VECTORS AND MATRICES

Preliminary Notes

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1 Complex numbers

And so we begin! I suppose this is a better opportunity than any for me to share a profound foreword to the erudite, learned journey of mathematical enlightenment we are about to embark on with linear algebra - namely, to answer the question: why learn linear algebra at all? There are only three reasons. First, you're a massive sci-fi movie nerd and have accidentally stumbled upon the far-inferior version of the Matrix. Second, you're a massive basketball fan and received the worst surprise of your entire life when you searched "Jordan form" on YouTube in hopes of basketball enlightenment. Third, you're a massive German and have taken linear algebra for the sole purpose of pronouncing *eigenvalue* "the German way". Either or, I'm glad you're here with me; after all, if you're destined to become a machine learning dev earning seven figures and sunbathing in a luxury yacht, then - in the eternal words of wisdom of r/animememes - "don't say you love the anime if you haven't read the manga".

(Of course, besides all these completely unhinged things I'm talking about, I suppose there's also a few nuggets of mathematical beauty to be found in these curious morsels we call vectors and matrices here and there.)

Let's start with a return to form: complex numbers. In the realm of linear algebra specifically, complex numbers are important for two reasons. First, in the set of complex numbers $\mathbb{C}$, we can guarantee that a polynomial of degree n will have n roots by the Fundamental Theorem of Algebra; never again will we have to worry about equations like $\lambda^2 + 1 = 0$ making us more confused than tasting a burger from Pizza Hut and finding it delicious. This becomes particularly relevant when we have to deal with these polynomials, which arise when we find the eigenvalues of a particular matrix - more on that later.

Definition 1.1. (Complex number). We define the imaginary unit i as satisfying $i^2 = -1$; as such, we also define the set of complex numbers $\mathbb{C}$ to encompass all numbers of the form

$$z = a + bi$$

where a and b are real. We write $a = \text{Re}(z)$, $b = \text{Im}(z)$, and the complex conjugate $\bar{z} = a - bi$ (a theorem in algebra will demonstrate that if z is a root of a polynomial, then $\bar{z}$ is too).

But second of all - and much more thematically - while real numbers are *one-dimensional,* all lying upon the same infinitely long number line, complex numbers are two-dimensional; it is useful to think of $z = a + bi$ as a vector in the *complex plane,* $\begin{bmatrix} a \\ b \end{bmatrix}$. The representation of complex numbers as vectors is done on an *Argand plane*, analogous to the Cartesian plane but with the x-axis representing the real part of z and the y-axis representing the imaginary part.

What follows is a carousel of important results for complex numbers which are truly astounding in their mind-numbingness, not because of what they are but because we've seen them all before:

Definition 1.2. (Modulus and argument). Define the modulus of $z = a + bi$ as $|z| = \sqrt{a^2 + b^2}$; this is analogous to the length of its vector representation on the Argand plane. Define its argument as the angle its vector makes with the real axis: $\arg z = \tan^{-1}(\frac{b}{a})$. The modulus-argument pair (r, θ) can uniquely describe a complex number z, but each z has infinitely many arguments $\theta + 2k\pi$ (a full revolution, but not the French kind). We often take only the principal argument - $-\pi < \theta < \pi$.

Proposition 1.3. We have

$$z\bar{z} = a^2 + b^2 = |z|^2$$

and

$$z^{-1} = \frac{\bar{z}}{|z|^2}$$

Theorem 1.4. (Triangle inequality). For any two complex numbers z_1 and z_2, we have

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

which can be shown by the geometrical interpretation of the two complex numbers as vectors representing sides of a triangle.

1.1 Complex exponentiation

To extend exponentiation to complex numbers, we use the Taylor series definition of exponentiation:

Definition 1.5. (Exponential function). Define

$$e^z = \sum_{n=0}^{\infty} \frac{x^z}{z!}$$

which can be verified to satisfy the properties we expect from the exponential function, including $e^a e^b = e^{a+b}$. We assume that this sum converges for all complex numbers z.

