Note 2 Solutions

1. Generalize the proof of Theorem 2.2 so that it works for any positive integer n. (Hint: Suppose n has k digits, and write a_i for the digits of n, so that $n = \sum_{i=0}^{k-1}(a_i \cdot 10^i)$.)

Theorem 2.9. *Let $n \in \mathbb{Z}^+$. If the sum of the digits of n is divisible by 9, then n is divisible by 9.*

Proof of Theorem 2.9. We proceed with direct proof. Suppose n has k digits and $n = \sum_{i=0}^{k-1}(a_i \cdot 10^i)$. i goes all the way up to $k-1$ since we start on index 0. We can rewrite n as

$$n = 10^0 \cdot a_0 + 10^1 \cdot a_1 + 10^2 \cdot a_2 + \cdots + 10^{k-2} \cdot a_{k-2} + 10^{k-1} \cdot a_{k-1}.$$

Assume that the sum of the digits of n is divisible by 9, i.e.

$$\exists l \in \mathbb{Z} \text{ such that } \sum_{i=0}^{k-1} a_i = 9l. \tag{1}$$

Adding $n = \sum_{i=1}^{k-1}(a_i \cdot [10^i - 1])$ to both sides of Equation (1), we have

$$\begin{aligned} &a_0 + 10a_1 + 100a_2 \cdots + 10^{k-1}a_{k-1} = n = 9l + 9a_1 + 99a_2 + \cdots + (10^{k-1} - 1)a_{k-1} \\ &= 9(l + a_1 + 9a_2 + \cdots + [10^{k-2} + 10^{k-3} + \cdots + 1]a_{k-1}). \end{aligned}$$

We conclude that n is divisible by 9. ☐

2. Prove Lemma 2.2. (Hint: First try a direct proof. Then, try contraposition. Which proof approach is better suited to proving this lemma?)

Lemma 2.2. *If a^2 is even, then a is even.*

Proof of Lemma 2.2 using direct proof. We proceed with direct proof. One thing to point out is that non-integer numbers cannot be even nor odd, so a and a^2 must be whole numbers, therefore meaning that whatever a^2 equals must be a perfect square. Assume that a^2 is even, i.e.,

$$\exists k \in \mathbb{Z}^+ \text{ such that } a^2 = (2k)^2.$$

Taking the square root of both sides yields

$$a = 2k.$$

Since k is divisible by 2, it holds that a is even, as desired. ☐

Proof of Lemma 2.2 using contraposition. We proceed by contraposition. This means that if a is not even, then a^2 is not even. Assume that a is not even, i.e.,

$$\exists k \in \mathbb{Z} \text{ such that } a = 2k + 1.$$

Squaring both sides yields

$$a^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1,$$

which is not even since there isn't a factor of two on the entire right side. Therefore a^2 is odd, as desired. ☐