Multiple Systems

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Chapter 1

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1.1 Classical Information

1.1.1 Classical states

Suppose that we have two systems:

- X is a system having classical state set Σ.
- Y is a system having classical state set Γ.

Imagine that X and Y are placed side-by-side, with X on the left and Y on the right, and viewed together as if they form a single system.

We denote this new compound system by (X, Y), or XY.

Question 1

What are the classical states of (X, Y)?

Solution: The classical state set of (X, Y) is the **Cartesian product**:

$$\Sigma \times \Gamma = \{(x, y) \mid x \in \Sigma, y \in \Gamma\}$$

Note:

It's not important that X is on the left and Y is on the right; we could have just as well placed Y on the left and X on the right. The important thing is that the two systems are distinguishable.

Example 1.1.1 (Card suits)

If $\Sigma = \{0, 1\}$ and $\Gamma = \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$, then:

$$\Sigma \times \Gamma = \{(0, \clubsuit), (0, \diamondsuit), (0, \heartsuit), (0, \spadesuit), (1, \clubsuit), (1, \diamondsuit), (1, \heartsuit), (1, \spadesuit)\}$$

This description generalizes to more than two systems in a natural way.

Suppose $X_1, \cdots, X_n$ are systems having classical state sets $\Sigma_1, \cdots, \Sigma_n$, respectively.

The classical state set of the n-tuple $(X_1, \cdots, X_n)$, viewed as a single compound system, is the Cartesian product:

$$\Sigma_1 \times \cdots \times \Sigma_n = \{(x_1, \cdots, x_n) \mid x_1 \in \Sigma_1, \cdots, x_n \in \Sigma_n\}$$

Example 1.1.2

If $\Sigma_1 = \Sigma_2 = \Sigma_3 = \{0, 1\}$, then the classical state set of (X_1, X_2, X_3) is:

$$\Sigma_1 \times \Sigma_2 \times \Sigma_3 = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$$

An n-tuple $(x_1, \cdots, x_n)$ may also be written as a string $x_1 \cdots x_n$.

Example 1.1.3 (Binary alphabet)

Suppose $X_1, \cdots, X_n$ are bits, so thei classical state sets are all the same:

$$\Sigma_1 = \Sigma_2 = \cdots = \Sigma_{10} = \{0, 1\}$$

The classical state set of $(X_1, \cdots, X_n)$ is the Cartesian product:

$$\Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_{10} = \{0, 1\}^{10}$$

You can also think about this as a 10-bit register in a classical computer. Written as strings, these classical states look like this:

$$\begin{gathered} 0000000000 \\ 0000000001 \\ 0000000010 \\ 0000000011 \\ \vdots \\ 1111111111 \end{gathered}$$

Note:

Cartesian products of classical state sets are ordered lexicographically (i.e., in dictionary order):

- We assume the individual classical state sets are already ordered.
- Significance decreases from left to right.

Example 1.1.4

The Cartesian product $\{1, 2, 3\} \times \{0, 1\}$ is ordered like this:

$$(1,0),(1,1),(2,0),(2,1),(3,0),(3,1)$$

When n-tuples are written as strings and ordered in this way, we observe familiar patterns, such as $\{0, 1\} \times \{0, 1\}$ being ordered as $00, 01, 10, 11$.

1.1.2 Probabilistic states

Probabilistic states of compound systems associates probabilities with the Cartesian product of the classical state sets of individual systems.

Example 1.1.5

This is a probabilistic state pair of bits (X, Y):

$$Pr((X,Y) = (0,0)) = \frac{1}{2}$$

$$Pr((X,Y) = (0,1)) = 0$$

$$Pr((X,Y) = (1,0)) = 0$$

$$Pr((X,Y) = (1,1)) = \frac{1}{2}$$

An alternate notation using vector notation:

$$\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} \begin{array}{l} \leftarrow \text{ probability associated with state } 00 \\ \leftarrow \text{ probability associated with state } 01 \\ \leftarrow \text{ probability associated with state } 10 \\ \leftarrow \text{ probability associated with state } 11 \end{array}$$

For a given probabilistic state of (X,Y), we say that X and Y are **statistically independent** if

$$Pr((X,Y) = (x,y)) = Pr(X = x)Pr(Y = y)$$

for all $x \in \Sigma$ and $y \in \Gamma$.

Suppose that a probabilistic state of (X,Y) is expressed as a vector:

$$|\pi\rangle = \sum_{(x,y)\in\Sigma\times\Gamma} p_{ab} |xy\rangle$$

The systems X and Y are independent if there exist probability vectors

$$|\phi\rangle = \sum_{x\in\Sigma} q_x |x\rangle \quad \text{and} \quad |\psi\rangle = \sum_{y\in\Gamma} r_y |y\rangle$$

such that $p_{xy} = q_x r_y$ for all $x \in \Sigma$ and $y \in \Gamma$.

Example 1.1.6 (Independent bits)

The probabilistic state of a pair of bits (X,Y) represented by the vector

$$|\pi\rangle = \frac{1}{6}|00\rangle + \frac{1}{12}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{4}|11\rangle$$

is one in which X and Y are independent. The required condition is true for these probability vectors:

$$|\phi\rangle = \frac{1}{4}|0\rangle + \frac{3}{4}|1\rangle \quad \text{and} \quad |\psi\rangle = \frac{2}{3}|0\rangle + \frac{1}{3}|1\rangle$$

Example 1.1.7 (Dependent bits)

For the probabilistic state

$$\frac{1}{2}|00\rangle + \frac{1}{2}|11\rangle$$

of two bits (X,Y), we have that X and Y are not independent.

