Brain-Inspired Computing
Class 5
From multi-option (event-based) representations to network (event-based) representations

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March 14, 2023

Contents

General design of multi-option event-based representations

General design of n-option flexible representations

- Abstract the categorization/decision making process into a symmetry group $\Gamma(= S_n$, the group of permutation of n elements, for indistinguishable options)
- Design a Γ-equivariant dynamics with tunable localized/state-dependent positive feedback

$n = 2$ (1D state space)

$$\tau \dot{x} = -x + \tanh\left(\tilde{k}(x)x + I\right)$$

$$\tilde{k}(x) = k_0 + \underline{K_x \cdot x^2}$$

Option 1 → x > 0

" 2 → x < 0

$n = 2$ (2D state space)

$$\tau \dot{x}_1 = -x_1 + \tanh\left(-\tilde{k}_1(x)x_2 + I_1\right)$$

$$\tau \dot{x}_2 = -x_2 + \tanh\left(-\tilde{k}_2(x)x_1 + I_2\right)$$

(2-option WTA with neutral state)

$$\tilde{k}_i(x) = k_0 + K_x \cdot x_i^2$$ local

but also (same symmetry)

$$\tilde{k}_i(x) = k_0 + K_x \cdot \left(x_1^2 + x_2^2\right)$$ global

Exercise: study the (k_0, x) and (I, x) bifurcation diagrams for varying K_x in both models.

n (nD state space)

$$\tau \dot{x}_i = -x_i + \tanh\left(-\tilde{k}_i(x)\sum_{j \neq i} x_j + I_i\right)$$

(n-option WTA with neutral state)

$$\tilde{k}_i(x) = k_0 + K_x \cdot x_i^2$$ local

but also (same symmetry)

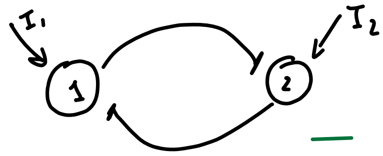
$$\tilde{k}_i(x) = k_0 + K_x \cdot \left(\sum_j x_j^2\right)$$

$\parallel$

$\tilde{k}_j(x)$

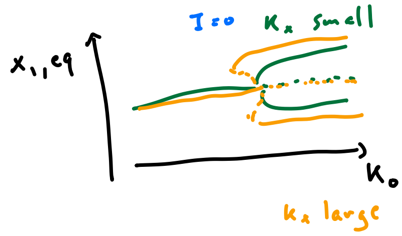
global

$K_0 > 0$, $K_x \geq 0$

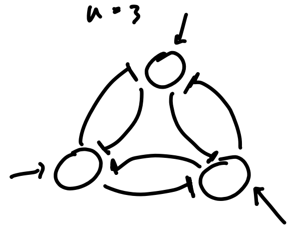


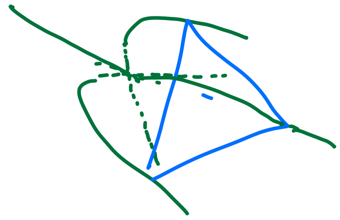
Option 1 $x_1 > x_2$

" 2 $x_2 > x_1$



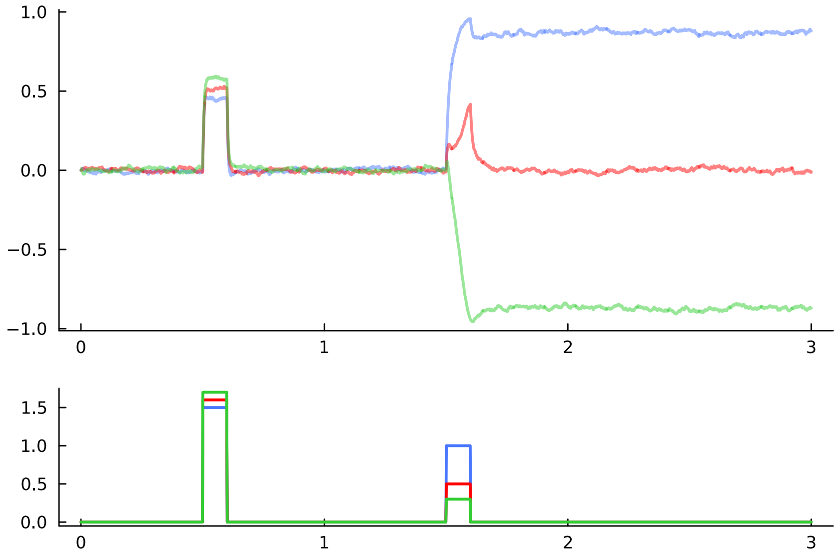
$n=3$



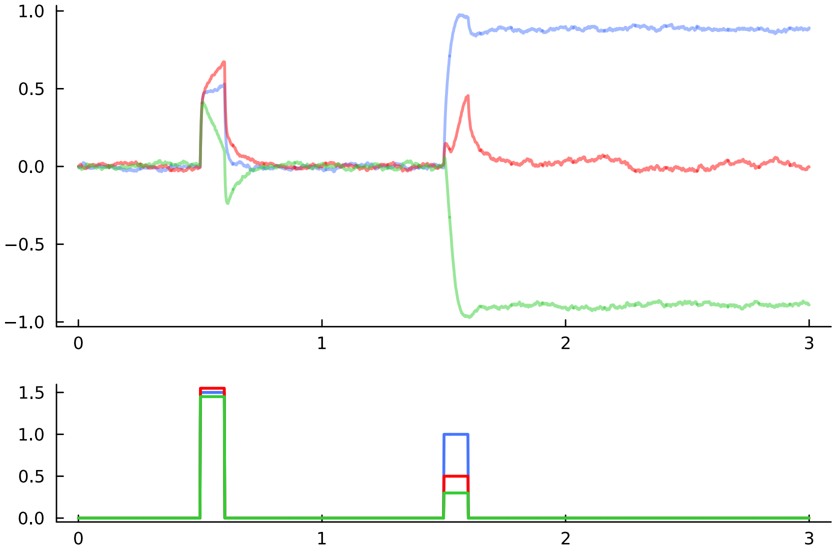


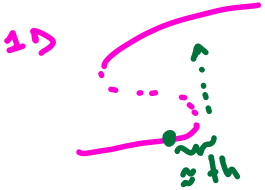
General design of n-option flexible representations

3-options, node-level fb



3-options, global fb





General design of n-option event-based representations

- Add **symmetry-preserving** slow negative feedback

$n = 2$ (1D state space)

$$\tau \dot{x} = -x + \tanh\left(\tilde{k}(x)x + I\right)$$

$$\tilde{k}(x) = k_0 + K_x \cdot x^2 \; -k_s$$

$$\dot{k}_s = \varepsilon\left((K \cdot x)^4 - k_s\right)$$

$n = 2$ (2D state space)

$$\tau \dot{x}_1 = -x_1 + \tanh\left(-\tilde{k}_1(x)x_2 + I\right)$$

$$\tau \dot{x}_2 = -x_2 + \tanh\left(-\tilde{k}_2(x)x_1 + I\right)$$

$$\tilde{k}_i(x) = k_0 + K_x \cdot x_i^2 \; -k_{s_i}$$

$$\dot{k}_{s_i} = \varepsilon\left((K \cdot x_i)^4 - k_{s_i}\right)$$

Local

but also (same symmetry)

$$\tilde{k}_i(x) = k_0 + K_x \cdot \left(x_1^2 + x_2^2\right) \; -k_s$$

$$\dot{k}_s = \varepsilon\left(K^4\left(x_1^4 + x_2^4\right) - k_s\right)$$

Glob al

n (nD state space)

$$\tau \dot{x}_i = -x_i + \tanh\left(-\tilde{k}_i(x)\sum_{j \neq i} x_j + I\right)$$

$$\tilde{k}_i(x) = k_0 + K_x \cdot x_i^2 \; -k_{s_i}$$

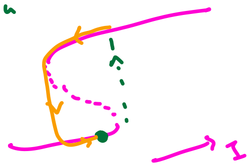
$$\dot{k}_{s_i} = \varepsilon\left((K \cdot x_i)^4 - k_{s_i}\right)$$

but also (same symmetry)

$$\tilde{k}_i(x) = k_0 + K_x \cdot \left(\sum_j x_j^2\right) \; -k_s$$

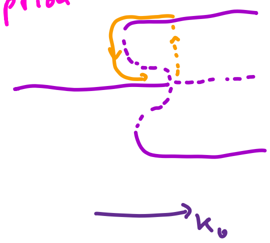
$$\dot{k}_s = \varepsilon\left(K^4\left(\sum_j x_j^4\right) - k_s\right)$$

