Note 3

Mathematical Induction

Induction is a powerful tool which is used to establish that a statement holds for *all* natural numbers. Of course, there are infinitely many natural numbers – induction provides a way to reason about them by finite means.

Suppose we wish to prove the statement: For all natural numbers n, $0+1+2+3+\cdots+n = \frac{n(n+1)}{2}$. More formally, we can write this as

$$\forall n \in \mathbb{N}, \sum_{i=0}^{n} i = \frac{n(n+1)}{2}. \tag{1}$$

How would you prove this? You could begin by checking that it holds for $n = 0, 1, 2$, and so forth, but there's an infinite number of values of n for which it needs to be checked. Moreover, checking just the first few values of n does not suffice to conclude the statement holds for all $n \in \mathbb{N}$.

Consider this statement that was shown in a previous note: $\forall n \in \mathbb{N}, n^2 - n + 41$ is a prime number. Check that it holds for the first few natural numbers. Now check the case for $n = 41$.

In mathematical induction, we instead make an interesting observation: Suppose the statement holds for some value $n = k$, i.e., $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$. (This is called the *induction hypothesis.* Then:

$$\left(\sum_{i=0}^{k} i\right) + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}, \tag{2}$$

i.e., the claim also holds for $n = k+1$! In other words, if the statement holds for some k, then it must also hold for $k+1$. Let us call the argument above the *inductive step.* If we can show that the statement holds for k, then the inductive step allows us to conclude that it also holds for $k+1$; but that it holds for $k+1$, the inductive step implies that it holds for $k+2$; and so on.