APPENDIX: COMPUTING (17) FROM (15)

This appendix to "Reconfigurable Intelligent Surfaces: A Signal Processing Perspective With Wireless Applications" will describe in detail how OFDM is utilized to reach from the system model in (15) with intersymbol interference, to the K parallel subcarriers in (17), which are free from intersymbol interference. The physical explanation for the intersymbol interference is that different propagation paths result in different delays. When the variation in delay is large compared to the symbol time, the symbol that is received over the fastest path will collide with older symbols that arrive over slower paths. From a mathematical standpoint, the reason behind intersymbol interference is the convolution in (15) between the signal sequence $x[m]$, for $m = -\infty, \ldots, -1, 0, 1, \ldots, \infty$, and the FIR filter $h_{\boldsymbol{\theta}}[0], \ldots, h_{\boldsymbol{\theta}}[M-1]$ describing the channel. To resolve this issue, we can make use of a property of the Fourier transform, namely that the convolution between a signal and a channel in the time-domain is equivalent to the multiplication of their Fourier transforms in the frequency domain. OFDM is designed to exploit this property.

We will now present a general result from signal processing theory and then apply it to compute (17). Let $\chi[0], \ldots, \chi[K-1]$ be a K-length signal sequence and let $h[0], \ldots, h[K-1]$ be a K-length FIR filter. The so-called *circular convolution* between the signal and the filter is defined as

$$(h \circledast \chi)[k] = \sum_{\ell=0}^{K-1} h[\ell]\chi[(k-\ell)_{\mathrm{mod}\, K}] \quad \text{for } k = 0, \ldots, K, \tag{A.1}$$

where "$\mathrm{mod}\, K$" is the modulo operation that adds K to $k-\ell$ whenever needed to give a value between 0 and $K-1$. It is called circular convolution since the modulo operation is providing indices from the end of the signal sequence when $k-\ell$ is negative; for example, $(-1)_{\mathrm{mod}\, K} = K-1$, $(-2)_{\mathrm{mod}\, K} = K-2$, etc. The K-point discrete-time Fourier transform of the signal is defined as

$$\bar{\chi}[\eta] = \mathcal{F}_d\{\chi[k]\} = \frac{1}{\sqrt{K}} \sum_{k=0}^{K-1} \chi[k] e^{-\mathrm{j}2\pi k\nu/K} \tag{A.2}$$

where the scaling factor $1/\sqrt{K}$ keeps the energy constant. The corresponding frequency-representation of the filter is

$$\bar{h}[\eta] = \sum_{k=0}^{K-1} h[k] e^{-\mathrm{j}2\pi k\nu/K} \tag{A.3}$$

where no scaling factor is used for notational convenience. Let $y[k] = (h \circledast \chi)[k]$ denote the output signal of the circular convolution. One can show that its K-point discrete-time Fourier transform is

$$\bar{y}[\eta] = \mathcal{F}_d\{y[k]\} = \bar{\chi}[\eta]\bar{h}[\eta]. \tag{A.4}$$

The proof is rather direct: one computes the Fourier transform of (A.1) and rewrites the expression to get a multiplication of (A.2) and (A.3). The conclusion from (A.4) is that, in the frequency domain, circular convolution in the time-domain turns into the multiplication of the frequency-representations of the signal and filter.

How can we apply this general result to our system model in (15)? There are two main differences:

1) We don't have a *circular* convolution but a regular *linear* convolution;
2) The signal and filter don't have length K.

We can deal with this by defining the signal and filter cleverly. Let us assume that $K > M$. The FIR filter is then shorter than it needs to be, but we can prolong it by adding zeros at the end:

$$h[k] = \begin{cases} h_{\boldsymbol{\theta}}[k] & k = 0, \dots, M-1 \\ 0 & k = M, \dots, K-1. \end{cases} \tag{A.5}$$

Next, let us design the signal to be transmitted as

$$x[k] = \begin{cases} \chi[k] & k = 0, \dots, K-1 \\ \chi[k+K] & k = -M+1, \dots, -1 \\ 0 & \text{elsewhere} \end{cases} \tag{A.6}$$

where $\chi[0], \dots, \chi[K-1]$ is a sequence of data signals that is repeated in a partially cyclic manner. By design, we have that $x[k] = \chi[(k)_{\mathrm{mod}\, K}] = \chi[k+K]$ for $k = -M+1, \dots, -1$. It also holds that $x[k] = \chi[(k)_{\mathrm{mod}\, K}] = \chi[k]$ for $k = 1, \dots, K-1$ since the modulo operation as no effect in this case. The signal part of (15) can be rewritten as follows:

$$\sum_{\ell=0}^{M-1} h_{\boldsymbol{\theta}}[\ell] x[k-\ell] = \sum_{\ell=0}^{M-1} h_{\boldsymbol{\theta}}[\ell] \chi[(k-\ell)_{\mathrm{mod}\, K}] = \sum_{\ell=0}^{K-1} h[\ell] \chi[(k-\ell)_{\mathrm{mod}\, K}] \tag{A.7}$$

where the first equality follows from (A.6) and the second equality follows from (A.5). When taking the Fourier transform, we can now apply (A.4) and obtain the signal term in (17).

