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Homework 8

Problem : Implementation of the Moonley-Rivlin model for nearly incompressible isotropic materials in FLagSHyp

A brief summary of the Moonley-Rivlin model is presented first, based on Cheng and Zhang's paper.

Consider a material domain $\vec{X}$ subject to a deformation field ϕ, that transforms to $\vec{x}$. The deformation gradient is thus $\boldsymbol{F} = \nabla\phi = \frac{\partial \vec{x}}{\partial \vec{X}}$.

For rubber-like material, the strain-energy function is postulated to a unique decoupled form:

$$\Psi(\boldsymbol{C}) = \Psi_{\text{vol}}(J) + \Psi_{\text{iso}}(\bar{\boldsymbol{C}})$$

where $\boldsymbol{C} = \boldsymbol{F^T F}$ is the Right-Cauchy Green tensor, $J = \det(\boldsymbol{F})$ is the Jacobian or the volume ratio, and $\bar{\boldsymbol{C}} = J^{-2/3}\boldsymbol{C}$ is the isochoric or deviatoric part of the Right-Cauchy Green tensor. `FLagSHyp`uses a mixed pressure/displacement formulation, and thus the volumetric part of the energy function is written as a Lagrangian to enforce the incompressibility condition in terms of the hydrostatic pressure p.

$$\Psi_{\text{vol}}(J) = p(J-1)$$

For isotropic materials, the isochoric part of the strain energy must depend only on the three invariants of the Cauchy-Green tensors:

$$\begin{aligned}
\Psi(\boldsymbol{C}) &= \Psi_{\text{vol}}(J) + \Psi_{\text{iso}}(\bar{I}_1, \bar{I}_2) \\
\bar{I}_1 &= \text{tr}(\bar{\boldsymbol{C}}) = \text{tr}(\bar{\boldsymbol{C}}) = J^{-2/3}\text{tr}(\boldsymbol{C}) = J^{-2/3} I_1 \\
\bar{I}_2 &= \frac{1}{2}(\bar{I}_1^2 - \text{tr}(\bar{\boldsymbol{C}}^{\boldsymbol{2}})) = J^{-4/3} I_2
\end{aligned}$$

The strain-energy is related to the work done on hyperelastic materials in a dynamic process:

$$\Psi(\boldsymbol{F}_1) - \Psi(\boldsymbol{F}_1) = \int_{t_1}^{t_2} \boldsymbol{S} : \dot{\boldsymbol{E}} dt = \int_{t_1}^{t_2} \boldsymbol{\sigma} : \dot{\boldsymbol{e}} dt \tag{1}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{S} = J^{-1}F^{-1}\sigma F^{-T}$ is the Second Piola-Kirchhoff tensor, $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - I)$ is the Green-Lagrange strain tensor and $\boldsymbol{e} = \frac{1}{2}(\boldsymbol{I} - \boldsymbol{B^{-1}})$ is the Euler-Almansi strain tensor. For isotropic material the PK2 stress tensor can determined by

$$\begin{aligned}
\boldsymbol{S} &= 2\frac{\partial \Psi(\boldsymbol{C})}{\partial \boldsymbol{C}} = \boldsymbol{S}_{\text{vol}} + \boldsymbol{S}_{\text{iso}} \\
\boldsymbol{S}_{\text{vol}} &= 2\frac{\partial \Psi_{\text{vol}}(\boldsymbol{C})}{\partial \boldsymbol{C}} = J^{1/3} p \bar{\boldsymbol{C}}^{-1} \\
\boldsymbol{S}_{\text{iso}} &= 2\frac{\partial \Psi_{\text{iso}}(\boldsymbol{C})}{\partial \boldsymbol{C}} = J^{-2/3}\mathcal{P} : \bar{\boldsymbol{S}}
\end{aligned}$$

where $\mathcal{P} = (\mathcal{I} - \frac{1}{3}\bar{\boldsymbol{C}}^{-1} \otimes \bar{\boldsymbol{C}})$ is the projection operator such that $\frac{\partial \bar{\boldsymbol{C}}}{\partial \boldsymbol{C}} = J^{-2/3}\mathcal{P}$ and $\bar{\boldsymbol{S}} = \frac{\partial \Psi_{\text{iso}}(\bar{\boldsymbol{C}})}{\partial \bar{\boldsymbol{C}}}$
The elasticy tensor:

