Note 0

Review of Sets and Notation

Set: well-defined collection of objects (**elements** or **members**)

- usually denoted by capital letters
- defined by listing its elements and surrounding the list by curly braces

$$A = \{2, 3, 5, 7, 11\}$$

If x is an element of A, then we write $x \in A$.
If y is not an element of A, then we write $y \notin A$.

Two sets A and B are said to be **equal**, written as $A = B$, if they have the same elements.

- order and repetition of elements do not matter
  - $\{\text{red}, \text{white}, \text{blue}\} = \{\text{blue}, \text{white}, \text{red}\} = \{\text{red}, \text{white}, \text{white}, \text{blue}\}$

More complicated sets might be defined using a different notation. The set of all rational numbers, denoted by $\mathbb{Q}$, can be written as:

$$\mathbb{Q} = \{\frac{a}{b} \mid a, b \text{ are integers}, b \neq 0\}$$

In English, this is read as "$\mathbb{Q}$ is the set of all fractions such that the numerator is an integer and the denominator is a non-zero integer."

Cardinality

Cardinality: the size of a set

If $A = \{1, 2, 3, 4\}$, then the cardinality of A, denoted by $|A|$, is 4. It is possible for the cardinality of a set to be 0. The **empty set**, denoted by the symbol $\emptyset$, is a unique such set.

A set can also have an infinite number of elements, such as the set of all integers, prime numbers, or odd numbers.

Subsets and Proper Subsets

If every element of a set A is also in set B, then A is a **subset** of B, written as $A \subseteq B$. Equivalently we can write $B \supseteq A$, or B is a **superset** of A.

A **proper subset** is a set A that is strictly contained in set B, written as $A \subset B$, meaning that A excludes at lest one element of B.

Consider the set $B = \{1, 2, 3, 4, 5\}$. Then $A = \{1, 2, 3\}$ is both a subset and a proper subset of B, while $C = \{1, 2, 3, 4, 5\}$ is a subset but not a proper subset of B.

Basic properties regarding subsets:

- The empty set, denoted by $\{\}$ or $\emptyset$, is a proper subset of any nonempty set A: $\emptyset \subset A$
- The empty set is a subset of every set B: $\emptyset \subseteq B$
- Every set A is a subset of itself: $A \subseteq A$

Intersections and Unions

The **intersection** of a set A with a set B, written as $A \cap B$, is the set containing all elements that are in both A and B.

Two sets are said to be **disjoint** if $A \cap B = \emptyset$.

The **union** of the set A with a set B, written as $A \cup B$, is the set of all elements which are in either A or B or both.

If A is the set of all positive even numbers, and B is the set of all positive odd integers, then $A \cap B = \emptyset$, and $A \cup B = \mathbb{Z}^+$, or the set of all positive integers.

Properties of intersections and unions:

- $A \cup B = B \cup A$
- $A \cup \emptyset = A$
- $A \cap B = B \cap A$
- $A \cap \emptyset = \emptyset$

Complements

If A and B are two sets, then the **relative complement** of A in B, or the **set difference** between B and A, written as $B - A$ or $B \setminus A$, is the set of elements in B, but not in A: $B \setminus A = \{x \in B \mid x \notin A\}$.

If $B = \{1, 2, 3\}$ and $A = \{3, 4, 5\}$, then $B \setminus A = \{1, 2\}$.
If $\mathbb{R}$ is the set of real numbers and $\mathbb{Q}$ is the set of rational numbers, then $\mathbb{R} \setminus \mathbb{Q}$ is the set of irrational numbers.

Properties of complements:

- $A \setminus A = \emptyset$
- $A \setminus \emptyset = A$
- $\emptyset \setminus A = \emptyset$

Significant Sets

In mathematics, some sets are referred to so commonly that they are denoted by special symbols. These include:

- $\mathbb{N}$ denotes the set of all natural numbers: $\{0, 1, 2, 3, \dots\}$
- $\mathbb{Z}$ denotes the set of all integer numbers: $\{\dots, -2, -1, 0, 1, 2, \dots\}$
- $\mathbb{Q}$ denotes the set of all rational numbers: $\{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$
- $\mathbb{R}$ denotes the set of all real numbers
- $\mathbb{C}$ denotes the set of all complex numbers

Products and Power Sets

The **Cartesian product** (also called the **cross product**) of two sets A and B, written as $A \times B$, is the set of all pairs whose first component is an element of A and whose second component is an element of B.

In set notation, $A \times B = \{(a, b) \mid a \in A, b \in B\}$.

If $A = \{1, 2, 3\}$ and $B = \{u, v\}$, then $A \times B = \{(1, u), (1, v), (2, u), (2, v), (3, u), (3, v)\}$.
$\mathbb{N} \times \mathbb{N} = \{(0, 0), (1, 0), (0, 1), (1, 1), (2, 0), \dots\}$ is the set of all pairs of natural numbers.

Given a set S, the **power set** of S, denotes by $\mathcal{P}(S)$, is the set of all subsets of S: $\{T \mid T \subseteq S\}$.

If $|S| = k$, then $|\mathcal{P}(S)| = 2^k$.

Mathematical Notation

Sums and Products

There is a compact notation for writing sums or products of large numbers of items.

To write $1 + 2 + \cdots + n$, we write $\sum_{i=1}^{n} i$. More generally we can write the sum $f(m) + f(m+1) + \cdots + f(n)$ as $\sum_{i=m}^{n} f(i)$. For example $\sum_{i=5}^{n} i^2 = 5^2 + 6^2 + \cdots + n^2$.

To write the product $f(m)f(m+1) + \cdots + f(n)$, we use the notation $\prod_{i=m}^{n} f(i)$. For example, $\prod_{i=1}^{n} i = 1 \cdot 2 \cdots n$ is the product of the first n positive integers.

Universal and Existential Quantifiers

$\forall$: universal quantifier ("for all")
$\exists$: existential quantifier ("there exists")

Consider the statement: For all natural numbers n, $n^2 + n + 41$ is prime. Here, n is quantified to any element of the set $\mathbb{N}$ of natural numbers.
In notation, we write $(\forall n \in \mathbb{N})(n^2 + n + 41 \text{ is prime})$.

$n^2 + n + 41$ is indeed prime for small values of n, but there are values that make it not prime ($n = 41$). Therefore, this statement is not true.

Consider this new statement: $(\exists x \in \mathbb{Z})(x < 2 \text{ and } x^2 = 4)$. The statement says that there is an integer x which is less than 2, but its square is equal to 4. This statement is true ($x = -2$).

Statements an be written using both kinds of quantifiers:

1. $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(y > x)$

For all x in the set of integers, there exists a y in the set of integers where y is greater than x (true).

2. $(\exists y \in \mathbb{Z})(\forall x \in \mathbb{Z})(y > x)$

There exists a y in the set of integers which is greater than x for all x in the set of integers (false).