EXERCISES

Exercise 1

(a) Prove the triangle inequality

$$||a| - |b|| \leq |a \pm b| \leq |a| + |b|$$

at least for $a, b \in \mathbb{R}$. (Hint: Start with obvious facts like $-|a| \leq a \leq |a|$.)

(b) What is true about $\vec{u}$ and $\vec{v}$ in the case that the triangle inequality for vectors is actually an *equality*?

Exercise 2 Let $\vec{u} = \langle 3, 2, 1\rangle$ and $\vec{v} = \langle 4, 5, 6\rangle$.

(a) Compute $\vec{u} \cdot \vec{v}$.

(b) Compute $\vec{u} \times \vec{v}$.

(c) Compute $|\vec{u}|$, $|\vec{v}|$, $|\vec{u} - \vec{v}|$, and $|\vec{u} + \vec{v}|$, and then compare these four quantities (lengths) in light of the triangle inequality.

Exercise 3 Confirm that the vector $\vec{u} \times \vec{v} = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1\rangle$ is perpendicular to both vectors $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$, by computing the two appropriate dot products.

Exercise 4 Verify that for any two three-dimensional vectors $\vec{u}$ and $\vec{v}$,

$$(|\vec{u} \times \vec{v}|)^2 + (\vec{u} \cdot \vec{v})^2 = (|\vec{u}|\,|\vec{v}|)^2\,,$$

i.e.,

$$\left(\frac{|\vec{u} \times \vec{v}|}{|\vec{u}|\,|\vec{v}|}\right)^2 + \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|}\right)^2 = 1.$$

What is the trigonometric relevance of this fact?

Exercise 5 At least in three dimensions, the angle θ between two vectors $\vec{u}$ and $\vec{v}$ can be computed using either $\theta = \sin^{-1}\left(\dfrac{|\vec{u} \times \vec{v}|}{|\vec{u}|\,|\vec{v}|}\right)$ or $\theta = \cos^{-1}\left(\dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|}\right)$. Why is the second preferable to the first?

Exercise 6 Determining that two vectors are perpendicular is simply a matter of confirming that their dot product is 0, because that is when the angle between them is $\theta = 90°$. Determining that two vectors are parallel is almost as easy, because that is when the angle between them is either $\theta = 0°$ or $\theta = 180°$.