Chapter 1

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1 Binomial Theorem

The Binomial Coefficient is defined by the expression :

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Pascal's Rule

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

This relation gives rise to the famous Pascal's Triangle, in which $\binom{n}{k}$ appears at (k + 1 $)^{th}$ index of the n^{th} row

Properties

1.0.1

This property can be proved by expansion

$$\binom{n}{k} \times \binom{k}{r} = \binom{n}{r} \times \binom{n-r}{k-r}$$

1.0.2

This property can be proved by using Property 1.0.1

$$\binom{n}{k} = \frac{(n-k+1)}{k} \times \binom{n}{k-1}$$

1.0.3

This property can be proved by expansion. This is valid for $n \geq 4$.

$$\binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}$$

1.0.4

$$\binom{2m}{2} = 2 \times \binom{m}{2} + m^2$$

2 Triangular Numbers

A triangular number is a positive integer n such that n = 1 + 2 + .. + k

Properties

2.0.1

For x to be triangular, x has to be of the form $\frac{n \times (n+1)}{2}$

Proof:

$$x = n + (n-1) + (n-2)... + 1 \tag{1}$$

$$x = 1 + 2 + 3... + n \tag{2}$$

Adding both 1 and 2, we get

$$2 \times x = (n+1) \times n \tag{3}$$

From this, we can derive that

$$x = \frac{(n+1) \times n}{2}$$

2.0.2

Integer x is triangular iff 8x + 1 is a perfect square
Part 1 :

Prove that if x is a triangular number, then 8x + 1 is a perfect square

From Property 2.0.1 we know that

$$x = \frac{n \times (n+1)}{2}$$

Hence, we know

$$2x = n \times (n+1)$$

$$8x = 4n^2 + 4n$$

$$8x + 1 = 4n^2 + 4n + 1$$

Hence we get at the final result

$$8x + 1 = (2n+1)^2 \tag{1}$$

Part 2 :

Prove that if 8x + 1 is a perfect square, x is triangular

$$8x + 1 = k^2$$

Since LHS is odd, this implies that k is odd. So let k = $2n + 1$

$$8x + 1 = (2n + 1)^2$$
$$8x + 1 = 4n^2 + 4n + 1$$
$$8x = 4n^2 + 4n$$
$$2x = n^2 + n$$
$$x = \frac{n \times (n + 1)}{2} \tag{2}$$

From Part 1 and Part 2, we can deduce that the property mentioned is true.

2.0.3

Property of any 2 consecutive triangular numbers is a perfect square.

From 2.0.1 we can say

$$x = \frac{n \times (n + 1)}{2} \tag{1}$$
$$y = \frac{(n + 1) \times (n + 2)}{2} \tag{2}$$

Where x and y are consecutive triangular numbers.

From 1 adding 2

$$x + y = \frac{(n + 1) \times (n + (n + 2))}{2}$$
$$x + y = (n + 1)^2$$

2.0.4

If number x is triangular, then 9x + 1, 25x + 3, 49x + 6 are also triangular.

From Property 2.0.2, we know

$$x = 8n + 1$$

Where x is the n^{th} triangular number. Now, replacing n with 9n + 1, we get

$$8 \times (9n + 1) + 1$$
$$72n + 9$$
$$9(8n + 1)$$

We can observe this is a perfect square by property 2.0.2. Hence this is a triangular number. We can do similarly for rest of it.

2.0.5

if t_n is the n^{th} triangular number, then

$$t_1 + t_2 ... + t_n = \frac{n \times (n+1) \times (n+2)}{6}$$

Proof:
Multiplying and dividing LHS by 2, we get

$$\frac{2t_1 + 2t_2 + 2t_3 ... 2t_n}{2}$$

Rearranging the terms

$$\frac{t_1 + (t_1 + t_2) + (t_2 + t_3) ... (t_{n-1} + t_n) + t_n}{2}$$

Using Property 2.0.3

$$\frac{t_1 + 2^2 + 3^2 ... n^2 + t_n}{2}$$

$$\frac{1^2 + 2^2 + 3^2 ... n^2}{2} + \frac{n(n+1)}{4}$$

$$\frac{n(n+1)(2n+1)}{12} \frac{n(n+1)}{4} \tag{1}$$

Simplifying 1, we get

$$\frac{n \times (n+1) \times (n+2)}{6}$$

2.0.6

$$1 \times 2 + 2 \times 3 + 3 \times 4 ... n \times (n+1) = \frac{n(n+1)(n+2)}{3}$$

Proof

We use the knowledge that $2 \times \binom{m}{2} = m^2$

$$2 \times \binom{2}{2} + 2 \times \binom{3}{2} ... 2 \times \binom{n+1}{2}$$

Taking 2 common and using Property 2.0.5, we get this simplified to

$$\frac{n(n+1)(n+2)}{3}$$

2.0.7

$$\binom{2}{2} + \binom{4}{2} + \binom{6}{2}...\binom{2n}{2} = \frac{n(n+1)(4n+1)}{6}$$

Proof

By Property 1.0.4, we get

$$\binom{2\times 1}{2} + \binom{2\times 2}{2} + \binom{2\times 3}{2}...\binom{2\times n}{2}$$

$$2\times\binom{1}{2} + 1^2 + 2\times\binom{2}{2} + 2^2...2\times\binom{n}{2} + n^2$$

Grouping

$$2(\binom{1}{2} + \binom{2}{2} + \binom{3}{2}..\binom{n}{2}) + (1^2 + 2^2 + 3^2.. + n^2)$$

$$2\times\frac{(n-1)n(n+1)}{6} + \frac{n(n+1)(2n-1)}{6}$$

Simplifying this, we get

$$\frac{n(n+1)(4n+1)}{6}$$

2.0.8

$$1^2 + 3^2 + 5^2...(2n-1)^2 = \binom{2n+1}{3}$$

Proof:

Using Property 2.0.3

$$k^2 = t_{k-1} + t_k$$

For all k belonging to odd integers specified above

$$t_1 + t_2 + t_3...t_{2n-2} + t_{2n-1}$$

Using Property 2.0.5

$$\frac{2n(2n+1)(2n-1)}{6}$$

Which is nothing but

$$\binom{2n+1}{3}$$