1 option

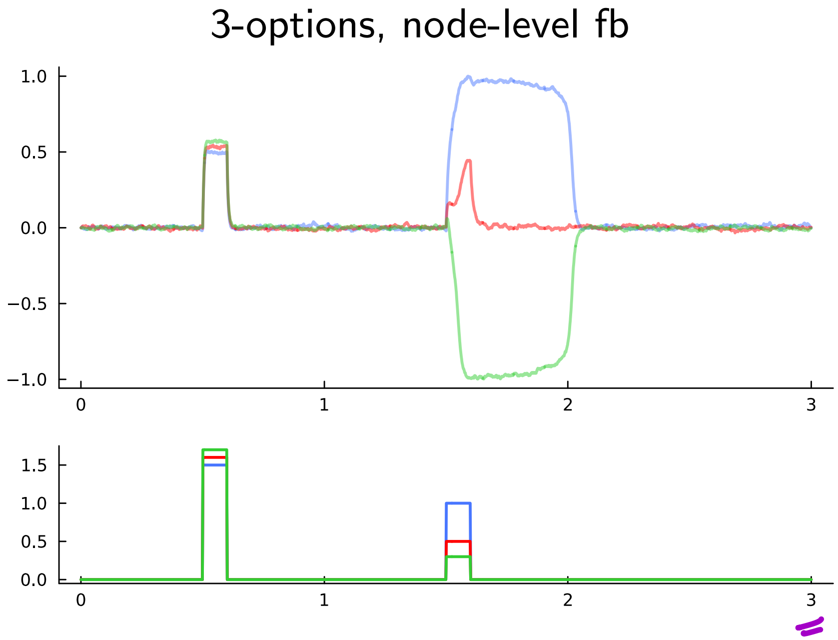


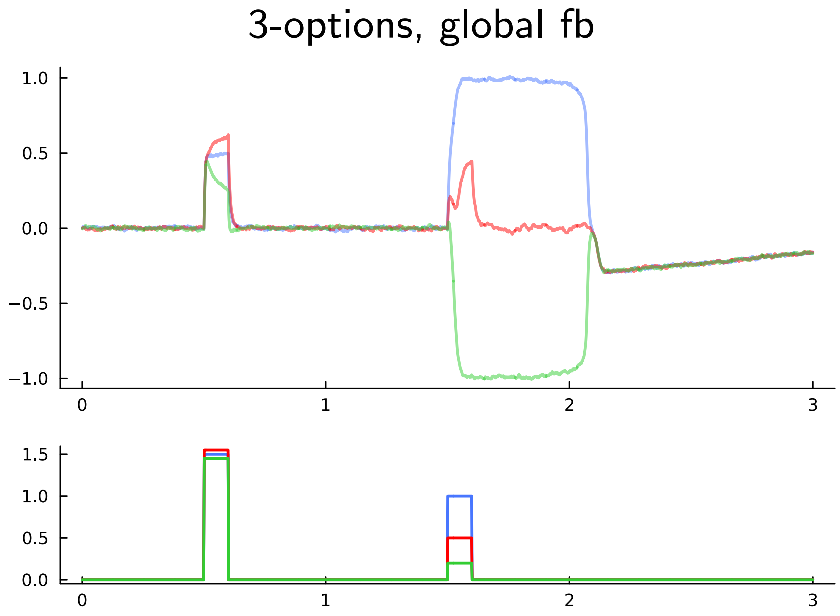
slow negative feedback

2 option

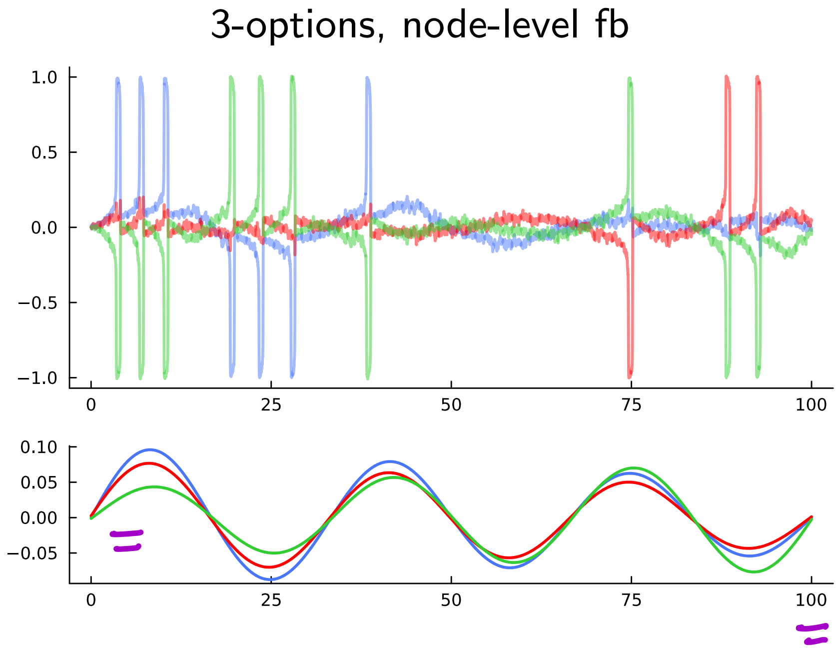


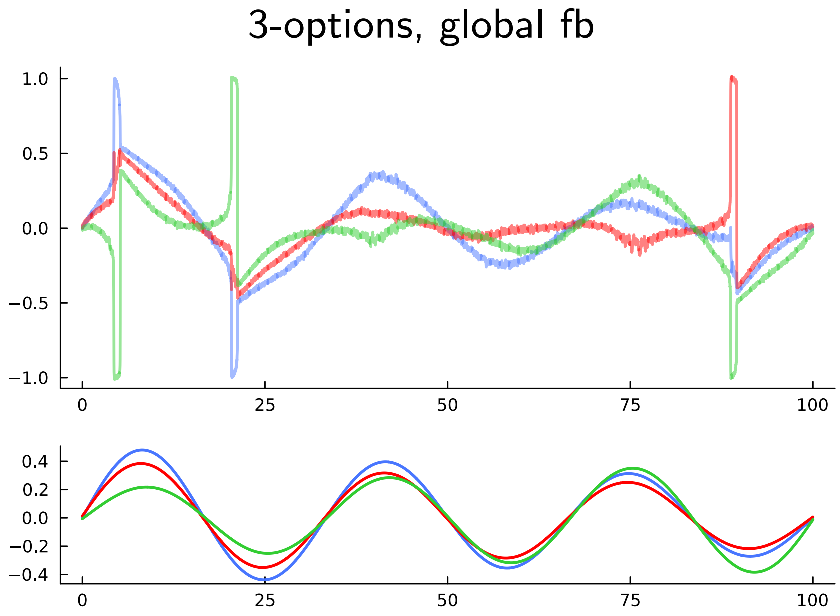
General design of n-option event-based representations





General design of n-option event-based representations





Exercise/project: Develop “MOSFET-realizable” versions of these models (using the $\mathcal{S}$ function) and draw connections with biological neuronal circuits.

A network viewpoint: 2 option case

The N agent, 2-option (≥ 0 or ≤ 0) network model

Projects $K_0 + K_i(x_r) - K_{s_i}$

$\dot{K}_{s_i} = \varepsilon(\cdots)$

$N > 1$, arbitrary otherwise

Consider the network model

Option 1: $x_i > 0$

$$\tau \dot{x}_i(t) = -x_i(t) + \sum_{j=1}^{N} A_{ij} \tanh\left(k_j \sum_{k=1}^{N} B_{jk} x_k(t) + \sum_{k=1}^{N} C_{jk} I_k(t)\right), \quad i = 1, \ldots, N$$

