Note 1

Propositional Logic

To become fluent in working with mathematical statements, you need to understand the basic framework of the language of mathematics.

Proposition: a statement which is either true or false

These statements are propositions:

1. $\sqrt{3}$ is irrational.
2. $1 + 1 = 5$.

These statements are not propositions:

1. $2 + 2$.
2. $x^2 + 3x = 5$. (since x is unknown)

Propositions should not include fuzzy terms, so these statements aren't propositions either (although some sources may say they are):

1. Arnold Schwarzenegger often eats broccoli. (”often” is fuzzy)
2. Henry VIII was unpopular. (”unpopular” is fuzzy)

Propositions may be joined together to form more complex statements. Let P, Q, and R be variables representing propositions. The simplest way of joining these propositions together is to use the connectives ”and,” ”or,” and ”not.”

1. **Conjunction**: $P \wedge Q$ (”P and Q”). True only when both P and Q are true.
2. **Disjunction**: $P \vee Q$ (”P or Q”). True when at least one of P and Q is true.
3. **Negation**: $\neg P$ (”not P”). True when P is false.

Statements with variables (like these) are called **propositional forms**.

A fundamental principle known as the **law of the excluded middle** says that, for any proposition P, either P is true or $\neg P$ is true (but not both). Thus $P \vee \neg P$ is always true, regardless of the truth value of P. A propositional form is always true regardless of its truth values is called a **tautology**. Conversely, a statement such as $P \wedge \neg P$, which is always false, is called a **contradiction**.

A **truth table** is used to describe the possible values of a propositional form. Truth tables are the same as function tables: you list all possible input values for the variables, and then list the outputs given for those inputs.

Here are the truth tables for conjunction, disjunction, and negation:

P	Q	$P \wedge Q$	$P \vee Q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

P	$\neg P$
T	F
F	T

The most important and subtle propositional form is an **implication**:

4. **Implication**: $P \Rightarrow Q$ ("P implies Q"). This is the same as "if P, then Q."

Here, P is called the **hypothesis** of the implication, and Q is the **conclusion**.

Here are some examples:

- If you stand in the rain, then you'll get wet.
- If you passed the class, you received a certificate.

An implication $P \Rightarrow Q$ is false only when P is true and Q is false.

Here is the truth table for $P \Rightarrow Q$ (along with an additional column):

P	Q	$P \Rightarrow Q$	$\neg P \vee Q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Note the $P \Rightarrow Q$ is always true when P is false. When an implication is stupidly true because the hypothesis is false, we say that it is **vacuously true**.

Note also that $P \Rightarrow Q$ is **logically equivalent** to $\neg P \vee Q$. We write this as $(P \Rightarrow Q) \equiv (\neg P \vee Q)$.

Here are some different ways of meaning $P \Rightarrow Q$:

1. if P, then Q
2. Q if P
3. P only if Q
4. P is sufficient for Q
5. Q is necessary for P
6. Q unless not P

If both $P \Rightarrow Q$ and $Q \Rightarrow P$ are true, then we say "P if and only if Q" (abbreviated "P iff Q"). Formally, we write $P \Leftrightarrow Q$. Note that $P \Leftrightarrow Q$ is true when P and Q have the same truth values (both true or false).

Given an implication $P \Rightarrow Q$, we can also define its

1. **Contrapositive**: $\neg Q \Rightarrow \neg P$
2. **Converse**: $Q \Rightarrow P$

Here are some truth tables:

P	Q	$\neg P$	$\neg Q$	$P \Rightarrow Q$	$Q \Rightarrow P$	$\neg Q \Rightarrow \neg P$	$P \Leftrightarrow Q$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	F	T	F	T	F
F	F	T	T	T	T	T	T

Note that $P \Rightarrow Q$ and its contrapositive have the same truth values, so they are logically equivalent: $(P \Rightarrow Q) \equiv (\neg Q \Rightarrow \neg P)$.

Also note that $P \Rightarrow Q$ and $Q \Rightarrow P$ are not logically equivalent.

Quantifiers

The mathematical statements you'll see in practice will look something like this:

1. For all natural numbers n, $n^2 + n + 41$ is prime.
2. If n is an odd integer, so is n^3.
3. There is an integer k that is both even and odd.

These statements assert something about lots of simple propositions all at once. For instance, the first statement is asserting that $0^2 + 0 + 41$ is prime, $1^2 + 1 + 41$ is prime, and so on.

The last statement says that as k ranges over all possible integers, we will find at least one value of k for which the statement is satisfied.

