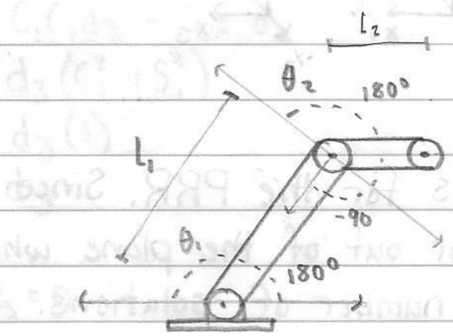
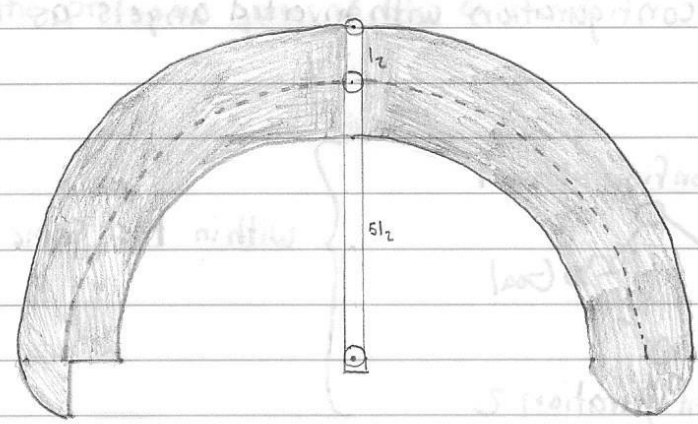
Problem # 1 (Exercise 4.9)

Sketch the approximate reachable workspace of the tip of link 2. The manipulator is a two-link planar arm with rotary joints. For this arm, the second link is one fifth as long as the first - that is, $l_1 = 5l_2$. The joint range limit in degrees are:

$$0 < \theta_1 < 180$$
$$-90 < \theta_2 < 180$$



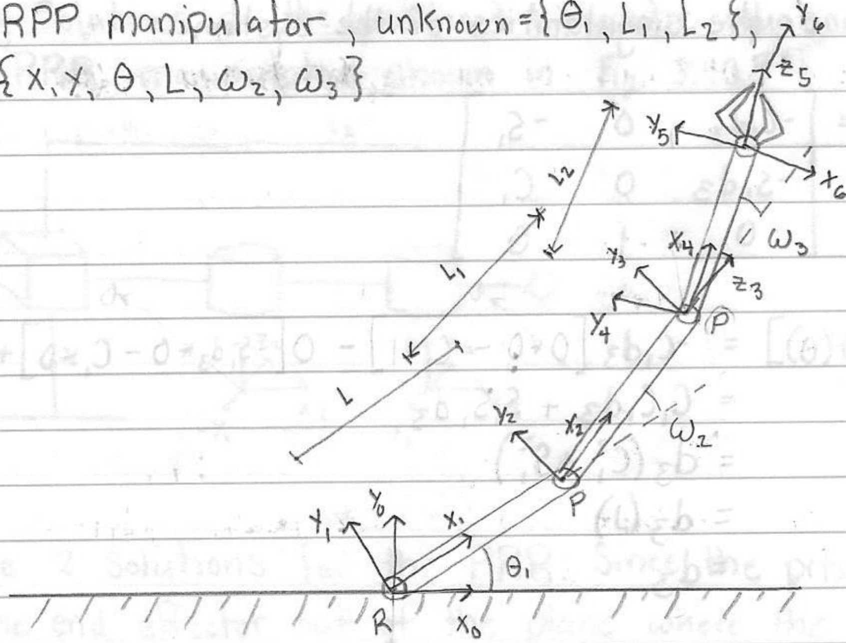
Workspace of link 2



Problem #2

Planar RPP manipulator, unknown $= \{\theta_1, L_1, L_2\}$,

Known $= \{X, Y, \theta, L, \omega_2, \omega_3\}$



$$\left.\begin{array}{ll} X = LC_1 + L_1C_{12} + L_2C_{123} & \theta = \theta_1 + \omega_2 + \omega_3 \\ Y = LS_1 + L_1S_{12} + L_2S_{123} & \end{array}\right\} \text{Geometric Solution}$$

DH Parameters

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	L	0	$\omega_2 - 90$
3	-90	0	L_1	0
4	90	0	0	ω_3
5	-90	0	L_2	0
6	90	0	0	90