Option 2: $x_i < 0$

which in vector form becomes

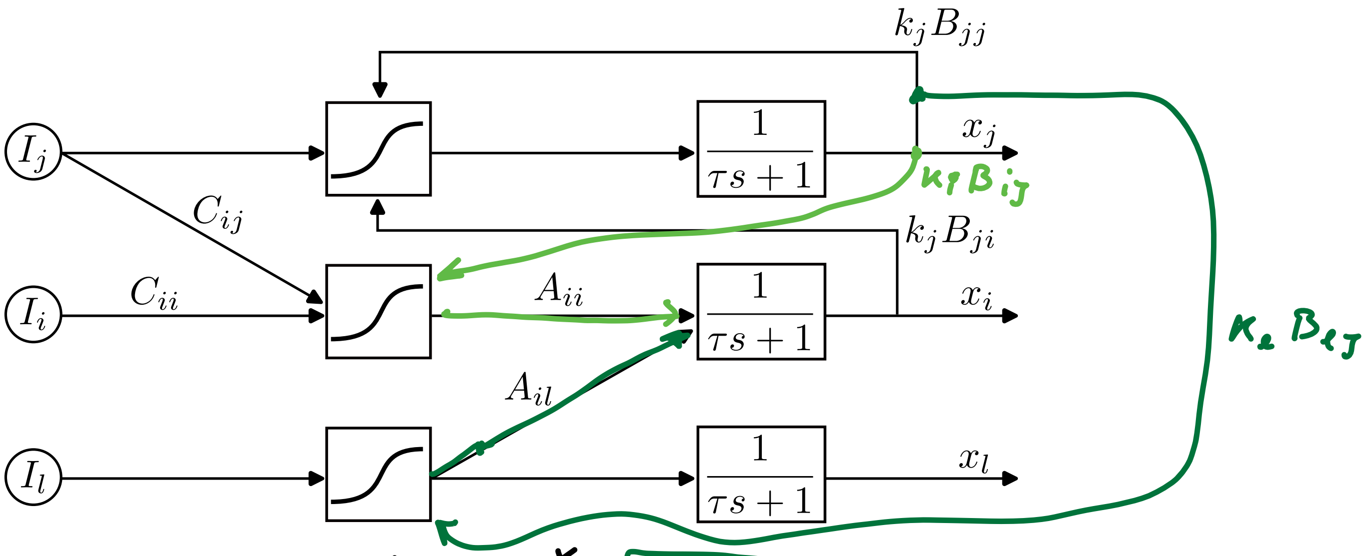
$$\tau \dot{\boldsymbol{x}}(t) = -\boldsymbol{x}(t) + A\boldsymbol{S}\left(KB\boldsymbol{x}(t) + C\boldsymbol{I}(t)\right)$$

where $\boldsymbol{x} = (x_1, \ldots, x_N)$, $\boldsymbol{I} = (I_1, \ldots, I_N)$, $K = \text{diag}(k_1, \ldots, k_N)$, $k_i \geq 0$, $\boldsymbol{S}(\boldsymbol{x}) = (\tanh(x_1), \ldots, \tanh(x_N))$.

It consists of an input layers ($\boldsymbol{I}$), a saturation layer ($\boldsymbol{S}$) with N saturating nodes, and an output layer ($\boldsymbol{x}$) . Nodes in the output layer are recurrently interconnected indirectly through the saturation layer. Adjacency matrix A defines the connections from the saturation to the output layer. Adjacency matrix B defines the connections from the output to the saturation layer. Adjacency matrix C defines the connection from the input to the saturation layer. The positive gain matrix K specifies the overall recurrent interconnection gain of each node in the saturation layer. The saturation layer gains can be interpreted as the **attention** paid by each layer in the node to its neighbors.

For specific choices of the various adjacency matrices, we recover (and generalize) the flexible multi-option representation dynamics above.

The N agent, 2-option (≥ 0 or ≤ 0) network model



Observe that the interconnection gain from node j to node i is $\boxed{\sum_l A_{il} k_l B_{lj}} = (AKB)_{ij}$.

AKB the effective recurrent interconnection matrix

A 2 agent, 2 option network: network pitchfork bifurcation

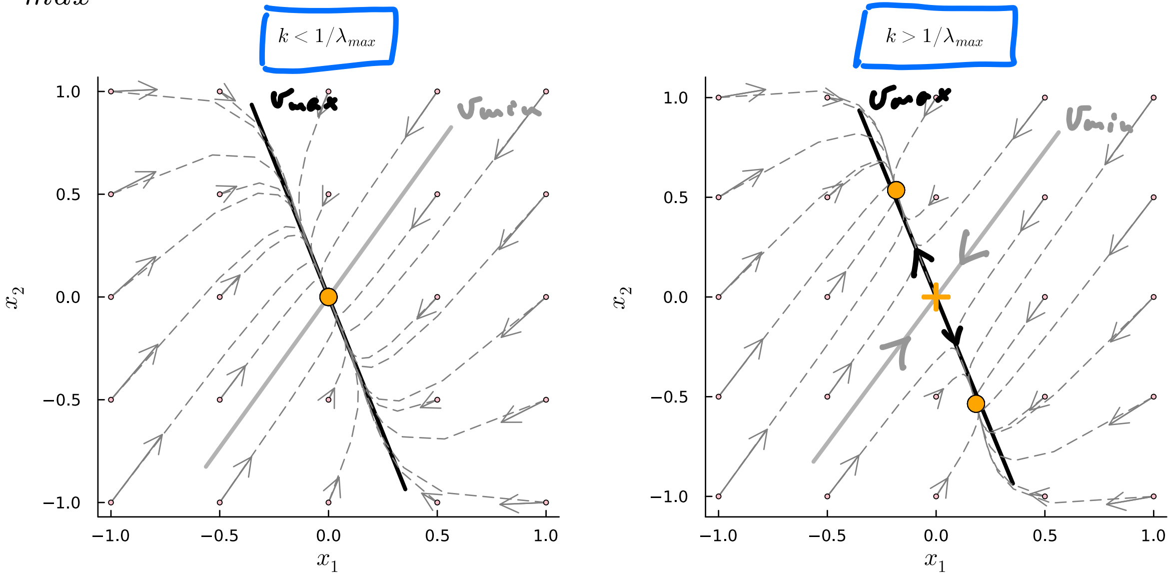
the "strength" of recurrent interconnections

Let's assume that $k_1 = k_2 = k$. Then $AKB = kAB$. Let's also assume that matrix AB has two distinct positive eigenvalues $0 < \lambda_{min} < \lambda_{max}$. Let $\boldsymbol{v}_{min}$ be the eigenvector associated to λ_{min} and $\boldsymbol{v}_{max}$ be the eigenvector associated to λ_{max}. We call λ_{max} the **dominant** eigenvalue and $\boldsymbol{v}_{max}$ the **dominant eigenvector**.

"topology" (who speaks with whom) of rec. int.

The two-agent, two option network model undergoes a qualitative change in its phase portrait for the critical parameter values $k_c = \lambda_{max}^{-1}$.

$I = 0$



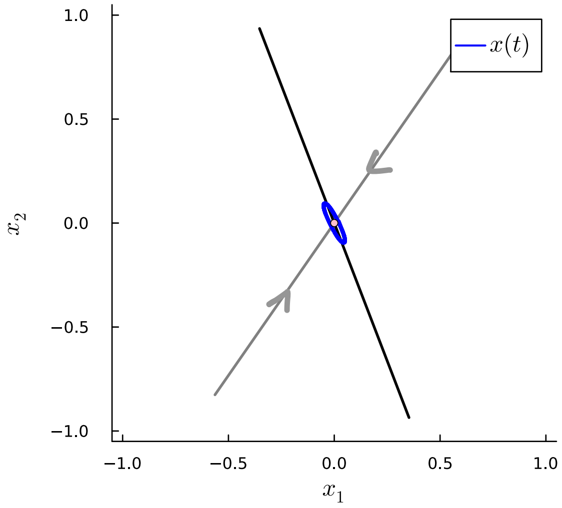
Exercise: Draw the $(k, \boldsymbol{x})$ bifurcation diagram and study how it changes as a function of I_1, I_2 (constant).

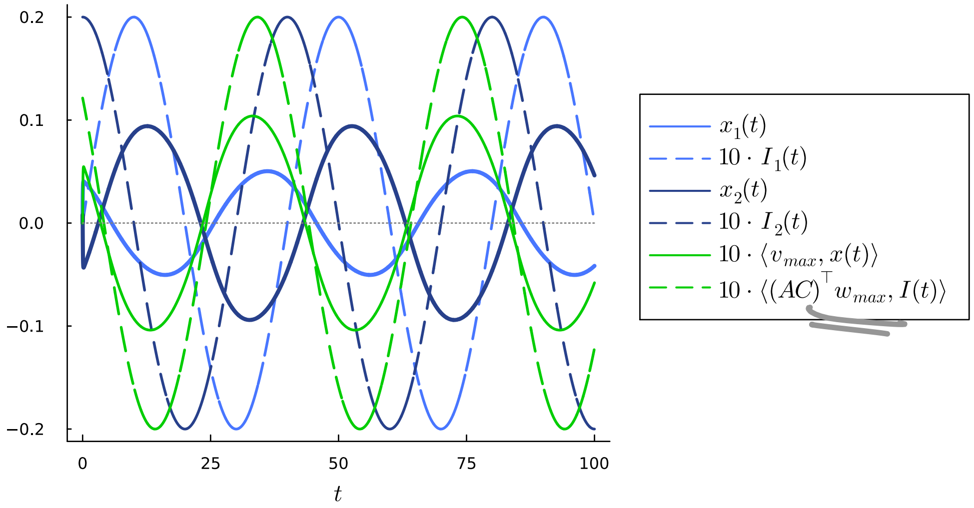
A 2 agent, 2 option network: signal representation

Let $\boldsymbol{w}_{max}$ be the eigenvector of $(AB)^\top$ associated to eigenvalue λ_{max} such that $\langle \boldsymbol{w}_{max}, \boldsymbol{v}_{max}\rangle = 1$. Let $I_{max}(t) = \langle (AC)^\top \boldsymbol{w}_{max}, \boldsymbol{I}(t)\rangle$. The projected input vector I_{max} determines how inputs affect the dynamics on the **dominant subspace**.