Similarly, we would like to extend the trigonometric functions to the complex realm, where a geometric definition fails due to the budding, unhinged insanity that underlines the words "an angle of $39 + 46\pi i$ degrees":

Definition 1.6. (Complex sine and cosine). Define

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...$$

and

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ...$$

From the two above results, we obtain a very important formula throughout all of math.

Theorem 1.7. (Euler's formula).

$$e^{iz} = \cos z + i \sin z$$

It almost feels like I ought to be wearing a suit and tie before I even dare to think about these symbols. We also note that any complex number can thus be written in terms of a complex exponential, as its modulus-argument form (r, θ) suggests it can be written as

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

which allows us to state that multiplication between two complex numbers $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ requires the multiplication of their moduli and addition of their arguments.

1.2 Roots of unity

Definition 1.8. (Roots of unity). We refer to the complex roots of the equation $\omega^n = 1$ as the *nth roots of unity*; as this is a polynomial of deg n, we have n roots of unity, which can be completely described by

$$\omega = e^{\frac{2\pi k i}{n}},\ k = 0, 1, 2, 3, ..., (n-1).$$

As a consequence of the above, we also have

$$\sum \omega = 1 + e^{\frac{2\pi i}{n}} + e^{\frac{4\pi i}{n}} + ... + e^{\frac{2(n-1)\pi i}{n}} = 0$$

(You may have noticed that we've reached a critical mass of handwaving away statements without proof in this section. De Moivre is surely spinning in his grave. The reason why is because I can't be bothered to prove any of this stuff, so the proofs are left as an exercise to the reader.)

1.3 Complex logarithms

Definition 1.9. (Complex logarithms). Define the complex logarithm $\omega = \ln z$ as the number which satisfies $e^{\omega} = z$. If z is a complex number $z = re^{i\theta}$, then we have $e^{\omega} = re^{i\theta}$ and thus $\ln \frac{1}{r} = i\theta - \omega$ and $\omega = \ln re^{i\theta} = i\theta + \ln r$. (I'm just now realizing we could've got here with $\ln ab = \ln a + \ln b$.)

Definition 1.10. (Complex powers). Define z^{α} for complex z as $e^{\alpha \ln z}$ where we insist that the argument used for z is $-\pi < \theta < \pi$, the principal argument.

Definition 1.11. (De Moivre's Theorem).

$$\cos n\theta + i \sin n\theta = (\cos \theta + i \sin n\theta)^n.$$

This can be proven by induction; it is functionally identical to stating that $e^{ni\theta} = (e^{i\theta})^n$, which is obvious over the reals but not so obvious over complex numbers.

1.4 Lines and circles

1.4.1 Complex equation of a line

The equation is not called "complex equation of a line" because it involves complex numbers, but because it is unnecessarily complex. Observe. What can we do if we want to find a line that goes through the point x_0 on the Argand diagram and is parallel to some complex number ω? By what we know of vector equations for lines, we can write the line as $x = x_0 + \lambda\omega$ for some real scalar λ. If we rewrite this as $\frac{x-x_0}{\omega} = \lambda$ and take the conjugate of both sides, we obtain

$$\frac{\bar{x} - \bar{x_0}}{\bar{\omega}} = \bar{\lambda} = \lambda$$

as λ is real. Thus the conjugate of this expression is equal to itself:

$$\frac{\bar{x} - \bar{x_0}}{\bar{\omega}} = \frac{x - x_0}{\omega}$$

which gives us the equation of a line parallel to ω and passing through x_0.

1.4.2 Complex equation of a circle

A circle with center x_0 and radius r, in abstract terms, is simply a locus of points a distance r away from the center x_0. (In less abstract terms, it can be referred to as "half a Venn diagram" or "the inferior donut".) This can be formulated as

$$|x - x_0| = r$$

or, squaring both sides,

$$|x - x_0|^2 = r^2$$
$$(x - x_0)(\bar{x} - \bar{x_0}) = r^2$$

2 Vectors