$\cos(\omega_2 - 90) = \sin(\omega_2)$

$\cos(\omega_2 + 90) = -\sin(\omega_2)$

$\sin(\omega_2 - 90) = -\cos(\omega_2)$

$\sin(\omega_2 + 90) = \cos(\omega_2)$

$${}^0_6T = {}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3_4T\,{}^4_5T\,{}^5_6T$$

$$\theta_1 = \theta - (\omega_2 + \omega_3)$$

$$L_1 = CSC_3(YC_{123} - XS_{123} + LS_{23})$$

$$L_2 = CSC_3(YC_{12} - XS_{12} + LS_2)$$

$${}^{0}_{1}T\,{}^{1}_{2}T = {}^{0}_{2}T =$$

$$\begin{bmatrix} C_1 & -S_1 & 0 & 0 \\ S_1 & C_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_2 & C_2 & 0 & L \\ -C_2 & S_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} S_{12} & C_{12} & 0 & LC_1 \\ -C_{12} & S_{12} & 0 & LS_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{0}_{2}T\,{}^{2}_{3}T = {}^{0}_{3}T =$$

$$\begin{bmatrix} S_{12} & C_{12} & 0 & LC_1 \\ -C_{12} & S_{12} & 0 & LS_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & L_1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} S_{12} & 0 & C_{12} & LC_1 + L_1C_{12} \\ -C_{12} & 0 & S_{12} & LS_1 + L_1S_{12} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{0}_{3}T\,{}^{3}_{4}T = {}^{0}_{4}T =$$

$$\begin{bmatrix} S_{12} & 0 & C_{12} & LC_1 + L_1C_{12} \\ -C_{12} & 0 & S_{12} & LS_1 + L_1S_{12} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_3 & -S_3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ S_3 & C_3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} S_{123} & C_{123} & 0 & LC_1 + L_1C_{12} \\ -C_{123} & S_{123} & 0 & LS_1 + L_1S_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{0}_{4}T\,{}^{4}_{5}T = {}^{0}_{5}T =$$

$$\begin{bmatrix} S_{123} & C_{123} & 0 & LC_1 + L_1C_{12} \\ -C_{123} & S_{123} & 0 & LS_1 + L_1S_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & L_2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} S_{123} & 0 & C_{123} & LC_1 + L_1C_{12} + L_2C_{123} \\ -C_{123} & 0 & S_{123} & LS_1 + L_1S_{12} + L_2S_{123} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{0}_{5}T\,{}^{5}_{6}T = {}^{0}_{6}T =$$

$$\begin{bmatrix} S_{123} & 0 & C_{123} & LC_1 + L_1C_{12} + L_2C_{123} \\ -C_{123} & 0 & S_{123} & LS_1 + L_1S_{12} + L_2S_{123} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_{123} & -S_{123} & 0 & LC_1 + L_1C_{12} + L_2C_{123} \\ S_{123} & C_{123} & 0 & LS_1 + L_1S_{12} + L_2S_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x = LC_1 + L_1C_{12} + L_2C_{123} \qquad \theta_1 = \phi - \omega_2 - \omega_3$$

$$y = LS_1 + L_1S_{12} + L_2S_{123}$$

$$\phi = \theta_1 + \underbrace{\omega_2 + \omega_3}_{\text{known}}$$

Problem # 2 : Inverse Kinematics

$\theta = \theta_1 + \omega_2 + \omega_3$ * $\omega_2, \omega_3,$ and θ are known

$\theta_1 = \theta - (\omega_2 + \omega_3)$

$X = LC_1 + L_1 C_{12} + L_2 C_{123}$

$Y = LS_1 + L_1 S_{12} + L_2 S_{123}$

$L_1 = \dfrac{X - LC_1 - L_2 C_{123}}{C_{12}}$ $L_2 = \dfrac{X - LC_1 - L_1 C_{12}}{C_{123}}$

$L_1 = \dfrac{Y - LS_1 - L_2 S_{123}}{S_{12}}$ $L_2 = \dfrac{Y - LS_1 - L_1 S_{12}}{S_{123}}$

$$\frac{X - LC_1 - L_2 C_{123}}{C_{12}} = \frac{Y - LS_1 - L_2 S_{123}}{S_{12}}$$

$$S_{12}(X - LC_1 - L_2 C_{123}) = C_{12}(Y - LS_1 - L_2 S_{123})$$

$$0 = YC_{12} - XS_{12} + LC_1 S_{12} - LS_1 C_{12} + L_2 S_{12} C_{123} - L_2 C_{12} S_{123}$$

$$0 = YC_{12} - XS_{12} + LS_2 - L_2 S_3$$

$$L_2 S_3 = YC_{12} - XS_{12} + LS_2 \rightarrow \boxed{L_2 = \csc_3 (YC_{12} - XS_{12} + LS_2)}$$

$$\frac{X - LC_1 - L_1 C_{12}}{C_{123}} = \frac{Y - LS_1 - L_1 S_{12}}{S_{123}}$$