$\tilde{w}_{max} = (AC)^\top w_{max}$

$k = 1/\lambda_{max}$





Exercise: Study flexible representation properties as a function of k. How would you include state-dependent feedback through k?

Network pitchfork bifurcation and resulting flexible representations

N agent, 2 option network: network pitchfork bifurcation (theory)

Let $k_1 = \ldots = k_N = k$. The Jacobian J_0 of the network model for zero input ($\boldsymbol{I} = 0$) and at the origin is

$$\boxed{J_0 = -I + kAB}$$

The basic **hypothesis** we need is that matrix AB **possesses a dominant eigenvalue** $\lambda_{max} > 0$, that is, a real eigenvalue that is larger of the real part of all other eigenvalues $\lambda_2, \ldots, \lambda_N$ of AB, i.e.

$$\lambda_{max} > \max_{i=2,\ldots,N} \Re(\lambda_i).$$

The Perron-Frobenius property

Let $\boldsymbol{v}_{max}$ be the **dominant eigenvector** associated to λ_{max}.

Connection with feedback theory: Sufficient conditions for the existence of a dominant eigenvector are:

1. AB defines a **strongly connected** graph. (AB is irreducible)
2. AB has all non-negative out-diagonal entries. $\rightarrow (AB)_{ij} > 0$

Then we can apply the Perron-Frobenius theorem [2, Theorem 8.4.4] to conclude the existence of a positive dominant eigenvalues with a positive dominant eigenvector.

Network interactions defined by a matrix AB with the properties above correspond to **purely positive feedback between the network nodes.**

N agent, 2 option network: network pitchfork bifurcation (theory)

For ①,② $\exists$ an m : $(AB)^m > 0$

No assumption on interconnection sign

Other sufficient conditions for the existence of a dominant eigenvalue/eigenvector:

- AB is **eventually positive**, i.e., there exists m such that $(AB)^m > 0$ [4, 1].
- AB is strongly connected and **similar to an eventually positive matrix**, i.e., there exists an eventually positive matrix $M^m > 0$ and an invertible matrix U such that $AKB = U^{-1}MU$.

Also in these case AB defines a network with **purely positive feedback between the network nodes**.

→ New parameterization For ARNN

N agent, 2 option network: network pitchfork bifurcation (theory)

Algorithmic construction of networks with a dominant eigenvalue/eigenvector:

See Notebook.

N agent, 2 option network: network pitchfork bifurcation (theory)

Under the hypothesis that AB has a positive dominant eigenvalue λ_{max} we can easily prove (for $\boldsymbol{I} = 0$) the existence of a **network pitchfork bifurcation** happening *along* $\boldsymbol{v}_{max}$ at the origin:

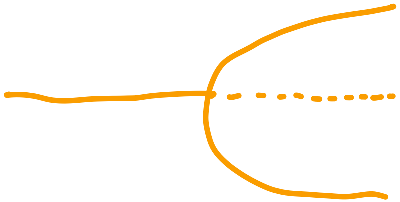
- Jacobian $J_0 = -I + kAB$ has dominant eigenvalue $\tilde{\lambda}_{max} = -1 + k\lambda_{max}$.
- Hence, J_0 has all eigenvalue with negative real part for $k < \lambda_{max}^{-1}$, it has a zero eigenvalue with null eigenvector $\boldsymbol{v}_{maxx}$ for $k = \lambda_{max}^{-1}$, it has an unstable eigenvalue with eigenvector $\boldsymbol{v}_{max}$ for $k > \lambda_{max}^{-1}$.
- By symmetry $\boldsymbol{x} \mapsto -\boldsymbol{x}$, the resulting bifurcation is a pitchfork.

$\Downarrow$

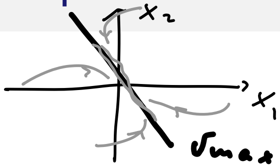
Bifurcation for $k = 1/\lambda_{max}$!!

$\Downarrow$

The bifurcation diagram must be symmetric !!



N agent, 2 option network: flexible representation along the pitchfork bifurcation



Incoming inputs are strongly amplified only along the dominant direction $\boldsymbol{v}_{max}$ because the dynamics is strongly exponentially contracting along all other directions. We can thus approximate $\boldsymbol{x}(t) \approxeq \chi(t)\boldsymbol{v}_{max}$ and study the resulting one-dimensional reduced dynamics. Let U ~~the~~ a matrix that put (AB) in the Jordan form

$$U^{-1}(AB)U = \begin{pmatrix} \lambda_{max} & \mathbf{0}_{1\times N-1} \\ \mathbf{0}_{N-1} & \tilde{A} \end{pmatrix},$$

where $\tilde{A}$ is $N-1 \times N-1$ whose eigenvalue are the non-dominant eigenvalues of AB. Observe that $U_{1\cdot}^{-1} = (\boldsymbol{w}_{max})^\top$, where $\boldsymbol{w}_{max}$ is the eigenvector of $(AB)^\top$ associated to eigenvalue λ_{max} and such that $\langle \boldsymbol{w}_{max}, \boldsymbol{v}_{max} \rangle = 1$. Let $\boldsymbol{z} = U^{-1}\boldsymbol{x}$. Then $\chi(t) = z_1(t)$, and, to linear order,

$$\begin{aligned} \tau\dot{\chi}(t) &= -\chi(t) + k\lambda_{max}\chi(t) + \overbrace{U_{1\cdot}^{-1}}^{\boldsymbol{w}_{max}} AC\boldsymbol{I}(t) \\ &= (-1 + k\lambda_{max})\chi(t) + \langle (AC)^\top \boldsymbol{w}_{max}, \boldsymbol{I}(t) \rangle \end{aligned}$$

$U_{1\cdot}^{-1} AC = y^T$

$\to y^T I(t) = \langle y, I(t) \rangle$

$y = (U_{1\cdot}^{-1} AC)^T = (AC)^T w_{max}$

N agent, 2 option network: flexible representation along the pitchfork center manifold

In the weakly nonlinear regime $k \leq \frac{1}{\lambda_{max}}$ the linearized reduced one-dimensional dynamics is stable and we can use linearized system analysis to derive the gain multi-input single-output gain from $\boldsymbol{I}$ to χ. The transfer function $H_{i\chi}(s)$ from I_i to χ is

$$\boxed{H_{i\chi}(s) = \frac{\left((AC)^\top \boldsymbol{w}_{max}\right)_i}{\tau s + 1 - k\lambda_{max}}}.$$

Observe that, as expected, the zero-frequency linear gains become infinite at bifurcation, i.e., for $k = \frac{1}{\lambda_{max}}$.

N agent, 2 option network: flexible representation along the pitchfork center manifold

FF

Recurrent $\Longrightarrow$ Nonlinear amplification (Lost in FF networks)

The vector $\tilde{\boldsymbol{w}}_{max} = (AC)^\top \boldsymbol{w}_{max}$ defines a preferred direction in the input space: when the input vector $\boldsymbol{I}(t)$ is roughly aligned with $\tilde{\boldsymbol{w}}_{max}$, inputs to single nodes cooperate to increase the response along the amplifying direction $\boldsymbol{v}_{max}$. In the limit of small τ (or slowly varying inputs), this means in the weakly nonlinear regime $k < \frac{1}{\lambda_{max}}$ the state $\boldsymbol{x}(t) \approx \boldsymbol{0}_N$ whenever $\langle \tilde{\boldsymbol{w}}_{max}, \boldsymbol{I}(t) \rangle \approx 0$ while $\boldsymbol{x}(t)$ amplifies inputs along $\boldsymbol{v}_{max}$ with gain $\frac{1}{1-k\lambda_{max}}$ when $\boldsymbol{I}(t)$ is roughly aligned with $\tilde{\boldsymbol{w}}_{max}$.

In analogy with neuronal systems, we can call the vector $\tilde{\boldsymbol{w}}_{max}$ the network's **receptive field**.

For larger τ (or more rapidly varying inputs), the state response along $\boldsymbol{v}_{max}$ is a lagged and amplified version of the projected input $\langle \tilde{\boldsymbol{w}}_{max}, \boldsymbol{I}(t) \rangle$.