Compared to the previous statement "$x^2 + 3x = 5$," these examples are quantified over a "universe." To express these statements mathematically, we need two **quantifiers**:

1. the universal quantifier $\forall$ ("for all")
2. the existential quantifier $\exists$ ("there exists")

Examples:

1. "Some mammals lay eggs."

Mathematically, "some" means "at least one," so the statement is actually saying "There exists a mammal x such that x lays eggs." If we let our universe U to be the set of mammals, then we can write: $(\exists x \in U)(x \text{ lays eggs})$.

2. "For all natural numbers n, $n^2 + n + 41$ is prime."

In this case, our universe becomes the set of natural numbers, denoted as $\mathbb{N}$:
$(\forall n \in \mathbb{N})(n^2 + n + 41 \text{ is prime})$.

We refer to a statement which refers to a variable as a **predicate** or as a **propositional formula** when replacing the variable with a value makes the statement true or false.

Note that in a finite universe, we can express existentially and universally quantified propositions without quantifiers, using disjunctions and conjunctions respectively.

For example, if our universe U is $\{1, 2, 3, 4\}$, then
$(\exists x \in U)P(x)$ is logically equivalent to $P(1) \vee P(2) \vee P(3) \vee P(4)$, and
$(\forall x \in U)P(x)$ is logically equivalent to $P(1) \wedge P(2) \wedge P(3) \wedge P(4)$.

However, in an infinite universe, this is not possible.

Negation

Let's look at how to negate conjunctions and disjunctions:

$$\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$$
$$\neg(P \vee Q) \equiv (\neg P \wedge \neg Q)$$

These two equivalences are known as **De Morgan's Laws**, and they are quite intuitive: for example, if it is not the case that $P \wedge Q$ is true, then either P or Q is false (and vice versa).

Negating propositions involving quantifiers actually follows analogous laws.

Assume that the universe is $\{1, 2, 3, 4\}$ and let $P(x)$ denote the propositional formula "$x^2 > 10$".
Check that $\exists x P(x)$ is true but $\forall x P(x)$ is false.
Observe that both $\neg(\forall x P(x))$ and $\exists x \neg P(x)$ are true because $P(1)$ is false.
Also note that both $\forall x \neg P(x)$ and $\neg(\exists x P(x))$ are false, because $P(4)$ is true.

The fact that each pair of statements had the same truth value, as the equivalences

$$\neg(\forall x P(x)) \equiv \exists x \neg P(x)$$
$$\neg(\exists x P(x)) \equiv \forall x \neg P(x)$$

are laws that hold for any proposition P quantified over any universe (including infinite ones).

It is helpful to think of English sentences to convince yourself (informally) that these laws are true.

For example, assume that we are working within the universe $\mathbb{Z}$ (the set of all integers), and that $P(x)$ is the proposition "x is odd." We know that the statement $(\forall x P(x))$ is false, since not every integer is odd.

Therefore, we expect its negation, $\neg(\forall x P(x))$, to be true. How would you say the negation in English? If it is not true that every integer is odd, then there must exist some integer which is not odd (i.e., even). $(\exists x \neg P(x))$

To see a more complex example, fix some universe and propositional formula $P(x, y)$. Assume we have the proposition $\neg(\forall x \exists y P(x, y))$ and we want to push the negation operator inside the quantifiers. By the above laws, we can do so:

$$\neg(\forall x \exists y P(x, y)) \equiv \exists x \neg(\exists y P(x, y)) \equiv \exists x \forall y \neg P(x, y)$$

Notice that we broke the complex negation into a smaller, easier problem as the negation propagated itself through the quantifiers. Note also that the quantifiers "flip" as we go.

Write the sentence "there are at least three distinct integers x that satisfy $P(x)$" as a proposition using quantifiers. One way to do it is

$$\exists x \exists y \exists z (x \neq y \wedge y \neq z \wedge x \neq z \wedge P(x) \wedge P(y) \wedge P(z))$$

Write the sentence "there are **at most** three distinct integers x that satisfy $P(x)$" as a proposition using quantifiers. One way to do it is

$$\exists x \exists y \exists z \forall d (P(d) \Rightarrow d = x \vee d = y \vee d = z)$$

Here is an equivalent way to do it:

$$\forall x \forall y \forall v \forall z ((x \neq y \wedge y \neq v \wedge v \neq x \wedge x \neq z \wedge y \neq z \wedge v \neq z) \Rightarrow \neg(P(x) \wedge P(y) \wedge P(v) \wedge P(z)))$$

What if we want to express the sentence "there are **exactly** three distinct integers x that satisfy $P(x)$?" We can use the conjunction of the two propositions above.