If they were, we would have numbers q_0, q_1, r_0, r_1 such that

$$q_0 r_0 = \frac{1}{2}$$

$$q_0 r_1 = 0$$

$$q_1 r_0 = 0$$

$$q_1 r_1 = \frac{1}{2}$$

But if $q_0 r_1 = 0$, then either $q_0 = 0$ or $r_1 = 0$ (or both), contradicting either the first or last equality since any number multiplied by 0 is 0.

1.1.3 Tensor products of vectors

The tensor product of two vectors

$$|\phi\rangle = \sum_{x\in\Sigma} \alpha_x |x\rangle \quad \text{and} \quad |\psi\rangle = \sum_{y\in\Gamma} \beta_y |y\rangle$$

is the vector

$$|\phi\rangle \otimes |\psi\rangle = \sum_{(x,y)\in\Sigma\times\Gamma} \alpha_x \beta_y |xy\rangle$$

Equivalently, the vector $|\phi\rangle \otimes |\psi\rangle$ is defined by this condition:

$$\langle xy \mid \pi\rangle = \langle x \mid \phi\rangle \langle y \mid \psi\rangle \qquad \text{(for all } x \in \Sigma \text{ and } y \in \Gamma)$$

Note:

In essense, this is the same operation for probabilistic states of a pair of independent bits; we are just giving a name to it now.

Following our convention for ordering the elements of Cartesian product sets, we obtain this specification for the tensor product of two column vectors:

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} \otimes \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_k \end{pmatrix} = \begin{pmatrix} \alpha_1\beta_1 \\ \vdots \\ \alpha_1\beta_k \\ \alpha_2\beta_1 \\ \vdots \\ \alpha_2\beta_k \\ \vdots \\ \alpha_m\beta_1 \\ \vdots \\ \alpha_m\beta_k \end{pmatrix}$$

Example 1.1.8 (Tensor product)

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \otimes \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} = \begin{matrix} \alpha_1 \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} \\ \alpha_2 \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} \\ \alpha_3 \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} \end{matrix} = \begin{matrix} \begin{pmatrix} \alpha_1\beta_1 \\ \alpha_1\beta_2 \\ \alpha_1\beta_3 \\ \alpha_1\beta_4 \end{pmatrix} \\ \begin{pmatrix} \alpha_2\beta_1 \\ \alpha_2\beta_2 \\ \alpha_2\beta_3 \\ \alpha_2\beta_4 \end{pmatrix} \\ \begin{pmatrix} \alpha_3\beta_1 \\ \alpha_3\beta_2 \\ \alpha_3\beta_3 \\ \alpha_3\beta_4 \end{pmatrix} \end{matrix} = \begin{pmatrix} \alpha_1\beta_1 \\ \alpha_1\beta_2 \\ \alpha_1\beta_3 \\ \alpha_1\beta_4 \\ \alpha_2\beta_1 \\ \alpha_2\beta_2 \\ \alpha_2\beta_3 \\ \alpha_2\beta_4 \\ \alpha_3\beta_1 \\ \alpha_3\beta_2 \\ \alpha_3\beta_3 \\ \alpha_3\beta_4 \end{pmatrix}$$

Observe the following expression for tensor products of standard basis vectors:

$$|x\rangle \otimes |y\rangle = |x\rangle |y\rangle = |xy\rangle$$

Alternatively, writing (x, y) as an ordered pair rather than a string, we could written

$$|x\rangle \otimes |y\rangle = |(x, y)\rangle$$

but it is more common to write

$$|x\rangle \otimes |y\rangle = |x, y\rangle$$

Note:

The tensor product of two vectors is **bilinear**.

1. Linearity in the first argument:

$$(|\phi_1\rangle + |\phi_2\rangle) \otimes |\psi\rangle = |\phi_1\rangle \otimes |\psi\rangle + |\phi_2\rangle \otimes |\psi\rangle$$

$$(\alpha |\phi\rangle) \otimes |\psi\rangle = \alpha(|\phi\rangle \otimes |\psi\rangle)$$

2. Linearity in the second argument:

$$|\phi\rangle \otimes (|\psi_1\rangle + |\psi_2\rangle) = |\phi\rangle \otimes |\psi_1\rangle + |\phi\rangle \otimes |\psi_2\rangle$$

$$|\phi\rangle \otimes (\alpha |\psi\rangle) = \alpha(|\phi\rangle \otimes |\psi\rangle)$$

Notice that scalars "float freely" within tensor products:

$$\alpha |\phi\rangle \otimes |\psi\rangle = \alpha(|\phi\rangle \otimes |\psi\rangle) = |\psi\rangle \otimes \alpha |\phi\rangle$$

There's no difference multiplying the first argument by the scalar α or multiplying the second argument by the scalar α.

Tensor products generalize to three or more systems.

If $|\phi_1\rangle, \cdots, |\phi_n\rangle$ are vectors, then the tensor product

$$|\psi\rangle = |\phi_1\rangle \otimes \cdots \otimes |\phi_n\rangle$$

is defined by the equation

$$\langle \alpha_1 \cdots \alpha_n \mid \psi\rangle = \langle \alpha_1 \mid \phi_1\rangle \cdots \langle \alpha_n \mid \phi_n\rangle$$

Equivalently, the tensor product of three or more vectors can be defined recursively:

$$|\phi_1\rangle \otimes \cdots \otimes |\phi_n\rangle = (|\phi_1\rangle \otimes \cdots \otimes |\phi_{n-1}\rangle) \otimes |\phi_n\rangle$$

We can think about the tensor product of n vectors ϕ_1 through ϕ_n as being the tensor product of the first $n-1$ vectors, followed by the tensor product of the result with the n-th vector.

To find the tensor product of the first $n-1$ vectors, we can apply the same recursive definition, and so on until we reach the tensor product of the first two vectors.