Statistical Signal Processing
Project

Submission deadline: 22 January 2025 by 23.59

- Hand in your Project as one PDF **structured as a report**. Make sure your report has sections, and is not structured like a homework set.
- The reports must contain enough information for the reader to understand and reproduce your results. Your reports will be graded based on your understanding of the problem, implementation and interpretation of results, and clarity of information in your text and figures.
- The code needs to be appended. No scans or pictures of handwritten solutions allowed. Make sure that your PDF is rendered correctly before submitting. Submit it strictly by the deadline. Later submissions will not be graded.
- You are not allowed to use any module/package (inbuilt functions) for the optimization problems or MCMC.
- Remember the ground rules: generative AI tools (like ChatGPT, etc.) are not allowed to be used for the project. Your goal for this course is to understand the translation of the mathematical processes discussed in the lectures into practical applications, and not to use the crutch of AI to skip over this understanding. The project can be discussed with each other, but you must submit your own work.
- Problem 6 is meant only for Masters students. Bachelors students can also attempt it but it will not be graded. The amount of points per question differs per type of student as well, and will be denoted with **B** for Bsc students and **M** for Msc students. The total amount of points is 100. This total number of points will be reduced if the work does not meet the standards of a report as mentioned above.
- Make sure your plots are correct and complete. Check your axis labels, add colourbars if needed and make sure the axis limits are correct.
- Add comments to your code for readability.

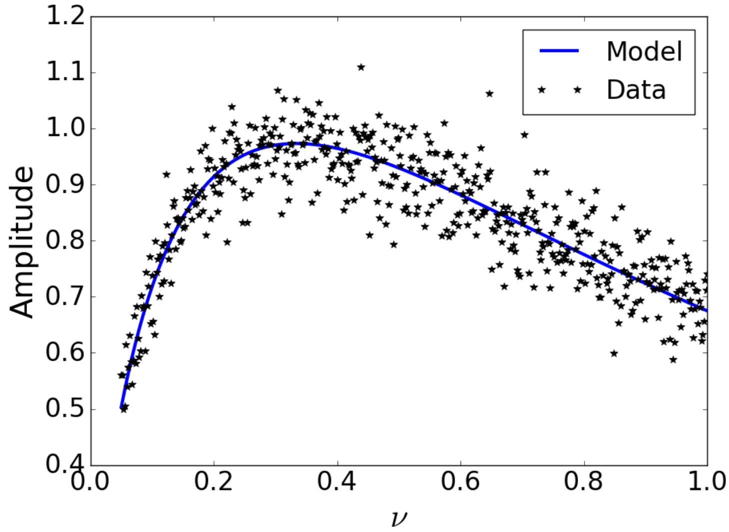
Load the CSV file `project_data.csv`, found on Brightspace. This file has three data columns:

(i) `nu`, which is the frequency.

(ii) `signal`, which is the power measured at each frequency.

(iii) `signal_plus_noise`, which is the signal with WGN ($\mu = 0,\ \sigma = 0.05$) added to it.

We'll use `nu` as the independent variable, and `signal_plus_noise` as the observed data. The Figure shows the plot of the `signal` (blue curve) and `signal_plus_noise` (points) vs. `nu`.



The observed data can be written in the following form:

$$\mathbf{x} = \mathbf{s} + \mathbf{w}$$

Where $\mathbf{x} = [x_1, x_2, ..., x_N]$ are the observed data points and $\mathbf{w} = [w_1, w_2, ..., w_N]$ is WGN with $\mathcal{N}(\mu = 0, \sigma^2 = 0.0025)$. We would like to fit the signal with the model of the following form:

$$s = A \times \left(\frac{\nu}{\nu_o}\right)^{\alpha} \left(1 + \frac{\nu}{\nu_o}\right)^{-4\alpha}$$

where ν is the frequency (the independent variable) and A, ν_o and α are the parameters to be estimated. Using the above data model, we can write each observed data point as

$$x_i = A \times \left(\frac{\nu_i}{\nu_o}\right)^{\alpha} \left(1 + \frac{\nu_i}{\nu_o}\right)^{-4\alpha} + w_i$$

1. [**M:** 10 pt, **B:** 15 pt] Plot the model function. What can you say about the probable values of the parameters from the plot. How does the model behave in the limits $\nu \gg \nu_o$ and $\nu \ll \nu_o$?

2. [**Both:** 20 pt] Derive the Maximum Likelihood Estimator (MLE) for the three parameters. Numerically evaluate the ML estimate of the parameters, for example using the Newton Raphson's optimization method (you can also use other optmization methods such as gradient descent etc.). Use $(A, \nu_o, \alpha)_{guess} = (6.0, 2.0, 1.0)$ as the initial guess parameter. Plot the residuals of the fit and their PDF.

3. [**M:** 15 pt, **B:** 20 pt] Generate 10,000 different Monte-Carlo realizations of $\mathbf{x}$ by adding different realizations of noise $\mathcal{N}(\sigma = 0.05, \mu = 0)$ to the `signal`

from the data-file and repeat the same process as in problem 2 for these realizations. Use these estimates from the Monte-Carlo simulations to compute the PDFs for the estimated parameters $\hat{A}, \hat{\nu_o}, \hat{\alpha}$. Find the mean and variance of these PDFs.

4. [**M:** 15 pt, **B:** 20 pt] Analytically derive the entries of the Fisher information matrix, and numerically compute their values and the CRLB matrix for the ML estimate values. How does this CRLB compare with the variance estimated from the Monte-Carlo simulations? Explain.