$$S_{123}(X - LC_1 - L_1 C_{12}) = C_{123}(Y - LS_1 - L_1 S_{12})$$

$$0 = YC_{123} - XS_{123} + LC_1 S_{123} - LS_1 C_{123} + L_1 C_{12} S_{123} - L_1 S_{12} C_{123}$$

$$0 = YC_{123} - XS_{123} + LS_{23} + L_1 S_3$$

$$\boxed{L_1 = \csc_3 (YC_{123} - XS_{123} + LS_{23})}$$

Problem # 2 : Jacobian

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = {}^0J \begin{bmatrix} \theta_1 \\ L_1 \\ L_2 \end{bmatrix} \qquad {}^0J = \begin{bmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial L_1} & \frac{\partial x}{\partial L_2} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial L_1} & \frac{\partial y}{\partial L_2} \\ \frac{\partial \theta}{\partial \theta_1} & \frac{\partial \theta}{\partial L_1} & \frac{\partial \theta}{\partial L_2} \end{bmatrix}$$

$\partial x / \partial \theta_1 : LC_1 + L_1 C_{12} + L_2 C_{123} \rightarrow -LS_1 - L_1 S_{12} - L_2 S_{123}$

$\partial y / \partial \theta_1 : LS_1 + L_1 S_{12} + L_2 S_{123} \rightarrow LC_1 + L_1 C_{12} + L_2 C_{123}$

$\partial \theta / \partial \theta_1 = 1$

$\partial x / \partial L_1 = C_{12}$

$\partial y / \partial L_1 = S_{12}$

$\partial \theta / \partial L_1 = 0$

$\partial x / \partial L_2 = C_{123}$

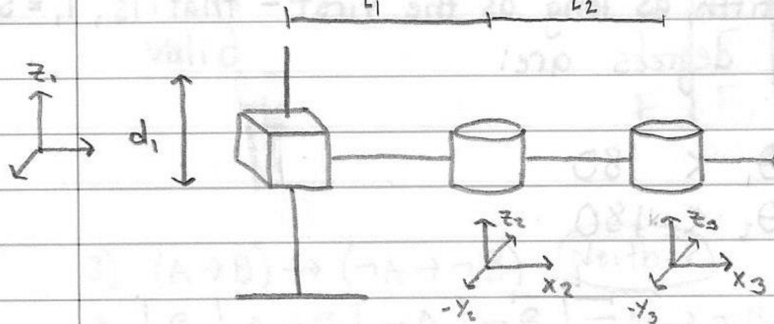
$\partial y / \partial L_2 = S_{123}$

$\partial \theta / \partial L_2 = 0$

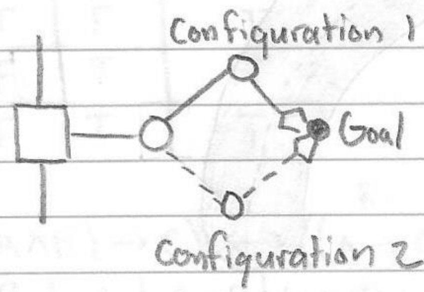
$${}^0J = \begin{bmatrix} -LS_1 - L_1 S_{12} - L_2 S_{123} & C_{12} & C_{123} \\ LC_1 + L_1 C_{12} + L_2 C_{123} & S_{12} & S_{123} \\ 1 & 0 & 0 \end{bmatrix}$$

Problem #3

How many Solutions do the (position) kinematic equations possess for the PRR manipulator shown in Fig. 3.40?



There are 2 Solutions for the PRR. Since the prismatic joint moves the end effector out of the plane where the goal is, it does not effect the number of solutions. Since there are two revolute joints that rotate in the same plane, the manipulator can take one configuration as one solution and another configuration with inverted angels as a second solution.



within the same plane

Problem #4

Determine the singularities of the system

$$J(\theta) = \begin{bmatrix} -C_1 d_3 & 0 & -S_1 \\ -S_1 d_3 & 0 & C_1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned}
DET[J(\theta)] &= -C_1 d_3 \left[\underbrace{0*0}_{\to 0} - C_1*1\right] - 0\left[\cancel{-S_1 d_3*0 - C_1*0}\right] + (-S_1)\left[-S_1 d_3*1 - \cancel{0*0}\right] \\
&= C_1 C_1 d_3 + S_1 S_1 d_3 \\
&= d_3 (C_1^2 + S_1^2) \\
&= d_3 (1) \\
&= d_3
\end{aligned}$$

$$d_3 = 0$$

A singularity of the mechism exists when d_3 is 0. Physically, when $d_3 = 0$, the prismatic joint is not extended.