In summary, physical propagation channels create a linear convolution of the kind in (15), but we want the received signal to contain a circular convolution so we can remove the intersymbol interference with the help of the Fourier transform. Therefore, we need to design the transmitted signal to make the received signal look like a circular convolution. This is what we did above. Even if it says "0 elsewhere" in (A.6), this property is not needed in practice since we only need a sequence of K received signals to look like a circular convolution. Moreover, since the FIR filter only has length $M < K$, we don't need to transmit the whole signal sequence twice to create a circular convolution, but only add a so-called cyclic prefix of length $M-1$ at the beginning of the signal sequence. By making $K \gg M$, we can make the cyclic prefix short compared to the rest of the transmission. In summary, we can obtain the OFDM system model in (17) with K parallel channels by dividing the transmission into blocks where one transmits signals of length $K+M-1$, and then apply the Fourier transform.

APPENDIX: COMPUTING (22) FROM (13)

This appendix to "Reconfigurable Intelligent Surfaces: A Signal Processing Perspective With Wireless Applications" will describe in detail how (22) is computed based on (13), for the channel and RIS model presented in (19)-(21). The term $(p * b_n * \vartheta_{n;\theta_n} * a_n * p)(t)$ appears in (13). We begin by inserting (19)-(21) into this expression to obtain

$$\begin{aligned}&(p * b_n * \vartheta_{n;\theta_n} * a_n * p)(t)\\&= \sum_{l=1}^{L_a}\sum_{\ell=1}^{L_b}\sqrt{\alpha_n^l \beta_n^\ell \gamma_{\theta_n}}\left(p(t) * e^{-\mathrm{j}2\pi f_c t}\delta(t-\tau_{n,b}^\ell) * e^{-\mathrm{j}2\pi f_c t}\delta(t-\tau_{\theta_n}) * e^{-\mathrm{j}2\pi f_c t}\delta(t-\tau_{n,a}^l) * p(t)\right),\end{aligned} \tag{A.8}$$

where we switched to another notation for the convolution, where the time variable t is written out in every function. Since convolution is a commutative operation, we can rearrange the order in (A.8) to obtain.

$$\begin{aligned}&(p * b_n * \vartheta_{n;\theta_n} * a_n * p)(t)\\&= \sum_{l=1}^{L_a}\sum_{\ell=1}^{L_b}\sqrt{\alpha_n^l \beta_n^\ell \gamma_{\theta_n}}\left(p(t) * p(t) * e^{-\mathrm{j}2\pi f_c t}\delta(t-\tau_{n,b}^\ell) * e^{-\mathrm{j}2\pi f_c t}\delta(t-\tau_{\theta_n}) * e^{-\mathrm{j}2\pi f_c t}\delta(t-\tau_{n,a}^l)\right),\end{aligned} \tag{A.9}$$

Let us begin by considering

$$p(t) * p(t) = \sqrt{B}\mathrm{sinc}(Bt) * \sqrt{B}\mathrm{sinc}(Bt) = \mathrm{sinc}(Bt). \tag{A.10}$$

To prove that the last equality holds, we will show that their Fourier transforms are equal. Many tables of Fourier transforms contain the following identity:

$$\mathcal{F}_c\left\{\mathrm{sinc}(Bt)\right\} = \frac{1}{B}\mathrm{rect}\left(\frac{f}{B}\right) \tag{A.11}$$

where $\mathrm{rect}(\cdot)$ is the rectangular function. The left-hand side of (A.10) has the Fourier transform

$$\begin{aligned}\mathcal{F}_c\left\{p(t) * p(t)\right\} &= B\mathcal{F}_c\left\{\mathrm{sinc}(Bt)\right\}\mathcal{F}_c\left\{\mathrm{sinc}(Bt)\right\}\\&= B\frac{1}{B}\mathrm{rect}\left(\frac{f}{B}\right)\frac{1}{B}\mathrm{rect}\left(\frac{f}{B}\right) && \text{(A.12)}\\&\overset{(a)}{=} \frac{1}{B}\mathrm{rect}\left(\frac{f}{B}\right) = \mathcal{F}_c\left\{\mathrm{sinc}(Bt)\right\} && \text{(A.13)}\end{aligned}$$

where we first utilized the fact that the Fourier transform of a convolution is the product of the individual Fourier transforms. Next, (a) follows from the fact that $\mathrm{rect}^2(\cdot) = \mathrm{rect}(\cdot)$ (since the function value is either zero or one). We have now proved the equality in (A.10).

