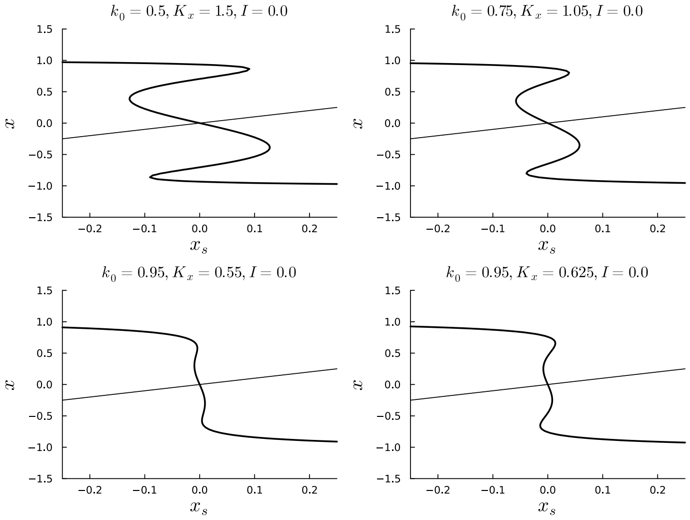
$$\tau\dot{x} = -x + \tanh\left(\tilde{k}(x)\cdot x + I(t)\right)$$

where $\tilde{k}(x) := k_0 + K_x \cdot x^2$, to ensure the existence of an excitable neutral state and two excited states (an up and a down one).

The first solution one might think about is using (as in one-option excitability) slow negative feedback on the state variable leading to

$$\begin{aligned}\tau\dot{x} &= -x + \tanh\left(\tilde{k}(x)\cdot x + I(t) - x_s\right)\\ \dot{x}_s &= \varepsilon(x - x_s)\end{aligned}$$

Nullcline analysis



Model stiffness

A fundamental property of excitable systems is the possibility of tuning their threshold. Let's explore if the proposed two-option excitable model possesses a tunable threshold.

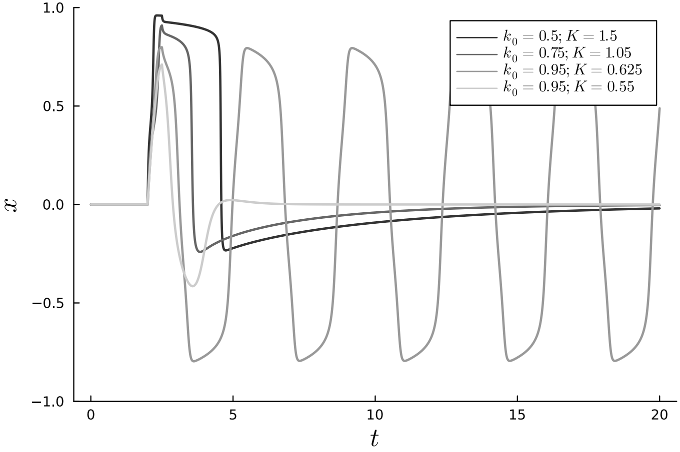
Note that to change the threshold we cannot bias the current as in the one-option case, because that would ...bias(!) event generation toward the up (down) category for $I_0 > 0$ ($I_0 < 0$). The only two parameters we can play with are k_0 and K_x.

The threshold tuner is the parameter k_0. The closer k_0 to 1, the smaller the threshold.
However to ensure that after a event is generated the model returns to the neutral equilibrium (instead of entering an oscillatory state between the up and down category) the size of the bistable region must also be changed. More precisely, order the four saddle-node bifurcation from the top to the bottom and let (x_{SN_i}, x_{s,SN_i}), $i = 1, \ldots, 4$, be their coordinates with $x_1 > x_2 > x_3 > x_4$. To ensure the state returns to the neutral equilibrium after each event is generated it must hold that

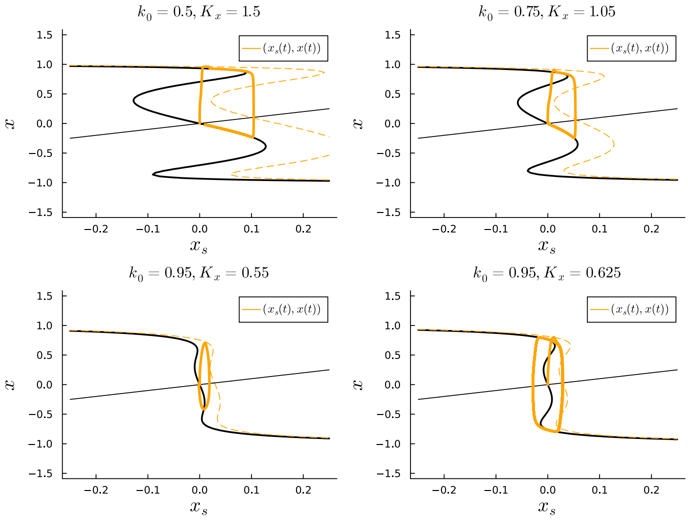
$$x_{s,SN_2} < \min(x_{s,SN_1}, x_{s,SN_4}) \leq \max(x_{s,SN_1}, x_{s,SN_4}) < x_{s,SN_3}.$$

As a result, excitability is progressively lost (in particular, events become smaller and shorter lasting) if we want our system to be arbitrarily close to threshold. There is no way to tune the threshold **and** the event size independently.

