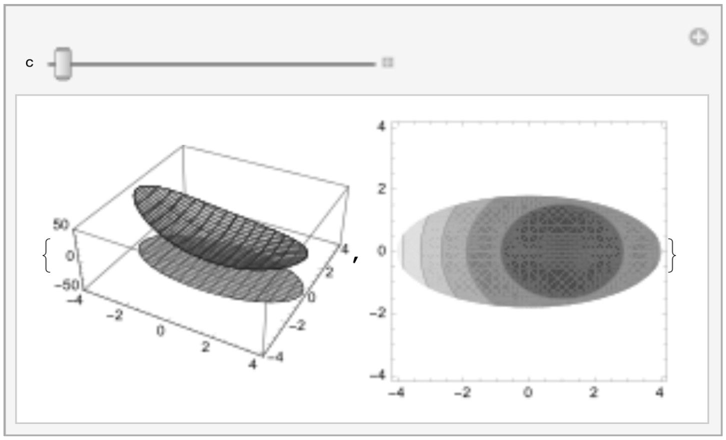

Figure 3.10

ANIMATED: CLICK!

Exercise 1 Maximize and minimize each function subject to the given constraint(s).

(a) $f(x, y) = 5x - 2y$ subject to $x^2 + y^2 \leq 9$

(b) $f(x, y) = 5x + 2y$ subject to $x^2 + y^2 \leq 9$

(c) $f(x, y) = 3x^2 + 2y^2 - 6x - 12y$ subject to $x^2 + y^2 \leq 9$

(d) $f(x, y) = 3x^2 + 2y^2 - 6x - 12y$ subject to $x^2 + y^2 \leq 16$

(e) $f(x, y) = x^2 + y^2$ subject to $3x^2 + 2y^2 - 6x - 12y \leq 30$

(f)

Exercise 2 Use one-variable calculus to minimize and maximize $f(x, y) = 2x + 3y$ subject to $x^2 + y^2 = 16$. (Hint: $y = \pm\sqrt{16 - x^2}$.) Is this method more or less preferable when compared to the method of Lagrange multipliers, done in an example in this section?

Exercise 3 Fix powers $p, q \geq 1$ and constant $m > 0$. Minimize and maximize $f(x, y) = x^p + y^p$ subject to the constraint $x^q + y^q = m$, as well as constraints $x, y \geq 0$ if either power p or q requires that. As in Examples 6(a) and 6(b) which this Exercise generalizes, there will be two cases: $1 \leq p < q$ and $p > q \geq 1$.

Exercise 4 Assume that $a_1, \ldots, a_n > 0$, $k > 0$, and $p > 1$. Maximize $f(x_1, \ldots, x_n) = a_1x_1 + \ldots + a_nx_n$ subject to $x_1^p + \ldots + x_n^p = k^p$ (and $x_1, \ldots, x_n > 0$, just so the powers make sense). (Hint: Just dig in and do it.)

Exercise 5 Assume that $a_1, \ldots, a_n > 0$, $k > 0$, and $p > 1$. In the previous Exercise 4, it was found that $k\left(\sum_{j=1}^{n} a_j^{p/(p-1)}\right)^{(p-1)/p}$ is the maximum value of $f(x_1,\ldots,x_n) = a_1x_1+\ldots+a_nx_n$ subject to $x_1^p+\ldots+x_n^p = k^p$ and $x_1,\ldots,x_n > 0$; this occurs when each $x_i = ka_i^{1/(p-1)}\left(\sum_{j=1}^{n} a_j^{p/(p-1)}\right)^{-1/p}$. Note how this obviously does not extend to the case $p = 1$. So here is that separate case $p = 1$:

Prove that $k\max\{a_1,\ldots,a_n\}$ is the maximum value of $f(x_1,\ldots,x_n) = a_1x_1+\ldots+a_nx_n$ subject to $x_1+\ldots+x_n = k$ where $k, a_1,\ldots,a_n > 0$, in two different ways:

(a) First prove this using only careful "common sense". Notice that the argument must have two parts: first, show how, given any $a_1,\ldots,a_n$, corresponding values $x_1,\ldots,x_n$ can be chosen so that $x_1+\ldots+x_n = k$ and $a_1x_1+\ldots+a_nx_n = kA$; second, show that for any values $x_1,\ldots,x_n$ with $x_1+\ldots+x_n = k$, it is always true that $a_1x_1+\ldots+a_nx_n \le kA$.

(b) Second prove this by carefully showing that $A = \lim_{p\to 1^+}\left(\sum_{j=1}^{n} a_j^{p/(p-1)}\right)^{(p-1)/p}$.

Outline of one possible proof for part (b):

- let $q = \frac{p}{p-1}$, and note that $q \to +\infty$ iff $p \to 1^+$;
- factor all the way out the quantity A, so that the expression inside becomes $\sum_{j=1}^{n} b_j^q$ where $b_j = \frac{a_j}{A}$, and carefully consider the range of possible values of $\sum_{j=1}^{n} b_j^q$; and
- use the usual "exp[ln( )]" trick to rewrite in order to cope with the outside power $1/q$.