$$
\begin{aligned}
\mathcal{C} &= \mathcal{C}_{\text{iso}} + \mathcal{C}_{\text{vol}} = 2\Big(\frac{\partial \boldsymbol{S}_{\text{vol}}(\boldsymbol{C})}{\partial \boldsymbol{C}} + \frac{\partial \boldsymbol{S}_{\text{iso}}(\boldsymbol{C})}{\partial \boldsymbol{C}}\Big) \\
\mathcal{C}_{\text{iso}} &= -\frac{2}{3}J^{-2/3}\Big(\boldsymbol{S}_{\text{iso}} \otimes \bar{\boldsymbol{C}}^{-1} + \bar{\boldsymbol{C}}^{-1} \otimes \boldsymbol{S}_{\text{iso}}\Big) + \mathcal{P} : \bar{\mathcal{C}} : \mathcal{P}^T + \frac{2}{3}\text{tr}(J^{-2/3}\boldsymbol{C} : \bar{\boldsymbol{S}})\hat{\mathcal{P}} \\
\mathcal{C}_{\text{vol}} &= \Big(p + J\frac{dp}{dJ})J^{-1/3}\bar{\boldsymbol{C}}^{-1} \otimes \bar{\boldsymbol{C}}^{-1} - 2pJ^{-2/3}\Big(-\frac{\partial(\bar{\boldsymbol{C}}^{-1})}{\partial \bar{\boldsymbol{C}}}\Big)
\end{aligned}
$$

where $\hat{\mathcal{P}} = J^{-4/3}\Big(-\frac{\partial(\bar{\boldsymbol{C}}^{-1})}{\partial \bar{\boldsymbol{C}}} - \frac{1}{3}\bar{\boldsymbol{C}}^{-1} \otimes \bar{\boldsymbol{C}}^{-1}\Big)$ and for any second order tensor, the first term is calculated from: $\boldsymbol{U}$, $\frac{\partial(\bar{\boldsymbol{C}}^{-1})}{\partial \boldsymbol{C}} : U = \frac{\partial(\bar{\boldsymbol{C}})}{\partial \boldsymbol{C}} : (\frac{\partial(\bar{\boldsymbol{C}}^{-1})}{\partial \bar{\boldsymbol{C}}} : U)$ with $\frac{\partial(\bar{\boldsymbol{C}}^{-1})}{\partial \bar{\boldsymbol{C}}} : U = -\bar{\boldsymbol{C}}^{-1}\boldsymbol{U}\bar{\boldsymbol{C}}^{-1}$
The Moonley-Rivlin model assumes a polynomial form:

$$
\begin{aligned}
\Psi_{\text{vol}} &= \frac{\kappa}{2}(J-1)^2 = \frac{1}{2}p(J-1) \\
\Psi_{\text{iso}} &= \frac{\mu_1}{2}(\bar{I}_1 - 3) + \frac{\mu_2}{2}(\bar{I}_2 - 3)
\end{aligned}
$$

κ, μ_1, μ_2 are material constants.