Model stiffness



Model stiffness



A solution based on symmetric bifurcations and its advantages

From bifurcation diagrams to excitable dynamics

Bifurcation diagrams with respect to the slow variable define the fast nullcline of the excitable dynamics. Because of time-scale separation between the master and adaptation variable, the fast nullcline largely determines the excitable model behavior.

Lack of symmetry of one-option excitability

Lack of symmetry of one-option excitability

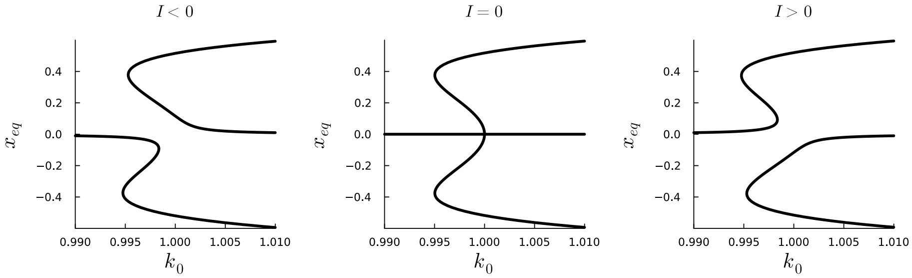
The hysteretic bifurcation diagram (and associated phase plane) can be understood to be a good model for tuneable one-option excitability in terms of symmetry. The two possible states of a one-option excitable system are asymmetric: the excitable state represents neutrality (lack of decision), the excited state represents a categorical decision. The lack of symmetry of the one-option excitable dynamics is reflected in the lack of symmetry of the hysteretic model that realizes it. Indeed, the hysteretic model of excitability (with or without state-dependent positive feedback) is not symmetric under $x \mapsto -x$. In particular, the slow adaptation dynamics "distinguishes" positive and negative values of x, as reflected by the fact that the neither nullcline of the the model is symmetric under $x \mapsto -x$.

Symmetries of $n \geq 2$-option representations

Two-option (and more generally n-option) categorical representations on the other hand posses symmetry because the two (n) categories should be intercheangeable (in the absence of inputs favoring one of them) and any category should be equally likely to be picked (depending on initial conditions and perturbations).

A bifurcation diagram respecting the symmetry of the two-option categorical representation is the (k_0, x) bifurcation diagram of the localized and state-dependent flexible representation model (similarly for the (k, x)-bifurcation diagram of its "engineering" version):

Symmetries of $n \geq 2$-option representations



We can use this bifurcation diagram as the geometrical template for a two-option slow-fast excitable phase portrait!

A symmetric 2-option excitable model

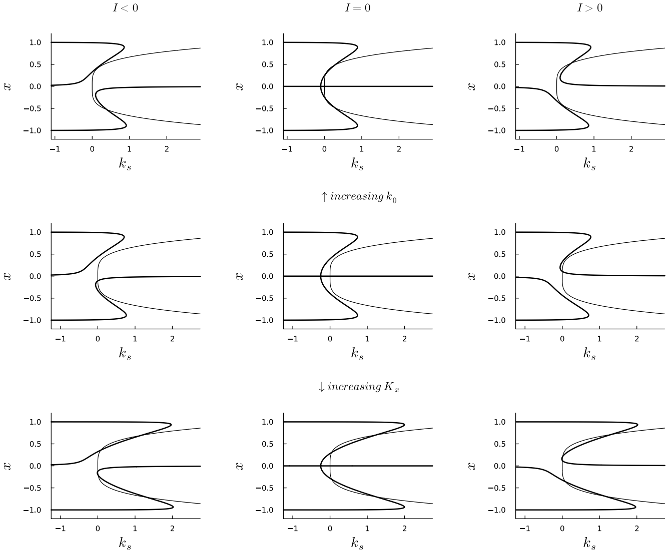
Let's consider the model

$$\tau\dot{x} = -x + \tanh\left(\left(k_0 + K_x x^2 - k_s\right)\cdot x + I(t)\right)$$
$$\dot{x}_s = \varepsilon\left((K\cdot x)^{2n} - k_s\right)$$

For $I(t) = 0$, it is symmetric with respect to $x \mapsto -x$!

Let's analyze its nullclines.