N agent, 2 option network: flexible representation along the pitchfork bifurcation

In the nonlinear regime $k > \frac{1}{\lambda_{max}}$ the reduced one-dimensional dynamics is unstable and we cannot neglect the effects of nonlinearities. Considering the effects of the saturation layer, we can approximate

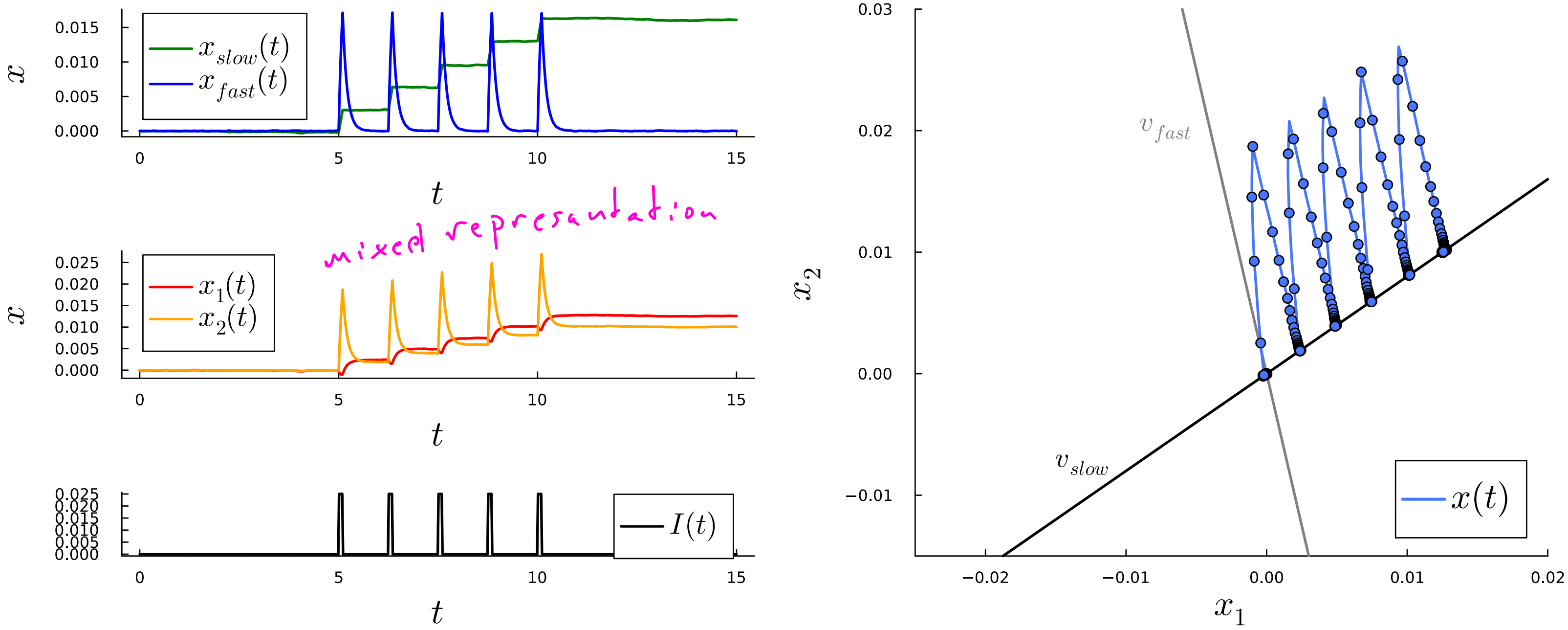
$$\tau \dot{\chi}(t) \approx -\chi(t) + \tanh\left(k\lambda_{max}\chi(t) + \langle (AC)^\top \boldsymbol{w}_{max}, \boldsymbol{I}(t)\rangle\right)$$

which exhibit categorical representation properties for $k > \frac{1}{\lambda_{max}}$.

The input category is determined by the network receptive field: when $\langle (AC)^\top \boldsymbol{w}_{max}, \boldsymbol{I}(t)\rangle > 0$ the network's state is pushed toward the up category, when $\langle (AC)^\top \boldsymbol{w}_{max}, \boldsymbol{I}(t)\rangle < 0$ the network's state is pushed toward the down category.

Simultaneous faithful and categorical representation along fast and slow directions

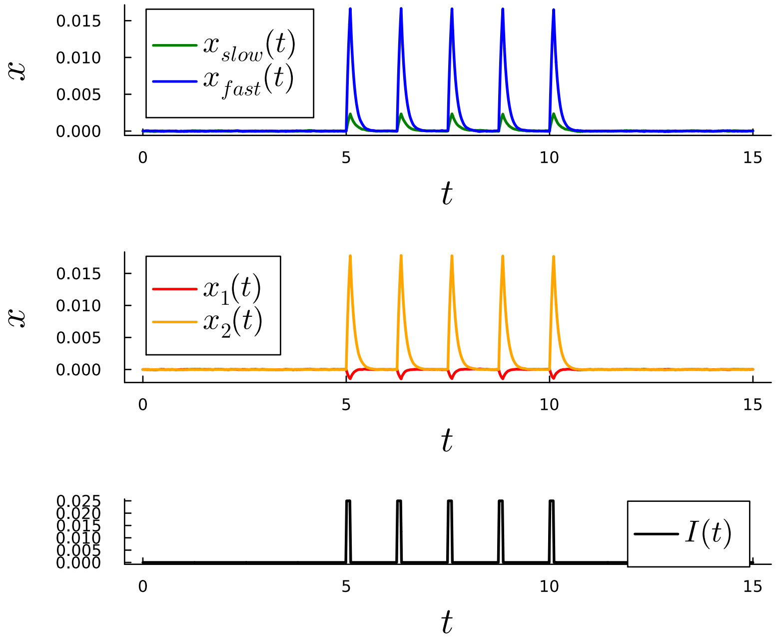
In the next example we study how a two-neuron network encodes an input made of 5 successive pulses. The behavior is shown for $k \lesssim \lambda_{max}^{-1}$ (strong recurrent interconnections).

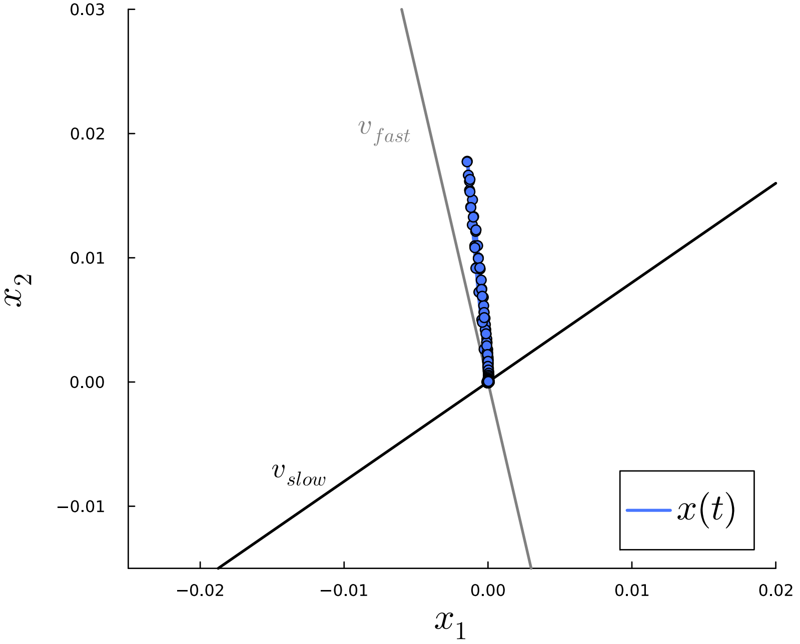


The stimulus is represented faithfully along the fast direction and categorically (pulse counting) along the slow direction.

Switching on and off categorical representations

Reducing the strength of recurrent interconnections ($k \ll \lambda_{max}^{-1}$), the dominant dynamics becomes faster and loses its categorical representation.





Hence, the overall strength of recurrent interconnection can be tuned switch on and off the network categorical representation while preserving the faithful representation component.

Switching on and off categorical representations

Ongoing work with Roman Rossi-Pool shows that this switching on and off of categorical representations might be happening in secondary sensory cortices: only when the animal is actively/attentively engaged in the task does the faithful representation emerge in S2 $\rightarrow$ Many projects (both in connection with neuroscience and ML) available here.

The connection between categorical representation, timescales, and strength of recurrent interconnections fits well a number of experimental evidences.

Shared representations

Observe that both the network's receptive field $(AC)^\top \boldsymbol{w}_{max}$ and the representation direction $\boldsymbol{v}_{max}$ are **shared** across the network nodes. In cognitive psychology [3], shared representations plays a key role in understanding **multi-tasking**.

[1] Claudio Altafini. "Investigating stability of Laplacians on signed digraphs via eventual positivity". In: *2019 IEEE 58th Conference on Decision and Control (CDC)*. IEEE. 2019, pp. 5044–5049.

