Perfect Writeup: Assignment 6 No. 4

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Question 4

Verify that for any two three-dimensional vectors $\vec{u}$ and $\vec{v}$,

$$(|\vec{u} \times \vec{v}|)^2 + (\vec{u} \cdot \vec{v})^2 = (|\vec{u}||\vec{v}|)^2$$

i.e.,

$$\left(\frac{|\vec{u} \times \vec{v}|}{|\vec{u}||\vec{v}|}\right)^2 + \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}\right)^2 = 1.$$

What is the trigonometric relevance of this fact?

Note that,

$$\begin{aligned}(|\vec{u} \times \vec{v}|)^2 + (\vec{u} \cdot \vec{v})^2 &= (|\vec{u}||\vec{v}|)^2 \\ &= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2).\end{aligned}$$

Thus,

$$\begin{aligned}(|\vec{u} \times \vec{v}|)^2 &= |\vec{u}|^2|\vec{v}|^2 - |\vec{u}|^2|\vec{v}|^2 \cos\theta \\ &= |\vec{u}|^2|\vec{v}|^2(1 - \cos\theta)\sin^2\theta.\end{aligned}$$

Thus,

$$(|\vec{u} \times \vec{v}|) = (|\vec{u}||\vec{v}|)\sin\theta.$$

$\therefore$

$$\sin\theta = \left(\frac{|\vec{u} \times \vec{v}|}{|\vec{u}||\vec{v}|}\right).$$

Note that,

$$|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta.$$

Thus,

$$\begin{aligned}2|\vec{u}||\vec{v}|\cos\theta &= |\vec{u}|^2 + |\vec{v}|^2 - |\vec{u} - \vec{v}|^2 \\ &= (u_1^2 + u_2^2 + u_3^2) + (v_1^2 + v_2^2 + v_3^2) - (u_1 - v_1)^2 - (u_2 - v_2)^2 - (u_3 - v_3)^2 \\ &= u_1^2 + u_2^2 + u_3^2 + v_1^2 + v_2^2 + v_3^2 - (u_1^2 - 2u_1v_1 + v_1^2) - (u_2^2 - 2u_2v_2 + v_2^2) - (u_3^2 - 2u_3v_3 + v_3^2) \\ &= 2u_1v_1 + 2u_2v_2 + 2u_3v_3 \\ |\vec{u}||\vec{v}|\cos\theta &= u_1v_1 + u_2v_2 + u_3v_3\end{aligned}$$

$\therefore$

$$\cos\theta = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}\right).$$

$$\left(\frac{|\vec{u} \times \vec{v}|}{|\vec{u}||\vec{v}|}\right)^2 + \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}\right)^2 = 1.$$

$$(\cos\theta)^2 + (\sin\theta)^2 = 1.$$