$$
\begin{aligned}
\mathcal{C}_{\text{iso}} &= 2\mu_2 J^{-4/3}(\boldsymbol{I} \otimes \boldsymbol{I} - \mathcal{I}) - \frac{2}{3}J^{-4/3}(\mu_1 + 2\mu_2\bar{I}_1)(\boldsymbol{I} \otimes \bar{\boldsymbol{C}}^{-1} + \bar{\boldsymbol{C}}^{-1} \otimes \boldsymbol{I}) \\
&+ \frac{4}{3}\mu_2 J^{-4/3}(\bar{\boldsymbol{C}} \otimes \bar{\boldsymbol{C}}^{-1} + \bar{\boldsymbol{C}}^{-1} \otimes \bar{\boldsymbol{C}}) + \frac{2}{9}J^{-4/3}(\mu_1\bar{I}_1 + 4\mu_2\bar{I}_2)\bar{\boldsymbol{C}}^{-1} \otimes \bar{\boldsymbol{C}}^{-1} \\
&+ \frac{2}{3}J^{-4/3}(\mu_1\bar{I}_1 2\mu_2\bar{I}_2)\Big(-\frac{\partial(\bar{\boldsymbol{C}}^{-1})}{\partial \bar{\boldsymbol{C}}}\Big) \\
\mathcal{C}_{\text{vol}} &= J^{-1/3}(2J-1)\kappa\bar{\boldsymbol{C}}^{-1} \otimes \bar{\boldsymbol{C}}^{-1} - 2J^{-1/3}(J-1)\kappa\Big(-\frac{\partial(\bar{\boldsymbol{C}}^{-1})}{\partial \bar{\boldsymbol{C}}}\Big)
\end{aligned}
$$

`FLagSHyP` uses an updated Lagrangian formulation by solving for the Cauchy-stresses in the deformed configuration instead of the Second Piola-Kirchhoff tensor. Therefore the Lagrangian elasticity tensor is needed for this implementation:

$$
\begin{aligned}
c^{CZ}_{ijkl} &= F_{iI}F_{jJ}F_{kK}F_{lL}\mathcal{C}_{IJKL} = c^{CZ}_{\text{vol},ijkl} + c^{CZ}_{\text{iso},ijkl} \\
c^{CZ}_{\text{vol},ijkl} &= \kappa\Big[J(2J-1)\delta_{ij}\delta_{kl} - J(J-1)(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{kj})\Big] \\
c^{CZ}_{\text{iso},ijkl} &= 2\mu_2\Big[\bar{B}_{ij}\bar{B}_{kl} - \frac{1}{2}\Big(\bar{B}_{ik}\bar{B}_{jl}+\bar{B}_{il}\bar{B}_{jk}\Big)\Big] - \frac{2}{3}(\mu_1+2\mu_2\bar{I}_1)(\bar{B}_{ij}\delta_{kl}+\bar{B}_{kl}\delta_{ij}) \\
&\quad + \frac{4}{3}\mu_2\Big((\bar{B}^2)_{ij}\delta_{kl}+(\bar{B}^2)_{kl}\delta_{ij}\Big) + \frac{2}{9}(\mu_1\bar{I}_1+4\mu_2\bar{I}_2)\delta_{ij}\delta_{kl} + \frac{1}{3}(\mu_1\bar{I}_1+2\mu_2\bar{I}_2)(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) \\
&= J^{-2/3}\Bigg[2\mu_2J^{-2/3}\Big(B_{ij}B_{kl} - \frac{1}{2}(B_{ik}B_{jl}+B_{il}B_{jk})\Big) - \frac{2}{3}(\mu_1+2\mu_2J^{-2/3}I_1)(B_{ij}\delta_{kl}+B_{kl}\delta_{ij}) \\
&\quad + \frac{4}{3}\mu_2J^{-2/3}\Big((B^2)_{ij}\delta_{kl}+(B^2)_{kl}\delta_{ij}\Big) + \frac{2}{9}(\mu_1I_1+4\mu_2J^{-2/3}I_2)\delta_{ij}\delta_{kl} \\
&\quad + \frac{1}{3}(\mu_1I_1+2\mu_2J^{-2/3}I_2)(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\Bigg]
\end{aligned}
$$

In the `FLagSHyP` code, the kinematics are written in terms of $\boldsymbol{F}$, $\boldsymbol{B}$. This can be written in tensor form:

$$
\begin{aligned}
c_{\text{vol}} &= \kappa\Big[J(2J-1)\boldsymbol{I}\otimes\boldsymbol{I} - 2J(J-1)i\Big] \\
c^{CZ}_{\text{iso}} &= J^{-2/3}\Bigg\{2\mu_2J^{-2/3}2\Big[\boldsymbol{B}\otimes\boldsymbol{B} - \Big(i:\boldsymbol{B}\Big)\otimes\boldsymbol{B}\Big] - \frac{2}{3}(\mu_1+2\mu_2J^{-2/3}I_1)(\boldsymbol{B}\otimes\boldsymbol{I}+\boldsymbol{I}\otimes\boldsymbol{B}) \\
&\quad + \frac{4}{3}J^{-2/3}\mu_2\Big(\boldsymbol{B}^2\otimes\boldsymbol{I}+\boldsymbol{I}\otimes\boldsymbol{B}^2\Big) + \frac{2}{9}(\mu_1I_1+4\mu_2J^{-2/3}I_2)\boldsymbol{I}\otimes\boldsymbol{I} + \frac{2}{3}(\mu_1I_1+2\mu_2J^{-2/3}I_2)i\Bigg\}
\end{aligned}
$$

where

$$
\begin{aligned}
\boldsymbol{I} &= \sum_{ij}\delta_{ij}\vec{e}_i\otimes\vec{e}_j = \sum_i \vec{e}_i\otimes\vec{e}_i \\
\boldsymbol{I}\otimes\boldsymbol{I} &= \sum_{ijkl}\delta_{ij}\delta_{kl}\vec{e}_i\otimes\vec{e}_j\otimes\vec{e}_k\otimes\vec{e}_l = \sum_{ij}\vec{e}_i\otimes\vec{e}_i\otimes\vec{e}_j\otimes\vec{e}_j \\
i &= \sum_{ijkl}\frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\vec{e}_i\otimes\vec{e}_j\otimes\vec{e}_k\otimes\vec{e}_l
\end{aligned}
$$

However, the push-forward operation is defined differently in `FLagSHyp`, as shown in the next equations:

$$
\begin{aligned}
c^{CZ}_{ijkl} &= \sum_{IJKL}F_{iI}F_{jJ}F_{kK}F_{lL}\mathcal{C}_{IJKL} = (\phi_*[\mathcal{C}])_{ijkl} \\
c^{\text{flagshyp}}_{ijkl} &= J^{-1}\mathcal{C}_{IJKL} = J^{-1}\sum_{IJKL}F_{iI}F_{jJ}F_{kK}F_{lL}\mathcal{C}_{IJKL}
\end{aligned}
$$

$$
\begin{aligned}
c^{\text{flagshyp}}_{\text{vol},ijkl} &= \kappa\Big[(2J-1)\delta_{ij}\delta_{kl} - (J-1)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})\Big] \\
&= \underbrace{\kappa(J-1)}_{=p}\Big[\delta_{ij}\delta_{kl} - (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj})\Big] + \kappa J(\delta_{ij}\delta_{kl}) \\
c^{\text{flagshyp}}_{\text{vol}} &= p(\boldsymbol{I}\otimes\boldsymbol{I} - i) + \bar{\kappa} i \qquad\qquad \bar{\kappa} = \kappa J \\
c^{\text{flagshyp}}_{\text{iso},ijkl} &= J^{-5/3}\Bigg[2\mu_2 J^{-2/3}\Big(B_{ij}B_{kl} - \frac{1}{2}(B_{ik}B_{jl} + B_{il}B_{jk})\Big) - \frac{2}{3}(\mu_1 + 2\mu_2 J^{-2/3} I_1)(B_{ij}\delta_{kl} + B_{kl}\delta_{ij}) \\
&\quad + \frac{4}{3}\mu_2 J^{-2/3}\Big((B^2)_{ij}\delta_{kl} + (B^2)_{kl}\delta_{ij}\Big) + \frac{2}{9}(\mu_1 I_1 + 4\mu_2 J^{-2/3} I_2)\delta_{ij}\delta_{kl} \\
&\quad + \frac{1}{3}(\mu_1 I_1 + 2\mu_2 J^{-2/3} I_2)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\Bigg]
\end{aligned}
$$

The difference could come from the fact that the paper calculates the Kirchhoff stresses from the Lagrangian strain, whereas the code calculates the Cauchy stresses. If $\mu_2 = 0$, it corresponds to an incompressible Neo-Hookean model, and $c^{\text{flagshyp}}_{\text{iso},ijkl}$ seems to match the function `ctens5.m` in the provided code.