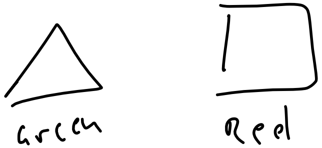
[2] Roger A Horn and Charles R Johnson. *Matrix analysis*. Cambridge university press, 2012.

[3] Sebastian Musslick and Jonathan D Cohen. "Rationalizing constraints on the capacity for cognitive control". In: *Trends in Cognitive Sciences* 25.9 (2021), pp. 757–775.

[4] Dimitrios Noutsos. "On Perron–Frobenius property of matrices having some negative entries". In: *Linear Algebra and its Applications* 412.2-3 (2006), pp. 132–153.

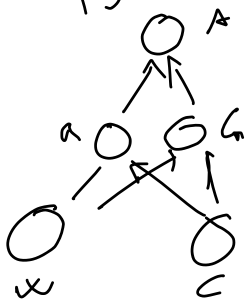
Networks with multiple flexible representations and their interactions

The Stroop effect: an example of representation interaction



Red Green Purple
Brown Blue Red

Jonhatan
Cohen
1990



Purple Red Brown
Red Green Blue

Categorical representation interaction in a 2D network

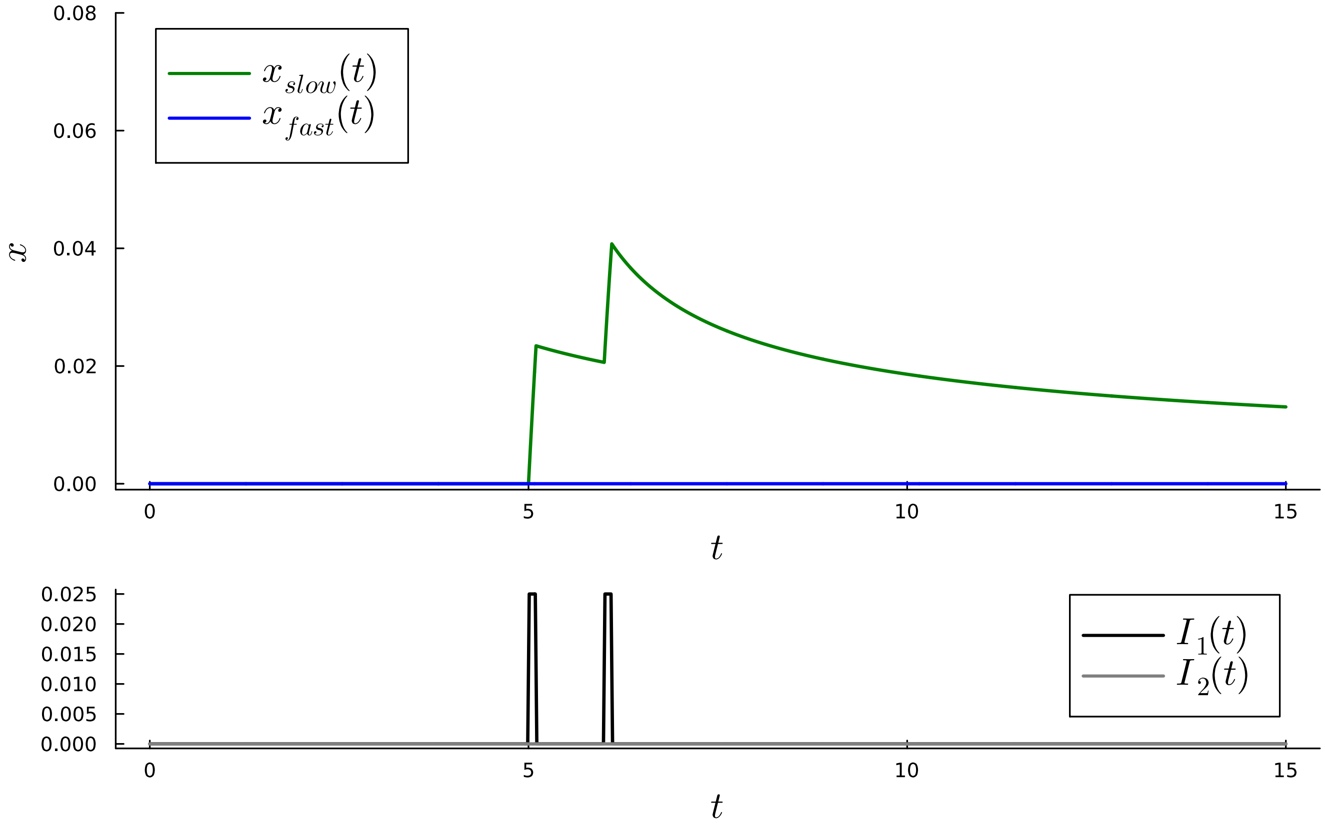
When some of the non-dominant eigenvalues are close to the dominant one (that is, when some non-dominant modes are almost as slow as the dominant one), multiple categorical representations can emerge and interact.

In the following example, we first illustrate the dominant and non-dominant representation properties when

1. The non-dominant mode is much faster then dominant one
2. The non-dominant mode is almost as slow as the dominant one

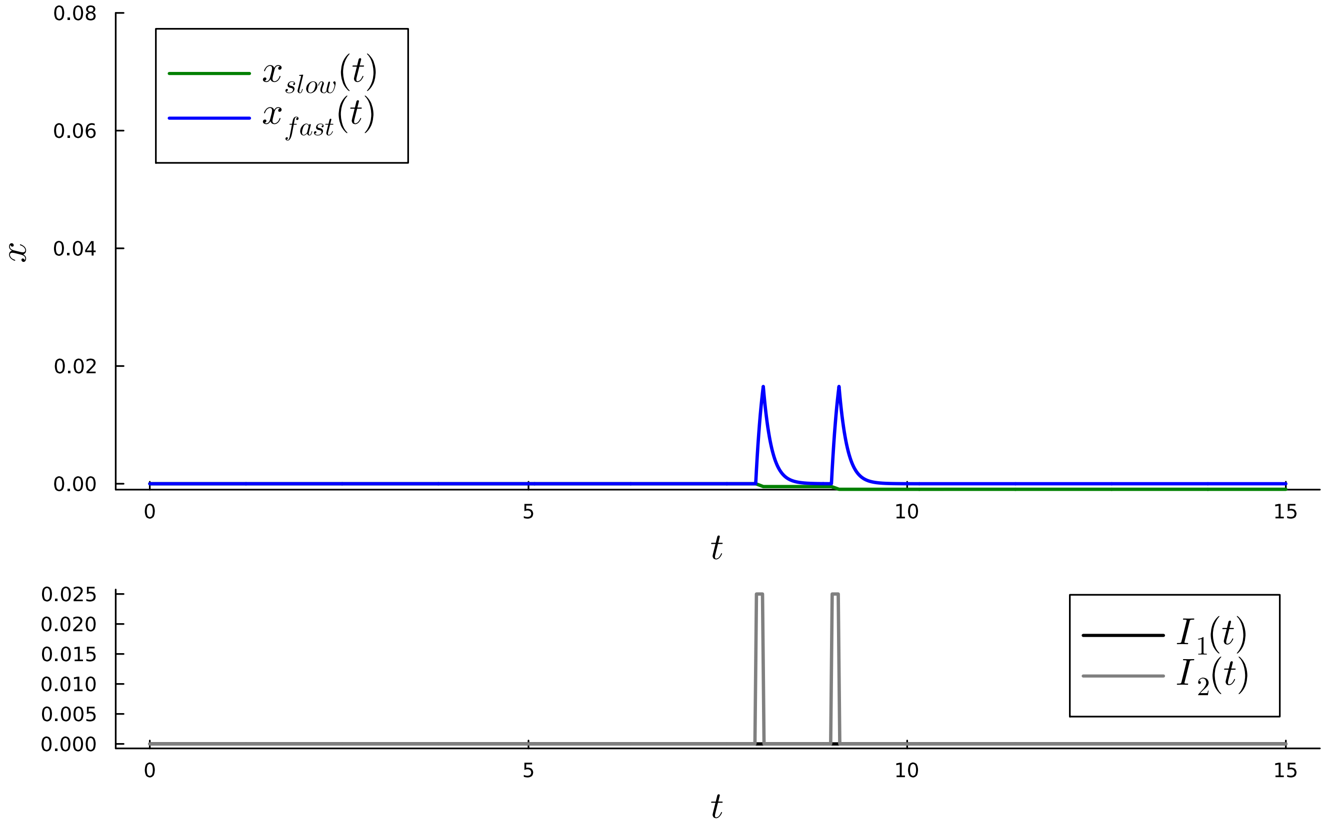
Categorical representation interaction in a 2D network
1. Non-dominant is much faster

Mode response to stimulus to Node 1.



