Note 1

Propositional Logic

To become fluent in working with mathematical statements, you need to understand the basic framework of the language of mathematics.

Proposition: a statement which is either true or false

These statements are propositions:

1. $\sqrt{3}$ is irrational.

2. $1 + 1 = 5$.

These statements are not propositions:

1. $2 + 2$.

2. $x^2 + 3x = 5$. (since x is unknown)

Propositions should not include fuzzy terms, so these statements aren't propositions either (although some sources may say they are):

1. Arnold Schwarzenegger often eats broccoli. ("often" is fuzzy)

2. Henry VIII was unpopular. ("unpopular" is fuzzy)

Propositions may be joined together to form more complex statements. Let P, Q, and R be variables representing propositions. The simplest way of joining these propositions together is to use the connectives "and," "or," and "not."

1. **Conjunction**: $P \wedge Q$ ("P and Q"). True only when both P and Q are true.

2. **Disjunction**: $P \vee Q$ ("P or Q"). True when at least one of P and Q is true.

3. **Negation**: $\neg P$ ("not P"). True when P is false.

Statements with variables (like these) are called **propositional forms**.

A fundamental principle known as the **law of the excluded middle** says that, for any proposition P, either P is true or $\neg P$ is true (but not both). Thus $P \vee \neg P$ is always true, regardless of the truth value of P. A propositional form is always true regardless of its truth values is called a **tautology**. Conversely, a statement such as $P \wedge \neg P$, which is always false, is called a **contradiction**.

A **truth table** is used to describe the possible values of a propositional form. Truth tables are the same as function tables: you list all possible input values for the variables, and then list the outputs given for those inputs.

Here are the truth tables for conjunction, disjunction, and negation:

P	Q	$P \wedge Q$	$P \vee Q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

P	$\neg P$
T	F
F	T

The most important and subtle propositional form is an **implication**:

4. **Implication**: $P \Rightarrow Q$ ("P implies Q"). This is the same as "if P, then Q."

Here, P is called the **hypothesis** of the implication, and Q is the **conclusion**.

Here are some examples:

- If you stand in the rain, then you'll get wet.
- If you passed the class, you received a certificate.

An implication $P \Rightarrow Q$ is false only when P is true and Q is false.

Here is the truth table for $P \Rightarrow Q$ (along with an additional column):

P	Q	$P \Rightarrow Q$	$\neg P \vee Q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Note the $P \Rightarrow Q$ is always true when P is false. When an implication is stupidly true because the hypothesis is false, we say that it is **vacuously true**.

Note also that $P \Rightarrow Q$ is **logically equivalent** to $\neg P \vee Q$. We write this as $(P \Rightarrow Q) \equiv (\neg P \vee Q)$.

Here are some different ways of meaning $P \Rightarrow Q$:

1. if P, then Q
2. Q if P
3. P only if Q
4. P is sufficient for Q
5. Q is necessary for P
6. Q unless not P

If both $P \Rightarrow Q$ and $Q \Rightarrow P$ are true, then we say "P if and only if Q" (abbreviated "P iff Q"). Formally, we write $P \Leftrightarrow Q$. Note that $P \Leftrightarrow Q$ is true when P and Q have the same truth values (both true or false).

Given an implication $P \Rightarrow Q$, we can also define its

1. **Contrapositive**: $\neg Q \Rightarrow \neg P$
2. **Converse**: $Q \Rightarrow P$

Here are some truth tables:

P	Q	$\neg P$	$\neg Q$	$P \Rightarrow Q$	$Q \Rightarrow P$	$\neg Q \Rightarrow \neg P$	$P \Leftrightarrow Q$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	F	T	F	T	F
F	F	T	T	T	T	T	T

Note that $P \Rightarrow Q$ and its contrapositive have the same truth values, so they are logically equivalent: $(P \Rightarrow Q) \equiv (\neg Q \Rightarrow \neg P)$.

Also note that $P \Rightarrow Q$ and $Q \Rightarrow P$ are not logically equivalent.

Quantifiers

The mathematical statements you'll see in practice will look something like this:

1. For all natural numbers n, $n^2 + n + 41$ is prime.
2. If n is an odd integer, so is n^3.
3. There is an integer k that is both even and odd.

These statements assert something about lots of simple propositions all at once. For instance, the first statement is asserting that $0^2 + 0 + 41$ is prime, $1^2 + 1 + 41$ is prime, and so on.

The last statement says that as k ranges over all possible integers, we will find at least one value of k for which the statement is satisfied.

Compared to the previous statement "$x^2 + 3x = 5$," these examples are quantified over a "universe." To express these statements mathematically, we need two **quantifiers**:

1. the universal quantifier $\forall$ ("for all")
2. the existential quantifier $\exists$ ("there exists")

Examples:

1. "Some mammals lay eggs."

Mathematically, "some" means "at least one," so the statement is actually saying "There exists a mammal x such that x lays eggs." If we let our universe U to be the set of mammals, then we can write: $(\exists x \in U)(x \text{ lays eggs})$.

2. "For all natural numbers n, $n^2 + n + 41$ is prime."

In this case, our universe becomes the set of natural numbers, denoted as $\mathbb{N}$:
$(\forall n \in \mathbb{N})(n^2 + n + 41 \text{ is prime})$.

We refer to a statement which refers to a variable as a **predicate** or as a **propositional formula** when replacing the variable with a value makes the statement true or false.

Note that in a finite universe, we can express existentially and universally quantified propositions without quantifiers, using disjunctions and conjunctions respectively.

For example, if our universe U is $\{1, 2, 3, 4\}$, then
$(\exists x \in U)P(x)$ is logically equivalent to $P(1) \vee P(2) \vee P(3) \vee P(4)$, and
$(\forall x \in U)P(x)$ is logically equivalent to $P(1) \wedge P(2) \wedge P(3) \wedge P(4)$.

However, in an infinite universe, this is not possible.