5. [**M:** 20 pt, **B:** 25 pt] Compute the likelihood function $\mathcal{L}(A, \nu_o, \alpha)$ on a three dimensional grid of values for A, ν_o, and α. Marginalize this function to compute the marginalized probability densities: $P(A, \nu_o)$, $P(A, \alpha)$, $P(\nu_o, \alpha)$, $P(A)$, $P(\nu_o)$ and $P(\alpha)$. Plot these distribution functions: Plot $P(A, \nu_o)$, $P(A, \alpha)$ and $P(\nu_o, \alpha)$ as colorplots (as images, avoid 3D plots) and $P(A)$, $P(\nu_o)$ and $P(\alpha)$ as histograms. Interpret your results.
Note: The values of the likelihood function can lead to numerical errors in Python due to the memory size of the floating point numbers. To circumvent this, you may ignore the $\sqrt{2\pi\sigma^2}$ term in your likelihood and either plot an unnormalized distribution or design your own normalization scheme. On some machines, you may also need to use `numpy.float128` for storing your likelihoods before plotting (convert back to `numpy.float64` when using `plt.imshow`).

6. [**M:** 20 pt] Use 10^5 Markov Chain Monte Carlo steps to sample the posterior $Pr(A, \nu_o, \alpha|\mathbf{X})$ assuming a flat prior $P(A, \nu_o, \alpha)$. In case of a flat prior, $Pr(A, \nu_o, \alpha|\mathbf{X}) = \mathcal{L}(\mathbf{X}|A, \nu_o, \alpha)$. Use a multivariate gaussian with $\mu = (A, \nu_o, \alpha)_{guess}$ (as used in Problem 2 and 3) and covariance matrix $\Sigma = 3 \times diag(\sigma_A^2, \sigma_{\nu_o}^2, \sigma_\alpha^2)_{MC}$ (results of Monte Carlo simulation) as a proposal distribution. Plot the histograms of the resulting PDFs $P(\hat{A})$, $P(\hat{\nu_o})$, $P(\hat{\alpha})$ and compute corresponding mean and variances. Also plot the joint distributions $P(\hat{A}, \hat{\nu_o})$, $P(\hat{\nu_o}, \hat{\alpha})$ and $P(\hat{A}, \hat{\alpha})$. Compare the MCMC results with MC results. What can you say about the choice of prior and the proposal function?

Hints:

1. Newton–Raphson method for multiple variables:
Let

$$\mathbf{F}(A, \nu_o, \alpha) = \begin{pmatrix} \dfrac{\partial ln\mathcal{L}}{\partial A} \\[2ex] \dfrac{\partial ln\mathcal{L}}{\partial \nu_o} \\[2ex] \dfrac{\partial ln\mathcal{L}}{\partial \alpha} \end{pmatrix} = \begin{pmatrix} f_1(A, \nu_o, \alpha) \\ f_2(A, \nu_o, \alpha) \\ f_3(A, \nu_o, \alpha) \end{pmatrix}$$

Let $\mathbf{\Theta} = [A\ \nu_o\ \alpha]^T$. Therefore, the Newton–Raphson iterative step can be written as

$$\mathbf{\Theta}_f = \mathbf{\Theta}_i - \gamma \mathbf{J}^{-1}\mathbf{F}(A_i, \nu_{oi}, \alpha_i)$$

Where $0 < \gamma \leq 1$ is a hyperparameter and $\mathbf{J}$ is the Jacobian matrix and is given by

$$\mathbf{J} = \begin{pmatrix} \dfrac{\partial f_1}{\partial A} & \dfrac{\partial f_1}{\partial \nu_o} & \dfrac{\partial f_1}{\partial \alpha} \\ \\ \dfrac{\partial f_2}{\partial A} & \dfrac{\partial f_2}{\partial \nu_o} & \dfrac{\partial f_2}{\partial \alpha} \\ \\ \dfrac{\partial f_3}{\partial A} & \dfrac{\partial f_3}{\partial \nu_o} & \dfrac{\partial f_3}{\partial \alpha} \end{pmatrix}$$

2. For calculating the second derivatives of the signal s_i to the parameters A, ν_0 and α, we first define:

$$\beta_i = 4\ln\left(1 + \frac{\nu_i}{\nu_0}\right) - \ln\left(\frac{\nu_i}{\nu_0}\right).$$

Then we can use:

$$\begin{aligned}
\frac{\partial^2 s_i}{\partial A^2} &= 0, \\
\frac{\partial^2 s_i}{\partial \nu_0^2} &= s_i \frac{\alpha\left[(1+\alpha)\nu_0^2 - 6(1+\alpha)\nu_i\nu_0 + 3(3\alpha - 1)\nu_i^2\right]}{\nu_0^2(\nu_0 + \nu_i)^2}, \\
\frac{\partial^2 s_i}{\partial \alpha^2} &= s_i \beta_i^2, \\
\frac{\partial^2 s_i}{\partial A \partial \alpha} &= -\frac{s_i \beta_i}{A}, \\
\frac{\partial^2 s_i}{\partial A \partial \nu_0} &= -\frac{s_i}{A}\frac{\alpha(\nu_0 - 3\nu_i)}{\nu_0(\nu_0 + \nu_i)}, \\
\frac{\partial^2 s_i}{\partial \alpha \partial \nu_0} &= s_i \frac{\nu_0 - 3\nu_i}{\nu_0(\nu_0 + \nu_i)}(\alpha\beta_i - 1).
\end{aligned}$$