Note 3

Mathematical Induction

Induction is a powerful tool which is used to establish that a statement holds for *all* natural numbers. Of course, there are infinitely many natural numbers – induction provides a way to reason about them by finite means.

Suppose we wish to prove the statement: For all natural numbers n, $0 + 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$. More formally, we can write this as

$$\forall n \in \mathbb{N}, \sum_{i=0}^{n} i = \frac{n(n+1)}{2}. \tag{1}$$

How would you prove this? You could begin by checking that it holds for $n = 0, 1, 2$, and so forth, but there's an infinite number of values of n for which it needs to be checked. Moreover, checking just the first few values of n does not suffice to conclude the statement holds for all $n \in \mathbb{N}$.

Consider this statement that was shown in a previous note: $\forall n \in \mathbb{N}, n^2 - n + 41$ is a prime number. Check that it holds for the first few natural numbers. Now check the case for $n = 41$.

In mathematical induction, we instead make an interesting observation: Suppose the statement holds for some value $n = k$, i.e., $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$. (This is called the *induction hypothesis*. Then:

$$\left(\sum_{i=0}^{k} i \right) + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}, \tag{2}$$

i.e., the claim also holds for $n = k + 1$! In other words, if the statement holds for some k, then it must also hold for $k + 1$. Let us call the argument above the *inductive step*. If we can show that the statement holds for k, then the inductive step allows us to conclude that it also holds for $k + 1$; but that it holds for $k + 1$, the inductive step implies that it holds for $k + 2$; and so on.

So have we proven Equation (1)? Not yet. The problem is that in order to apply the inductive step, we first have to establish that Equation (1) holds for some initial value of k. Since our aim is to prove the statement for all natural numbers, the obvious choice is $k = 0$. We call this choice of k the *base case*. Then, if the base case holds, the axiom of mathematical induction says that the inductive step allows us to conclude that Equation (1) indeed holds for all $n \in \mathbb{N}$.

Let's formally rewrite this proof.

Theorem 3.1. $\forall n \in \mathbb{N}, \displaystyle\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.

Proof of Theorem 3.1. We proceed by induction on the variable n.

Base case (n=0): Here, we have $\displaystyle\sum_{i=0}^{0} i = 0 = \frac{0(0+1)}{2}$. Thus, the base case is correct.

Induction Hypothesis: For arbitrary $n = k \geq 0$, assume that $\sum_{i=0}^{k} i = \dfrac{k(k+1)}{2}$. In words, the induction hypothesis says "let's assume we have proved the statement for an arbitrary value of $n = k \geq 0$."

Inductive Step: Prove the statement for $n = (k+1)$, i.e., show that $\sum_{i=0}^{k+1} i = \dfrac{(k+1)(k+2)}{2}$:

$$\sum_{i=0}^{k+1} i = \left(\sum_{i=0}^{k} i \right) + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}, \quad (3)$$

where the second equality follows from the induction hypothesis. By the principle of mathematical induction, the claim follows. □

Recap.

1. **Base Case**: Prove that $P(0)$ is true.

2. **Induction Hypothesis**: For arbitrary $k \geq 0$, assume that $P(k)$ is true.

3. **Inductive Step**: With the assumption of the induction hypothesis in hand, show that $P(k+1)$ is true.

Finally, a word about choosing a base case – in general, the choice of base case will naturally depend on the claim you with to prove.

Let us do another proof by induction. Recall that for integers a and b, we say that a divides b, denoted $a \mid b$, iff there exists an integer q satisfying $b = aq$.

Theorem 3.2. *For all $n \in \mathbb{N}, n^3 - n$ is divisible by 3.*