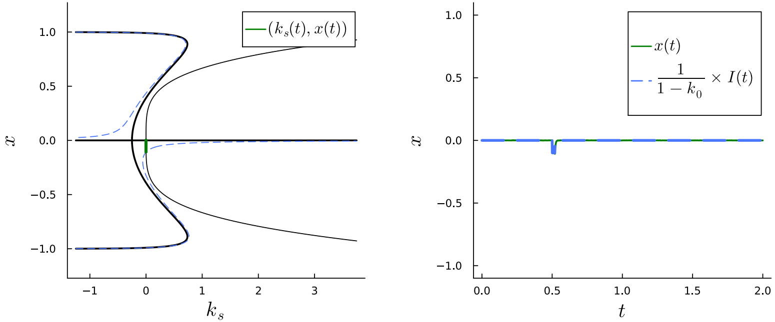
Nullcline analysis



- k_0 tunes the excitability threshold
- K_x (mostly) tunes the width of the multi-stable region (and therefore the event size)

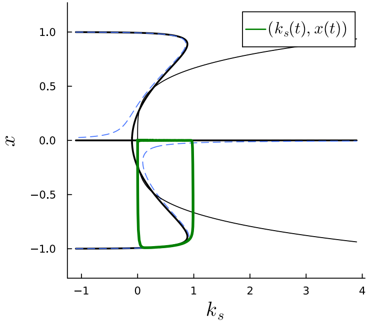
Simulation results

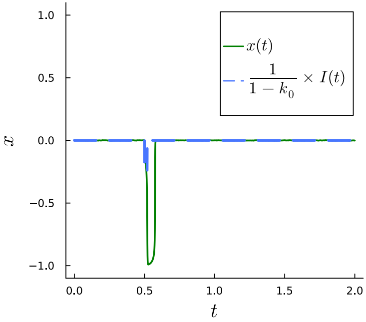
Nominal parameters



Simulation results

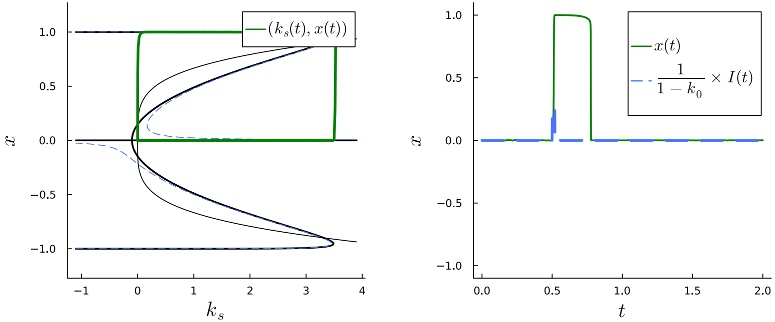
Increasing k_0





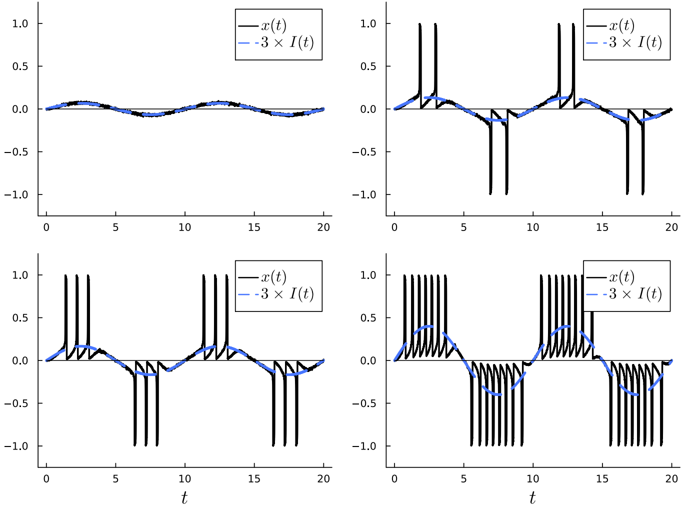
Simulation results

Increasing k_0 and K_x



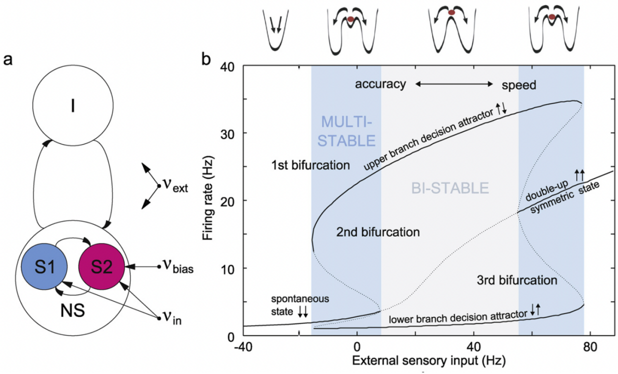
Simulation results

Sinusoidal forcing (For fast events, event rate encode for input strength!)



References?

Unfortunately, this is all completely new material. However, given the many examples of neuroscience model exhibiting a subcritical pitchfork:



From [1]

See also [2, Chapter 6]

[1] G. Deco et al. "Brain mechanisms for perceptual and reward-related decision-making". In: *Progress in Neurobiology* 103 (2013), pp. 194–213.

[2] X-J. Wang. *Theoretical Neuroscience of Cognition*. 2023.