Formation of Differential Equation

Differential Equation: An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation.

Example: For examples of differential equation we list the following**:**

1. $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0$

2. $\frac{d^4u}{dt^4}+5\frac{d^2u}{dt^2}+3u=\sin t$

3. $\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{\partial^2u}{\partial z^2}=0$

4. $\frac{\partial u}{\partial s}+\frac{\partial v}{\partial t}=v$

Classification: Differential equations are classified on the basis of type as follows**:**

1. Ordinary Differential Equation (ODE),
2. Partial Differential Equation (PDE).

Ordinary Differential Equation (ODE): A differential equation involving derivatives of one or more dependent variables with respect to only one independent variable is called an ordinary differential equation.

Example: 1. $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$

2. $\frac{d^3y}{dx^3}+5\frac{d^2y}{dx^2}+2y=\sin x$

Partial Differential Equation (PDE): A differential equation involving derivatives of one or more dependent variables with respect to more than one independent variable is called a partial differential equation.

Example: 1. $\frac{\partial u}{\partial s}+\frac{\partial v}{\partial t}=v$

2. $\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}=0$

Order of a differential equation: The order of the highest ordered derivative involved in a differential equation is called the order of the differential equation.

Example: 1. $\frac{dy}{dx}+y=0$ is a first order differential equation**.**

2. $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0$ is a second order differential equation**.**

Degree of a differential equation: The power of the highest ordered derivative involved in a differential equation is called the degree of the differential equation, after the equation is freed from radicals and fractions in its derivatives.

Example: 1. $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0$ is a differential equation of first degree**.**

2. $\left(\frac{d^2y}{dx^2}\right)^2+\left(\frac{dy}{dx}\right)^4+y=0$ is a differential equation of second degree.

3. $\sqrt{\frac{d^3y}{dx^3}+4\frac{d^2y}{dx^2}}+y=\left(\frac{dy}{dx}\right)^2$ is a differential equation of first degree.

Linear ordinary differential equation: An ordinary differential equation of order n is called a linear ordinary differential equation of order n if it does not contain,

1. the transcendental functions of dependent variable,
2. the product of dependent variable and
3. the product of the derivatives of dependent variable.

It can be expressed as

$$a_0(x)\frac{d^n y}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\dots\dots\dots+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y=b(x)$$

where, a_0 is not identically zero.

Example: 1. $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=0$

2. $\frac{d^3y}{dx^3}+x^2\frac{d^2y}{dx^2}+xy=xe^x$

Nonlinear ordinary differential equation: A nonlinear ordinary differential equation is an ordinary differential equation that is not linear**.**

Example: 1. $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y^2=0$

2. $\frac{d^3y}{dx^3}+e^y\frac{d^2y}{dx^2}+xy=xe^x$

3. $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+6y=0$

Application of differential equation: Some applications of differential equation are given below:

1. The problem of determining the motion of a projectile, rocket, satellite or planet.
2. The problem of the conduction of heat in a rod or wire in a slab.
3. The problem of determining the vibrations of a wire or a membrane.
4. The study of the rate of decomposition of a radioactive substance or the rate of growth of a population.

Problem-01: Form the differential equation of the complete integral $y=ax+bx^2$**.**

Solution: Given that,

$$y=ax+bx^2\dots\dots\dots\quad(1)$$

Differentiating (1) with respect to x we get,

$$\frac{dy}{dx}=a+2bx\dots\dots\dots(2)$$

$$\Rightarrow a=\frac{dy}{dx}-2bx$$

Again differentiating (2) with respect to x we get,

$$\frac{d^2y}{dx^2}=2b$$

$$\Rightarrow b=\frac{1}{2}\frac{d^2y}{dx^2}$$

Putting these values of 'a' and 'b' in the equation (1) we have,

$$y=x\left(\frac{dy}{dx}-2x.\frac{1}{2}\frac{d^2y}{dx^2}\right)+\frac{x^2}{2}\frac{d^2y}{dx^2}$$

$$\Rightarrow y=x\frac{dy}{dx}-x^2\frac{d^2y}{dx^2}+\frac{x^2}{2}\frac{d^2y}{dx^2}$$

$$\Rightarrow y=x\frac{dy}{dx}-\frac{x^2}{2}\frac{d^2y}{dx^2}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0$$

which is a differential equation of second order and first degree.

Problem-02: Form the differential equation of the complete integral $c(y+c)^2 = x^3$.