To compute the final result we will utilize the following "translation" result related to the convolution between an arbitrary function (in this case a sinc-function) and a time-delayed delta function:

$$\begin{aligned}\mathrm{sinc}(Bt) * e^{-\mathrm{j}2\pi f_c t}\delta(t-\tau) &= \int_{-\infty}^{\infty} \mathrm{sinc}(Bu) e^{-\mathrm{j}2\pi f_c (t-u)}\delta(t-u-\tau)du \\ &= \int_{-\infty}^{\infty} \mathrm{sinc}(Bu) e^{-\mathrm{j}2\pi f_c (t-u)}\delta(u-(t-\tau))du \\ &= \mathrm{sinc}(B(t-\tau)) e^{-\mathrm{j}2\pi f_c \tau}. \end{aligned} \tag{A.14}$$

By utilizing (A.14) three times, we can simplify the chain of convolutions in (A.9) progressively as follows:

$$\underbrace{p(t) * p(t)}_{=\mathrm{sinc}(Bt)} * e^{-\mathrm{j}2\pi f_c t}\delta(t-\tau_{n,b}^{\ell}) * e^{-\mathrm{j}2\pi f_c t}\delta(t-\tau_{\theta_n}) * e^{-\mathrm{j}2\pi f_c t}\delta(t-\tau_{n,a}^{l}) \tag{A.15}$$

$$= \mathrm{sinc}(B(t-\tau_{n,b}^{\ell})) e^{-\mathrm{j}2\pi f_c \tau_{n,b}^{\ell}} * e^{-\mathrm{j}2\pi f_c t}\delta(t-\tau_{\theta_n}) * e^{-\mathrm{j}2\pi f_c t}\delta(t-\tau_{n,a}^{l}) \tag{A.16}$$

$$= \mathrm{sinc}(B(t-\tau_{n,b}^{\ell}-\tau_{\theta_n})) e^{-\mathrm{j}2\pi f_c \tau_{n,b}^{\ell}} e^{-\mathrm{j}2\pi f_c \tau_{\theta_n}} * e^{-\mathrm{j}2\pi f_c t}\delta(t-\tau_{n,a}^{l}) \tag{A.17}$$

$$= \mathrm{sinc}(B(t-\tau_{n,b}^{\ell}-\tau_{\theta_n}-\tau_{n,a}^{l})) e^{-\mathrm{j}2\pi f_c \tau_{n,b}^{\ell}} e^{-\mathrm{j}2\pi f_c \tau_{\theta_n}} e^{-\mathrm{j}2\pi f_c \tau_{n,a}^{l}}. \tag{A.18}$$

By inserting this result back into (A.9), we obtain

$$\begin{aligned}&(p * b_n * \vartheta_{n;\theta_n} * a_n * p)(t) \\ &= \sum_{l=1}^{L_a}\sum_{\ell=1}^{L_b} \sqrt{\alpha_n^l \beta_n^{\ell} \gamma_{\theta_n}} e^{-\mathrm{j}2\pi f_c (\tau_{n,a}^{l}+\tau_{n,b}^{\ell}+\tau_{\theta_n})} \mathrm{sinc}(B(t-\tau_{n,a}^{l}-\tau_{n,b}^{\ell}-\tau_{\theta_n})),\end{aligned} \tag{A.19}$$

We are now ready to compute the final result

$$\begin{aligned}h_{\boldsymbol{\theta}}[k] &= \sum_{n=1}^{N} (p * b_n * \vartheta_{n;\theta_n} * a_n * p)(t)\Big|_{t=k/B+\eta} \\ &= \sum_{n=1}^{N}\sum_{l=1}^{L_a}\sum_{\ell=1}^{L_b} \sqrt{\alpha_n^l \beta_n^{\ell} \gamma_{\theta_n}} e^{-\mathrm{j}2\pi f_c (\tau_{n,a}^{l}+\tau_{n,b}^{\ell}+\tau_{\theta_n})} \mathrm{sinc}(B(t-\tau_{n,a}^{l}-\tau_{n,b}^{\ell}-\tau_{\theta_n}))\Big|_{t=k/B+\eta} \\ &= \sum_{n=1}^{N}\sum_{l=1}^{L_a}\sum_{\ell=1}^{L_b} \sqrt{\alpha_n^l \beta_n^{\ell} \gamma_{\theta_n}} e^{-\mathrm{j}2\pi f_c (\tau_{n,a}^{l}+\tau_{n,b}^{\ell}+\tau_{\theta_n})} \mathrm{sinc}(k+B(\eta-\tau_{n,a}^{l}-\tau_{n,b}^{\ell}-\tau_{\theta_n})).\end{aligned} \tag{A.20}$$