Greatest Integer Function

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1 Introduction

Any number x can be represented as

$$x = \lfloor x \rfloor + \theta$$

Where $\lfloor x \rfloor$ represents integer part of x whereas θ represents the fractional part of x

It is given that

$$x \geq \lfloor x \rfloor$$

2 Theorems

2.1 Legendre Formula

If n is a positive integer, and p is a prime, then the exponent of the highest power of p that divides n! is

$$\sum_{k=1}^{\infty} = \lfloor \frac{n}{p^k} \rfloor$$

Proof

Assume p, 2p, 3p . .. tp all exist before n, that implies they all divide n!

$$tp \leq n$$

Which implies that there are exactly $\lfloor \frac{n}{p} \rfloor$ multiplies of p withing n!

Similarly we can do this for $p^2, p^3 ... p^k$

And thus we conclude that

$$\sum_{k=1}^{\infty} = \lfloor \frac{n}{p^k} \rfloor$$

This can be extended to write

$$n! = \prod_{p \leq n} p^{\sum_{k=1}^{\infty} \lfloor \frac{n}{p^k} \rfloor}$$

2.2

Let F and f be number theoretic function such that

$$F(n) = \sum_{d|n} f(d)$$

Then

$$\sum_{n=1}^{N} F(n) = \sum_{k=1}^{N} f(k) \times \lfloor \frac{N}{k} \rfloor$$

Proof:
We know that

$$\sum_{n=1}^{N} F(n) = \sum_{n=1}^{N} \sum_{d|n} f(d) \tag{1}$$

Let there be some $k \leq N$, therefore there must be multiples of k occurring less than N which are $k, 2k ... \lfloor \frac{N}{k} \rfloor$
So if we can find number of times each f(k) occurs, what that would mean is that we would be finding number of times each number less than N occurs in the second summation of RHS
Thus we can transform equation 1 into

$$\sum_{n=1}^{N} F(n) = \sum_{n=1}^{N} \sum_{d|n} f(d) = \sum_{k=1}^{N} f(k) \times \lfloor \frac{N}{k} \rfloor$$

2.3

Let n be represented in p-nary, such that

$$n = a_k p^k + a_{k-1} p^{k-1} + ... + a_0$$

The exponent of highest power of p occurring in n! is

$$\frac{n - (a_k + a_{k-1} + .. + a_0)}{p - 1}$$

Proof:

$$answer = \sum_{k=1}^{\infty} \lfloor \frac{n}{p} \rfloor$$

$$answer = (a_k p^{k-1} + a_{k-1} p^{k-2} + ..a_1) + (a_k p^{k-2} + ... + a_2)....(a_k)$$

$$answer = a_k (p^{k-1} + p^{k-2} + ..1) + a_{k-1} (p^{k-2} + p^{k-3} + ..1)..a_1$$

$$answer = a_k \frac{p^k - 1}{p - 1} + a_{k-1} \frac{p^{k-1} - 1}{p - 1} + ... + a_1 \frac{p - 1}{p - 1} + a_0 \frac{1 - 1}{p - 1}$$

$$answer = \frac{n - (a_k + a_{k-1} + ... + a_0)}{p - 1}$$