

# Note 0

## Review of Sets and Notation

**Set:** well-defined collection of objects (**elements** or **members**)

- usually denoted by capital letters
- defined by listing its elements and surrounding the list by curly braces

$$A = \{2, 3, 5, 7, 11\}$$

If  $x$  is an element of  $A$ , then we write  $x \in A$ .

If  $y$  is not an element of  $A$ , then we write  $y \notin A$ .

Two sets  $A$  and  $B$  are said to be **equal**, written as  $A = B$ , if they have the same elements.

- order and repetition of elements do not matter
  - $\{\text{red, white, blue}\} = \{\text{blue, white, red}\} = \{\text{red, white, white, blue}\}$

More complicated sets might be defined using a different notation. The set of all rational numbers, denoted by  $\mathbb{Q}$ , can be written as:

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \text{ are integers, } b \neq 0 \right\}$$

In English, this is read as "  $\mathbb{Q}$  is the set of all fractions such that the numerator is an integer and the denominator is a non-zero integer."

## Cardinality

**Cardinality:** the size of a set

If  $A = \{1, 2, 3, 4\}$ , then the cardinality of  $A$ , denoted by  $|A|$ , is 4. It is possible for the cardinality of a set to be 0. The **empty set**, denoted by the symbol  $\emptyset$ , is a unique such set.

A set can also have an infinite number of elements, such as the set of all integers, prime numbers, or odd numbers.

## Subsets and Proper Subsets

If every element of a set  $A$  is also in set  $B$ , then  $A$  is a **subset** of  $B$ , written as  $A \subseteq B$ .

Equivalently we can write  $B \supseteq A$ , or  $B$  is a **superset** of  $A$ .

A **proper subset** is a set  $A$  that is strictly contained in set  $B$ , written as  $A \subset B$ , meaning that  $A$  excludes at least one element of  $B$ .

Consider the set  $B = \{1, 2, 3, 4, 5\}$ . Then  $A = \{1, 2, 3\}$  is both a subset and a proper subset of  $B$ , while  $C = \{1, 2, 3, 4, 5\}$  is a subset but not a proper subset of  $B$ .

Basic properties regarding subsets:

- The empty set, denoted by  $\{\}$  or  $\emptyset$ , is a proper subset of any nonempty set  $A$ :  $\emptyset \subset A$
- The empty set is a subset of every set  $B$ :  $\emptyset \subseteq B$
- Every set  $A$  is a subset of itself:  $A \subseteq A$

## Intersections and Unions

The **intersection** of a set  $A$  with a set  $B$ , written as  $A \cap B$ , is the set containing all elements that are in both  $A$  and  $B$ .

Two sets are said to be **disjoint** if  $A \cap B = \emptyset$ .

The **union** of the set  $A$  with a set  $B$ , written as  $A \cup B$ , is the set of all elements which are in either  $A$  or  $B$  or both.

If  $A$  is the set of all positive even numbers, and  $B$  is the set of all positive odd integers, then  $A \cap B = \emptyset$ , and  $A \cup B = \mathbb{Z}^+$ , or the set of all positive integers.

Properties of intersections and unions:

- $A \cup B = B \cup A$
- $A \cup \emptyset = A$
- $A \cap B = B \cap A$
- $A \cap \emptyset = \emptyset$

## Complements

If  $A$  and  $B$  are two sets, then the **relative complement** of  $A$  in  $B$ , or the **set difference** between  $B$  and  $A$ , written as  $B - A$  or  $B \setminus A$ , is the set of elements in  $B$ , but not in  $A$ :  $B \setminus A = \{x \in B \mid x \notin A\}$ .

If  $B = \{1, 2, 3\}$  and  $A = \{3, 4, 5\}$ , then  $B \setminus A = \{1, 2\}$ .

If  $\mathbb{R}$  is the set of real numbers and  $\mathbb{Q}$  is the set of rational numbers, then  $\mathbb{R} \setminus \mathbb{Q}$  is the set of irrational numbers.

Properties of complements:

- $A \setminus A = \emptyset$
- $A \setminus \emptyset = A$
- $\emptyset \setminus A = \emptyset$

## Significant Sets

In mathematics, some sets are referred to so commonly that they are denoted by special symbols. These include:

- $\mathbb{N}$  denotes the set of all natural numbers:  $\{0, 1, 2, 3, \dots\}$
- $\mathbb{Z}$  denotes the set of all integer numbers:  $\{\dots, -2, -1, 0, 1, 2, \dots\}$
- $\mathbb{Q}$  denotes the set of all rational numbers:  $\{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$
- $\mathbb{R}$  denotes the set of all real numbers
- $\mathbb{C}$  denotes the set of all complex numbers

## Products and Power Sets

The **Cartesian product** (also called the **cross product**) of two sets  $A$  and  $B$ , written as  $A \times B$ , is the set of all pairs whose first component is an element of  $A$  and whose second component is an element of  $B$ .

In set notation,  $A \times B = \{(a, b) \mid a \in A, b \in B\}$ .

If  $A = \{1, 2, 3\}$  and  $B = \{u, v\}$ , then  $A \times B = \{(1, u), (1, v), (2, u), (2, v), (3, u), (3, v)\}$ .  $\mathbb{N} \times \mathbb{N} = \{(0, 0), (1, 0), (0, 1), (1, 1), (2, 0), \dots\}$  is the set of all pairs of natural numbers.

Given a set  $S$ , the **power set** of  $S$ , denoted by  $\mathcal{P}(S)$ , is the set of all subsets of  $S$ :  $\{T \mid T \subseteq S\}$ .

If  $|S| = k$ , then  $|\mathcal{P}(S)| = 2^k$ .

## Mathematical Notation

### Sums and Products

There is a compact notation for writing sums or products of large numbers of items.

To write  $1 + 2 + \dots + n$ , we write  $\sum_{i=1}^n i$ . More generally we can write the sum  $f(m) + f(m+1) + \dots + f(n)$  as  $\sum_{i=m}^n f(i)$ . For example  $\sum_{i=5}^n i^2 = 5^2 + 6^2 + \dots + n^2$ .

To write the product  $f(m)f(m+1) + \dots + f(n)$ , we use the notation  $\prod_{i=m}^n f(i)$ . For example,  $\prod_{i=1}^n i = 1 \cdot 2 \cdot \dots \cdot n$  is the product of the first  $n$  positive integers.

### Universal and Existential Quantifiers

$\forall$ : universal quantifier ("for all")

$\exists$ : existential quantifier ("there exists")

Consider the statement: For all natural numbers  $n$ ,  $n^2 + n + 41$  is prime. Here,  $n$  is quantified to any element of the set  $\mathbb{N}$  of natural numbers.

In notation, we write  $(\forall n \in \mathbb{N})(n^2 + n + 41 \text{ is prime})$ .

$n^2 + n + 41$  is indeed prime for small values of  $n$ , but there are values that make it not prime ( $n = 41$ ). Therefore, this statement is not true.

Consider this new statement:  $(\exists x \in \mathbb{Z})(x < 2 \text{ and } x^2 = 4)$ . The statement says that there is an integer  $x$  which is less than 2, but its square is equal to 4. This statement is true ( $x = -2$ ).

Statements can be written using both kinds of quantifiers:

1.  $(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(y > x)$

For all  $x$  in the set of integers, there exists a  $y$  in the set of integers where  $y$  is greater than  $x$  (true).

2.  $(\exists y \in \mathbb{Z})(\forall x \in \mathbb{Z})(y > x)$

There exists a  $y$  in the set of integers which is greater than  $x$  for all  $x$  in the set of integers (false).