(d)

$$\frac{\partial^2 z}{\partial \theta^2} \overset{\text{(b)}}{=} \frac{\partial}{\partial \theta}\left[\frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)\right]$$
$$= \tfrac{\partial}{\partial \theta}\left[\tfrac{\partial z}{\partial x}\right](-r\sin\theta) + \tfrac{\partial z}{\partial x}(-r\cos\theta) + \tfrac{\partial}{\partial \theta}\left[\tfrac{\partial z}{\partial y}\right](r\cos\theta) + \tfrac{\partial z}{\partial y}(-r\sin\theta)$$
$$\overset{\text{C.R.}}{=} \left\{\tfrac{\partial}{\partial x}\left[\tfrac{\partial z}{\partial x}\right]\tfrac{\partial x}{\partial \theta} + \tfrac{\partial}{\partial y}\left[\tfrac{\partial z}{\partial x}\right]\tfrac{\partial y}{\partial \theta}\right\}(-r\sin\theta) + \tfrac{\partial z}{\partial x}(-r\cos\theta)$$
$$+ \left\{\tfrac{\partial}{\partial x}\left[\tfrac{\partial z}{\partial y}\right]\tfrac{\partial x}{\partial \theta} + \tfrac{\partial}{\partial y}\left[\tfrac{\partial z}{\partial y}\right]\tfrac{\partial y}{\partial \theta}\right\}(r\cos\theta) + \tfrac{\partial z}{\partial y}(-r\sin\theta)$$
$$=$$

(e)

$$\frac{\partial^2 z}{\partial r \partial \theta} = \frac{\partial^2 z}{\partial \theta \partial r} \overset{\text{(a)}}{=} \frac{\partial}{\partial \theta}\left[\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right]$$
$$= \tfrac{\partial}{\partial \theta}\left[\tfrac{\partial z}{\partial x}\right]\cos\theta + \tfrac{\partial z}{\partial x}(-\sin\theta) + \tfrac{\partial}{\partial \theta}\left[\tfrac{\partial z}{\partial y}\right]\sin\theta + \tfrac{\partial z}{\partial y}\cos\theta$$
$$\overset{\text{C.R.}}{=} \left\{\tfrac{\partial}{\partial x}\left[\tfrac{\partial z}{\partial x}\right]\tfrac{\partial x}{\partial \theta} + \tfrac{\partial}{\partial y}\left[\tfrac{\partial z}{\partial x}\right]\tfrac{\partial y}{\partial \theta}\right\}\cos\theta - \tfrac{\partial z}{\partial x}\sin\theta$$
$$+ \left\{\tfrac{\partial}{\partial x}\left[\tfrac{\partial z}{\partial y}\right]\tfrac{\partial x}{\partial \theta} + \tfrac{\partial}{\partial y}\left[\tfrac{\partial z}{\partial y}\right]\tfrac{\partial y}{\partial \theta}\right\}\sin\theta + \tfrac{\partial z}{\partial y}\cos\theta$$
$$=$$

$$=$$

Alternatively, $\dfrac{\partial^2 z}{\partial r \partial \theta} \overset{\text{(b)}}{=} \dfrac{\partial}{\partial r}\left[\dfrac{\partial z}{\partial x}(-r\sin\theta) + \dfrac{\partial z}{\partial y}(r\cos\theta)\right] = \cdots$, etc., yielding the same result; see Exercise 1.

When these are combined in various ways, simplification often occurs because of basic trigonometry (most often, $\sin^2\theta + \cos^2\theta = 1$).

Example 4 (a relationship between all four first derivatives)

Prove that $\left(\dfrac{\partial z}{\partial x}\right)^2 + \left(\dfrac{\partial z}{\partial y}\right)^2 = \left(\dfrac{\partial z}{\partial r}\right)^2 + \dfrac{1}{r^2}\left(\dfrac{\partial z}{\partial \theta}\right)^2$: see Exercise 2.

Example 5 (a relationship between four second derivatives, and one pesky first derivative)

Prove that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial^2 z}{\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^2 z}{\partial \theta^2} + \dfrac{1}{r}\dfrac{\partial z}{\partial r}$: see Exercise 3.