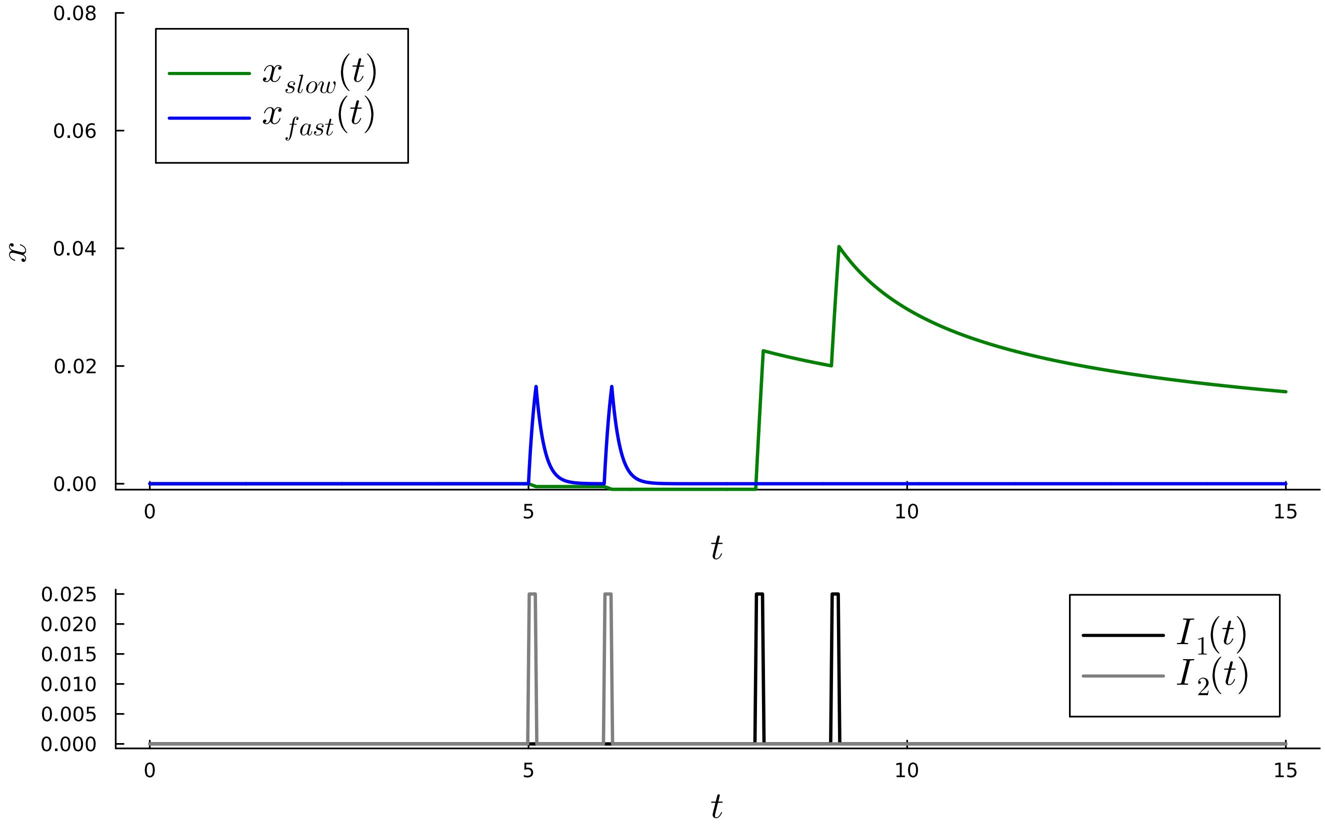
Categorical representation interaction in a 2D network
1. Non-dominant is much faster

Mode response to stimulus to Node 2.



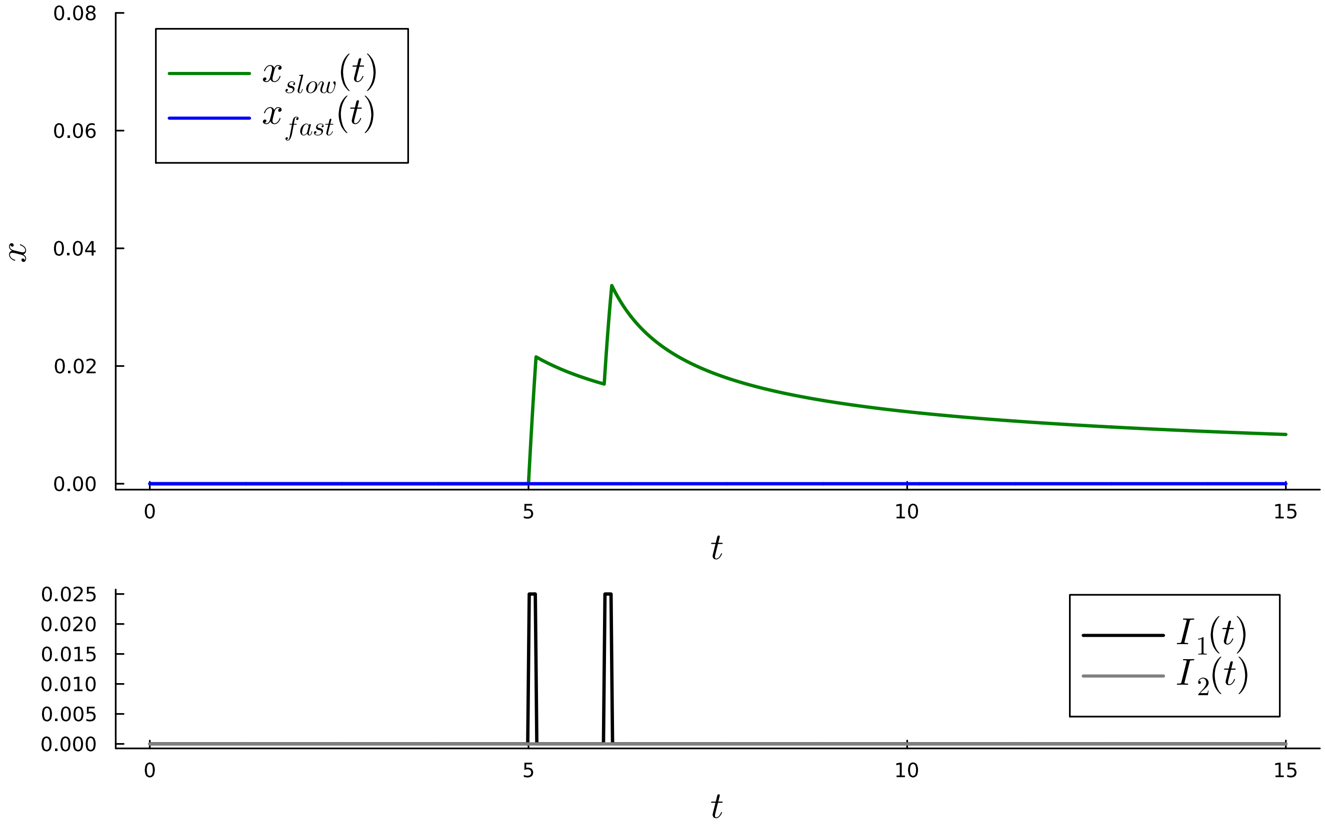
Categorical representation interaction in a 2D network
1. Non-dominant is much faster

Mode response to stimulus to Node 2 and then Node 1.



