Perfect Writeup: Assignment 13 Extra Credit

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Exercise 5

Assume that $a_1, \ldots, a_n > 0, k > 0$, and $p > 1$. In the previous Exercise 4, it was found that $k\left(\sum_{j=1}^{n} a_j^{p/(p-1)}\right)^{(p-1)/p}$ is the maximum value of $f\left(x_1, \ldots, x_n\right) = a_1 x_1 + \ldots + a_n x_n$ subject to $x_1^p + \ldots + x_n^p = k^p$ and $x_1, \ldots, x_n > 0$; this occurs when each $x_i = k a_i^{1/(p-1)}\left(\sum_{j=1}^{n} a_j^{p/(p-1)}\right)^{-1/p}$. Note how this obviously does not extend to the case $p = 1$. So here is that separate case $p = 1$:

Prove that $k \max\left\{a_1, \ldots, a_n\right\}$ is the maximum value of $f\left(x_1, \ldots, x_n\right) = a_1 x_1 + \ldots + a_n x_n$ subject to $x_1 + \ldots + x_n = k$ where $k, a_1, \ldots, a_n > 0$, in two different ways:

(a) First prove this using only careful "common sense". Notice that the argument must have two parts: first, show how, given any $a_1, \ldots, a_n$, corresponding values $x_1, \ldots, x_n$ can be chosen so that $x_1 + \ldots + x_n = k$ and $a_1 x_1 + \ldots + a_n x_n = kA$; second, show that for any values $x_1, \ldots, x_n$ with $x_1 + \ldots + x_n = k$, it is always true that $a_1 x_1 + \ldots + a_n x_n \leq kA$.

(b) Second prove this by carefully showing that $A = \lim_{p \to 1^+}\left(\sum_{j=1}^{n} a_j^{p/(p-1)}\right)^{(p-1)/p}$.

(a) The first case is, given $a_1, \ldots, a_n$, corresponding values $x_1, \ldots, x_n$ can be chosen. By setting all $a_i = A$, the maximum value of $f(x)$ occurs when $x_1, \ldots, x_n$ is as large as possible while still under the constraints of $x_1 + \cdots + x_n = k$, so $x_i = \frac{k}{n}$.

$$
\begin{aligned}
f(x) &= x_1 a_1 + \cdots + x_n a_n \\
&= a(x_1 + \cdots + x_n) \\
&= Ak \\
&= kA.
\end{aligned}
$$

The second case is, given $x_1, \ldots, x_n$ with $x_1 + \cdots + x_n = k$, it is always true that $a_1 x_1 + \cdots + a_n x_n \leq kA$. This means that not all $a_i = A$. Without loss of generality, suppose that $a_i = A$ and $a_i = 0$ for all $i > 1$,

$$
\begin{aligned}
f(x) &= A * x_1 + a_2 x_2 + \cdots + a_n x_n \\
&= Ak \\
&= kA.
\end{aligned}
$$

(b) We want to prove that $A = \lim_{p \to 1^+} \left(\sum_{j=1}^{n} a_j^{p/(p-1)} \right)^{(p-1)/p}$. To do this, we start by letting $q = \frac{p}{p-1}$, where $q \to +\infty$ iff $p \to 1^+$,

$$A = \lim_{q \to +\infty} \left(\sum_{j=1}^{n} a_j^q \right)^{\frac{1}{q}}.$$

Dividing both sides by A,

$$1 = \lim_{q \to +\infty} \left(\frac{a_1^q}{A^q} + \frac{a_2^q}{A^q} + \cdots + \frac{a_n^q}{A^q} \right)^{\frac{1}{q}}.$$

Let $b_j = \frac{a_j}{A}$. Now, taking the natural log of both sides,

$$\ln 1 = \lim_{q \to +\infty} \frac{\ln \left(b_1^q + \cdots + b_n^q \right)}{q}.$$

Taking the limit as q approaches $+\infty$, the expression goes to the value of $\ln 1$, which is 0.

$$\therefore A = \lim_{p \to 1^+} \left(\sum_{j=1}^{n} a_j^{p/(p-1)} \right)^{(p-1)/p}.$$