Perfect Writeup: Assignment 23 Exercise 7

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Exercise 7

For fixed positive integer k, define the function

$$f_k(x) = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{k\pi}\Gamma\left(\frac{k}{2}\right)}\left(1+\frac{x^2}{k}\right)^{-\left(\frac{k+1}{2}\right)} \quad \text{for all } x \in \mathbb{R}$$

Prove rigorously but concisely that $\lim_{k\to\infty} f_k(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$.

To begin, we want to find the limit of $f_k(x)$ as $k \to \infty$:

$$\lim_{k\to\infty} f_k(x) = \lim_{k\to\infty} \frac{\Gamma(\frac{k+1}{2})}{\sqrt{k\pi}\Gamma(\frac{k}{2})} \cdot \lim_{k\to\infty}\left(1+\frac{x^2}{k}\right)^{-\frac{k+1}{2}}.$$

Solving for the left limit,

$$\begin{aligned}
\lim_{k\to\infty} \frac{\Gamma(\frac{k+1}{2})}{\sqrt{k\pi}\Gamma(\frac{k}{2})} &= \frac{\left(\frac{k-1}{2e}\right)^{\frac{k-1}{2}}\sqrt{2\pi\left(\frac{k-1}{2}\right)}}{\left(\frac{k-2}{2e}\right)^{\frac{k-2}{2}}\sqrt{2k\pi^2\left(\frac{k-2}{2}\right)}} \\
&= \frac{\left(\frac{k-1}{2e}\right)^{\frac{k-1}{2}}}{\left(\frac{k-2}{2e}\right)^{\frac{k-2}{2}}} \cdot \sqrt{\frac{k-1}{k\pi(k-2)}} \\
&= \left(\frac{\frac{k-1}{2e}}{\frac{k-2}{2e}}\right)^{\frac{k-2}{2}} \cdot \sqrt{\frac{k-1}{2e}} \cdot \sqrt{\frac{k-1}{k\pi(k-2)}} \\
&= \left(\frac{k-1}{k-2}\right)^{\frac{k-2}{2}} \cdot \frac{k-1}{\sqrt{2ek\pi(k-2)}} \\
&= \left(\frac{k-1}{k-2}\right)^{\frac{k-2}{2}} \cdot \frac{\sqrt{k-2}}{\sqrt{2\pi}\sqrt{ek}} \\
&= \left[\sqrt{\left(1+\frac{1}{k-2}\right)^{k-2}} \cdot 1 + \frac{1}{k-2}\right] \cdot \frac{\sqrt{k-2}}{\sqrt{2\pi}\sqrt{ek}} \\
&= \sqrt{e} \cdot \frac{1}{\sqrt{2\pi}\sqrt{e}} \\
&= \frac{1}{2\pi}.
\end{aligned}$$

The right limit of the function looks similar to a definition of e^x:

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n.$$

$$\lim_{k\to\infty}\left(1+\frac{x^2}{k}\right)^{-\frac{k+1}{2}} = e^{-\frac{x^2}{2}}.$$

Combining the values of the left and right limits,

$$f_k(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$