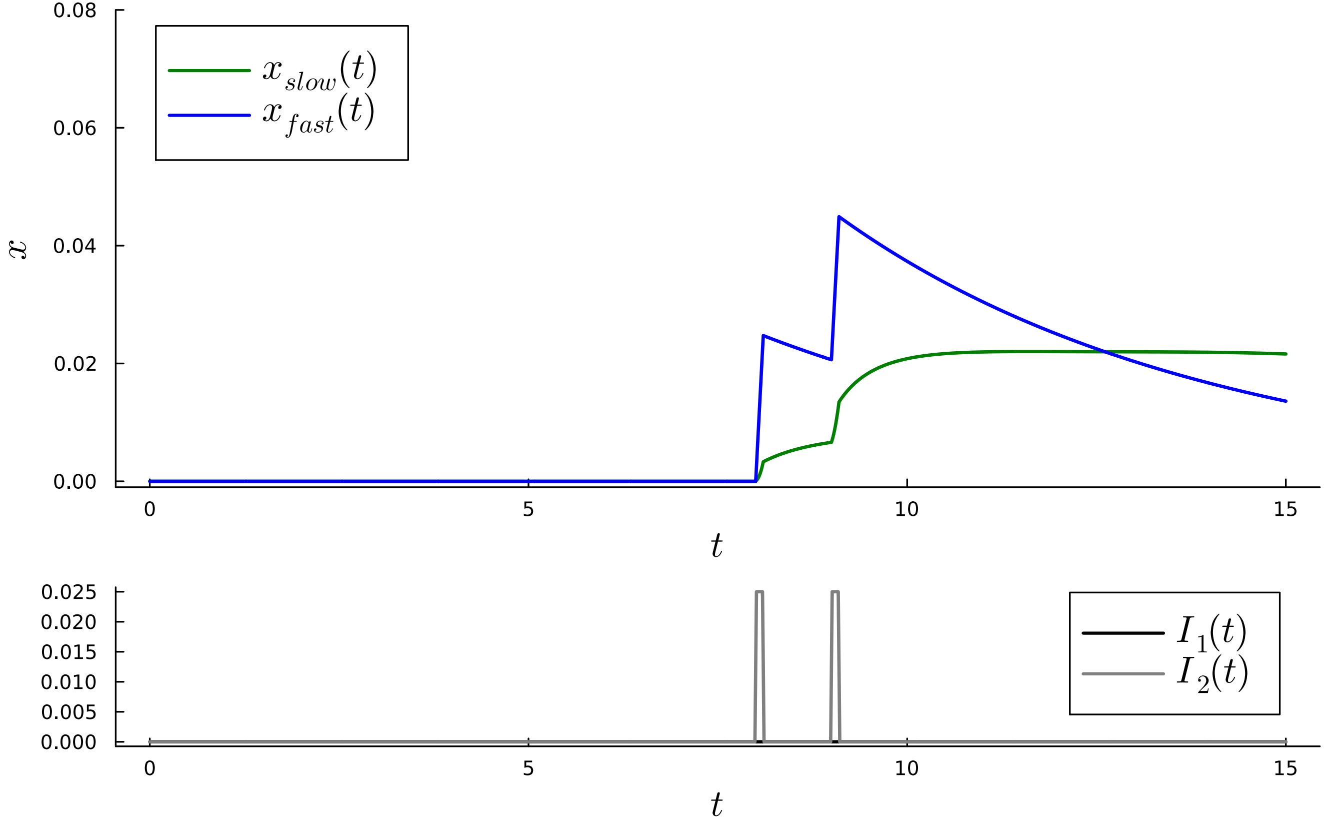
Categorical representation interaction in a 2D network
2. Non-dominant is almost as slow

Mode response to stimulus to Node 1.



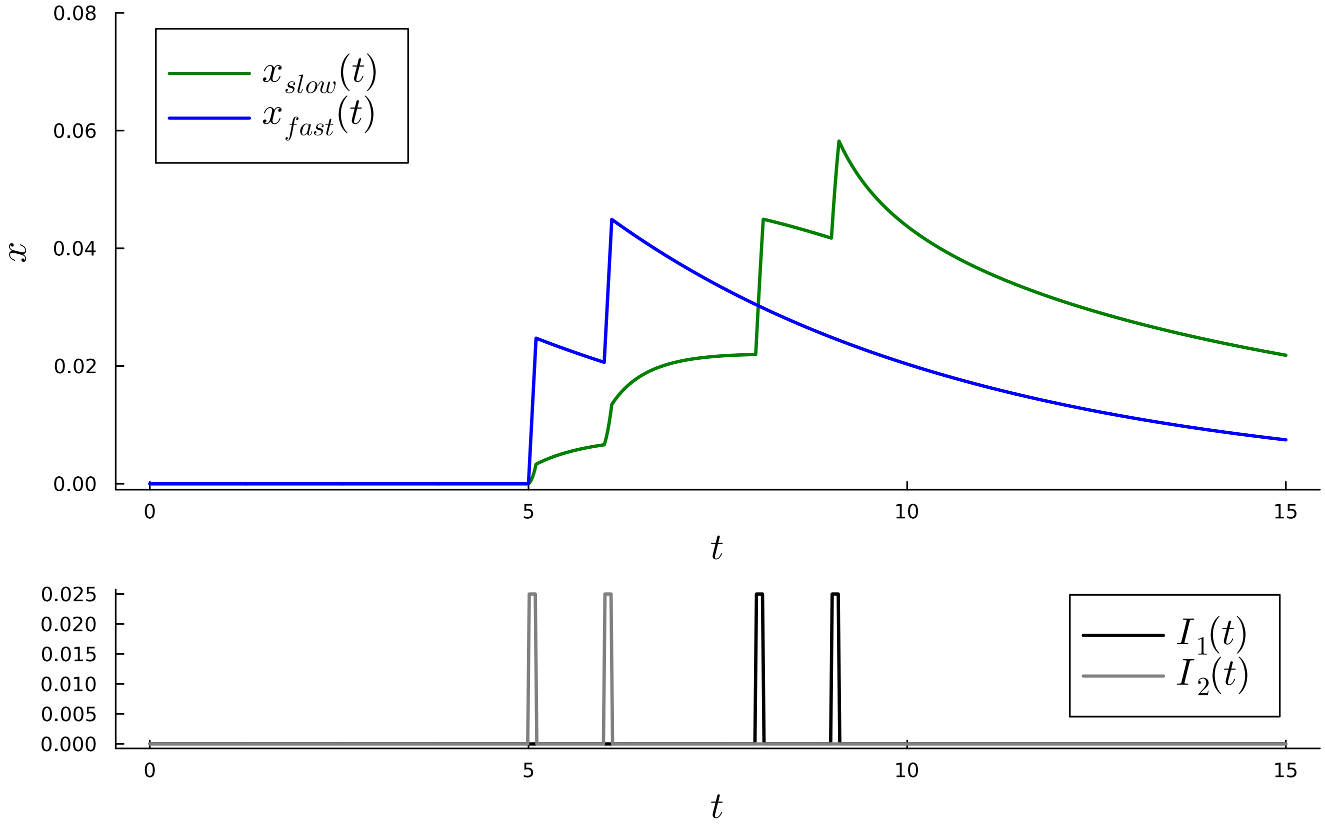
Categorical representation interaction in a 2D network
2. Non-dominant is almost as slow

Mode response to stimulus to Node 2.



Categorical representation interaction in a 2D network
2. Non-dominant is almost as slow

Mode response to stimulus to Node 2 and then Node 1.



Shared representations and representation interaction

Sharing representation across network nodes is key for their interaction through recurrent network interactions.

Representation interaction might be key for learning and multi-tasking by sharing information about incoming inputs across multiple computational modes.

It is a very active area of research [1] and another place for projects at the intersection between cognitive neuroscience/psychology and machine learning.

[1] Sebastian Musslick and Jonathan D Cohen. "Rationalizing constraints on the capacity for cognitive control". In: *Trends in Cognitive Sciences* 25.9 (2021), pp. 757–775.

Network state-dependent feedback and network excitability

Network state-dependent feedback and network excitability

In the same way as we introduced state-depend positive feedback and excitability in symmetric network models of n-options representations, we can introduce these elements in our network model.

Studying and applying the resulting computational capabilities can be the subject of research projects.

Generalization to $N_o > 2$ options

Network state-dependent feedback and network excitability

Suppose we want ot model a network of N_a nodes in which each node implement an N_o-option flexible representation.

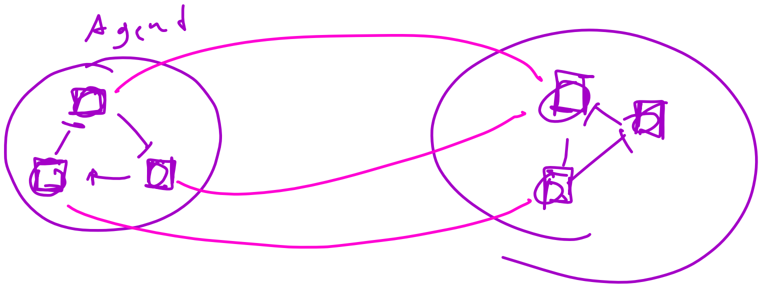
We can achieve this by interconnecting N_a copies of an N_o-option flexible representation, a **network of networks**. Possible equations are [1]

Intra-agent   different-option

$$\dot{z}_{ij} = -z_{ij} + u\left(S_1\left(\alpha z_{ij} + \gamma \sum_{\substack{k=1\\k\neq i}}^{N_a} (A_a)_{ik} z_{kj}\right) + \sum_{\substack{l\neq j\\l=1}}^{N_o} S_2\left(\beta (A_o)_{jl} z_{il} + \delta (A_o)_{jl} \sum_{\substack{k=1\\k\neq i}}^{N_a} (A_a)_{ik} z_{kl}\right)\right) + b_{ij}$$

Bifurcation theory can be applied also here to study, design, and tune the resulting representation behavior.

Same option network   Inter-agent



options / categories

[1] Anastasia Bizyaeva, Alessio Franci, and Naomi Ehrich Leonard. "Nonlinear opinion dynamics with tunable sensitivity". In: *IEEE Transactions on Automatic Control* (2022).