Solution: Given that,

$$c(y+c)^2 = x^3 \ldots \ldots \ldots (1)$$

Differentiating (1) with respect to x we get,

$$2c(y+c)\frac{dy}{dx} = 3x^2 \ldots \ldots \ldots (2)$$

Dividing (1) by (2) we get,

$$\frac{y+c}{2\frac{dy}{dx}} = \frac{x}{3}$$

$$\Rightarrow 3(y+c) = 2x\frac{dy}{dx}$$

$$\Rightarrow c = \frac{1}{3}\left(2x\frac{dy}{dx} - 3y\right)$$

Putting this value of 'c' in the equation (2) we get,

$$\frac{2}{3}\left(2x\frac{dy}{dx} - 3y\right)\left\{y + \frac{1}{3}\left(2x\frac{dy}{dx} - 3y\right)\right\}\frac{dy}{dx} = 3x^2$$

$$\Rightarrow \left(\frac{4x}{3}\frac{dy}{dx} - 2y\right)\left\{y + \frac{2x}{3}\frac{dy}{dx} - y\right\}\frac{dy}{dx} = 3x^2$$

$$\Rightarrow \left(\frac{4x}{3}\frac{dy}{dx} - 2y\right)\frac{2x}{3}\left(\frac{dy}{dx}\right)^2 = 3x^2$$

$$\Rightarrow \frac{8x^2}{9}\left(\frac{dy}{dx}\right)^3 - \frac{4xy}{3}\left(\frac{dy}{dx}\right)^2 = 3x^2$$

$$\Rightarrow 8x\left(\frac{dy}{dx}\right)^3 - 12y\left(\frac{dy}{dx}\right)^2 = 27x$$

which is a differential equation of first order and third degree.

Problem-03: Find the differential equation from the relation $y = A\cos x + B\sin x$ where A and B are arbitrary constants.

Solution: Given that,

$$y = A\cos x + B\sin x \ \ldots \ldots \ldots (1)$$

Differentiating (1) with respect to x we get,

$$\frac{dy}{dx} = -A\sin x + B\cos x \ \ldots \ldots \ldots (2)$$

Again, differentiating (2) with respect to x we get,

$$\frac{d^2y}{dx^2} = -A\cos x - B\sin x$$

$$\Rightarrow \frac{d^2y}{dx^2} = -(A\cos x + B\sin x)$$

$$\Rightarrow \frac{d^2y}{dx^2} = -y$$

which is a differential equation of second order and first degree.

Problem-04: Find the differential equation of the family of curves , $y = Ae^{2x} + Be^{-2x}$ for different values of A and B.

Solution: Given that,

$$y = Ae^{2x} + Be^{-2x} \quad \dots \dots \dots (1)$$

Differentiating (1) with respect to x we get,

$$\frac{dy}{dx} = 2Ae^{2x} - 2Be^{-2x} \quad \dots \dots \dots (2)$$

Again, differentiating (2) with respect to x we get,

$$\frac{d^2y}{dx^2} = 4Ae^{2x} + 4Be^{-2x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = 4\left(Ae^{2x} + Be^{-2x}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = 4y$$

which is a differential equation of second order and first degree.

Problem-05: Find the differential equation from the relation $y = e^x\left(A\cos x + B\sin x\right)$ where A and B are arbitrary constants.

Solution: Given that,

$$y = e^x\left(A\cos x + B\sin x\right) \quad \dots \dots \dots (1)$$

Differentiating (1) with respect to x we get,

$$\frac{dy}{dx} = e^x\left(-A\sin x + B\cos x\right) + e^x\left(A\cos x + B\sin x\right)$$

$$\Rightarrow \frac{dy}{dx} = e^x\left(-A\sin x + B\cos x\right) + y$$

$$\Rightarrow \frac{dy}{dx} - y = e^x\left(-A\sin x + B\cos x\right) \quad \dots \dots \dots (2)$$

Again, differentiating (2) with respect to x we get,

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = e^x\left(-A\cos x - B\sin x\right) + e^x\left(-A\sin x + B\cos x\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} = -e^x\left(A\cos x + B\sin x\right) + e^x\left(-A\sin x + B\cos x\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} = -y + \left(\frac{dy}{dx} - y\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

which is a differential equation of second order and first degree.

Problem-06: Find the differential equation of the complete integral $Ax^2 + By^2 = 1$ for different values of A and B.

Solution: Given that,

$$Ax^2 + By^2 = 1 \quad \dots \dots \dots (1)$$

Differentiating (1) with respect to x we get,

$$2Ax + 2By\frac{dy}{dx} = 0$$

$$\Rightarrow Ax + By\frac{dy}{dx} = 0 \quad \dots \dots \dots (2)$$

Again, differentiating (2) with respect to x we get,

$$A+B\left[\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}\right]=0$$

Multiply above equation by x we get,

$$Ax+Bx\left[\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}\right]=0 \ldots \ldots \ldots (3)$$

Subtracting equation (2) from equation (3) we get,

$$Ax+Bx\left[\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}\right]-\left(Ax+By\frac{dy}{dx}\right)=0$$

$$\Rightarrow Bx\left[\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}\right]-By\frac{dy}{dx}=0$$

$$\Rightarrow x\left[\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}\right]-y\frac{dy}{dx}=0$$

which is a differential equation of second order and first degree.

Exercise: Form the differential equations for the following equations:

1. $xy=ae^{x}+be^{-x}$
2. $y=ae^{3x}+be^{-2x}+\sin 5x$
3. $y=a\cos(mx+b)$
4. $(x-h)^2=4a(y-k)$
5. $y=axe^{x}$
6. $y=ae^{-4x}+be^{-6x}$