The Cauchy stress tensor is:

$$
\begin{aligned}
\boldsymbol{\sigma} &= J^{-1}\boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^{\boldsymbol{T}} = \boldsymbol{\sigma}_{\text{vol}} + \boldsymbol{\sigma}_{\text{iso}} \\
\boldsymbol{\sigma}_{\text{vol}} &= p\boldsymbol{I} = \kappa(J-1)\boldsymbol{I} \\
\boldsymbol{\sigma}_{\text{iso}} &= J^{-1}\Big(-\frac{1}{3}(\mu_1\bar{I}_1 + 2\mu\bar{I}_2)\boldsymbol{I} - \mu_2\bar{\boldsymbol{B}}^2 + (\mu_1 + \mu_2\bar{\boldsymbol{I}}_1)\bar{\boldsymbol{B}}\Big)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\sigma}_{\text{iso}} &= J^{-1}\Big(-\frac{1}{3}(\mu_1\bar{I}_1 + 2\mu\bar{I}_2)\boldsymbol{I} - \mu_2\bar{\boldsymbol{B}}^2 + (\mu_1 + \mu_2\bar{\boldsymbol{I}}_1)\bar{\boldsymbol{B}}\Big) \\
&= J^{-1}\Big(-\frac{1}{3}(\mu_1 J^{-2/3} I_1 + 2\mu J^{-4/3} I_2)\boldsymbol{I} - \mu_2(J^{-2/3}\boldsymbol{B})^2 + (\mu_1 + \mu_2 J^{-2/3} I_1)J^{-2/3}\boldsymbol{B}\Big) \\
&= J^{-5/3}\Big(-\frac{1}{3}(\mu_1 I_1 + 2\mu J^{-2/3} I_2)\boldsymbol{I} - \mu_2 J^{-2/3}\boldsymbol{B}^2 + (\mu_1 + \mu_2 J^{-2/3} I_1)\boldsymbol{B}\Big)
\end{aligned}
$$

In the `FLagSHyP` code, the tensor $\boldsymbol{I}\otimes\boldsymbol{I}$ is stored in the variable `CONS.IDENTITY_TENSORS.c1`; whereas the identity tensor tensor is such that $i = \frac{1}{2}\hat{i}$ and the tensor $\hat{i}$ is stored in the variable `CONS.IDENTITY_TENSORS.c2`. Both variables are defined in the initialization step, by the function `flagshyp/code/support/constant_entities.m`. Additionally, the identity matrix $\boldsymbol{I}$ is stored in the variable `cons.I`. These tensors are needed for the implementation of the elasticity tensor.

The second invariant $I_2 = \frac{1}{2}(I_1^2 - \text{tr}(B^2))$ could be calculated in the function `gradients.m`, at the same time as the other invariant. I chose to calculate the other form of this invariant $\text{tr}(B^2)$ to be consistent with the textbook.

For a material type "**", the functions `ctens**.m` and `stress**.m` calculate the isochoric elasticity tensor and Cauchy stresses respectively. The Moonley-Rivlin model is called material 18. For incompressible or nearly incompressible materials, the pressure is calculated in `mean_dilatation_pressure.m` as $p = \kappa(J-1)$ and the mean dilatation volumetric matrix is $\kappa J^2 \delta_{ij}\delta_{kl}$

List of files modified / added for the implementation:

- Created functions `ctens18.m` and `stress18.m` which calculate elasticity tensor and the Cauchy stress tensor respectively.
- In `gradients.m`: Added the calculation of $\mathrm{tr}(\boldsymbol{B}^2)$ in the variable `KINEMATICS.trb2`. Added the initialization of the variable in `initialisation/kinematics_initialisation.m`
- In `element_force_and_stiffness.m`, added the case of `matyp`=18 for the calculation of the mean dilatation pressure.
- In `mean_dilatation_pressure.m`: Added the case `matyp`=18: κ is the 4th material properties, the pressure is calculated as $p = \kappa(J-1)$ and the coefficient for the mean dilatation volumetric matrix is $\bar{\kappa} = \kappa J$.
- In `mean_dilatation_pressure_addition.m`: Added the case `matyp`=18 for the addition of pressure effects to Cauchy stresses and the elasticity tensor.
- In `/input_reading/matprop.m`: Added a case in the local function `material_choice` (matype): For `matyp`=18, 4 properties are input: ρ, μ_1, μ_2 and κ.
- I modified the file `stress_output.m` so that the output (for Paraview) would be the isochoric Cauchy stresses, to try to compare to the results from the paper

I tried to implement the example of a biaxial tension as presented in the paper but with only 1 hexahedron element (1m x 1 m x 1m): the boundary conditions are summarized on Figures 1 and 2. The bar is stretched in the x and y directions while compressed in the z direction to maintain the incompressibility condition. The deformation tensor is

$$\boldsymbol{F} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}$$

Assuming the deformation tensor has this form, We need to calculate the corresponding displacement from the initial configuration $\vec{X}$ for the boundary conditions such that in the x and y directions the final configuration is $x_{1,2} = \lambda X_{1,2} = X_{1,2} + \Delta u_{1,2}$. Therefore, $\Delta u_{1,2} = (\lambda - 1)/X_{1,2}$. Likewise, in the vertical direction, $\Delta u_3 = (1 - \lambda^{-2})X_3$. where X represents the initial position of the nodes. I have attached an input file for the the case of $\lambda = 1.4$: the setup is such that the final configuration is achieved in 10 displacement

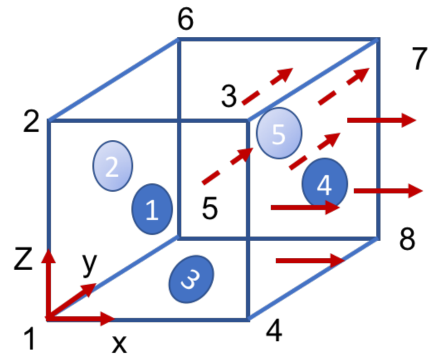

Figure 1: Boundary Conditions for 1 Element test

	Faces	Fixed BC	BC Code	Nonzero BC
Node 1	1,2,3	x=y=z=0	7	-
Node 2	1,2	x=y=0, z is free	3	-
Node3	1,4	y=0, z is free	3	Direction 1, k
Node 4	1,3,4	y=z=0	7	Direction 1, k
Node 5	2,3,5	x=z=0	7	Direction 2, k
Node 6	2,5	x=0, z is free	3	Direction 2, k
Node 7	5,4	z is free	3	Direction 1 & 2, k
Node 8	3,4,5	z=0	8	Direction 1 & 2, k

Figure 2: Boundary Conditions for 1 Element test

increments. Therefore, for nodes 3, 4, 7 and 8 located at $X = 1$ the incremental displacement BC in the x-direction is $u_{x0} = 0.04$m ; for nodes 5, 6, 7 and 8 located at $Y = 1$ the incremental displacement BC in the y-direction is $u_{y0} = 0.04$m ; and finally for nodes 2, 3, 6, 7, the incremental displacement is $u_{y0} = 0.04898$m. The material properties are set to $\rho = 1.0$kg/m3, $\mu_1 = 0.595522$MPa, $\mu_2 = 0.050381$MPa and $\kappa = 1e5$MPa.

I tested the input file with the Neo-Hookean model first and then the Moonley-Rivlin model. It can be seen on Figure 3 that the deformation tensor has the desired form and the volume ratio is equal to 1. The isochoric Cauchy stresses are shown on Figure 4: please note that I modified the `stress_output.m` file to write out only the isochoric stresses. To obtain the nominal Piola-Kirchhoff stress 1 of the paper, we can use the fact that $\boldsymbol{P} = J\sigma F^{-T}$ which gives $P_{xx}^{MR} = P_{yy}^{MR} = 0.27857$ MPa and $P_{xx}^{NH} = P_{yy}^{NH} = 0.248286$ MPa for the Moonley-Rivlin and Neo-Hookean material model respectively. This does not quite match the results of the

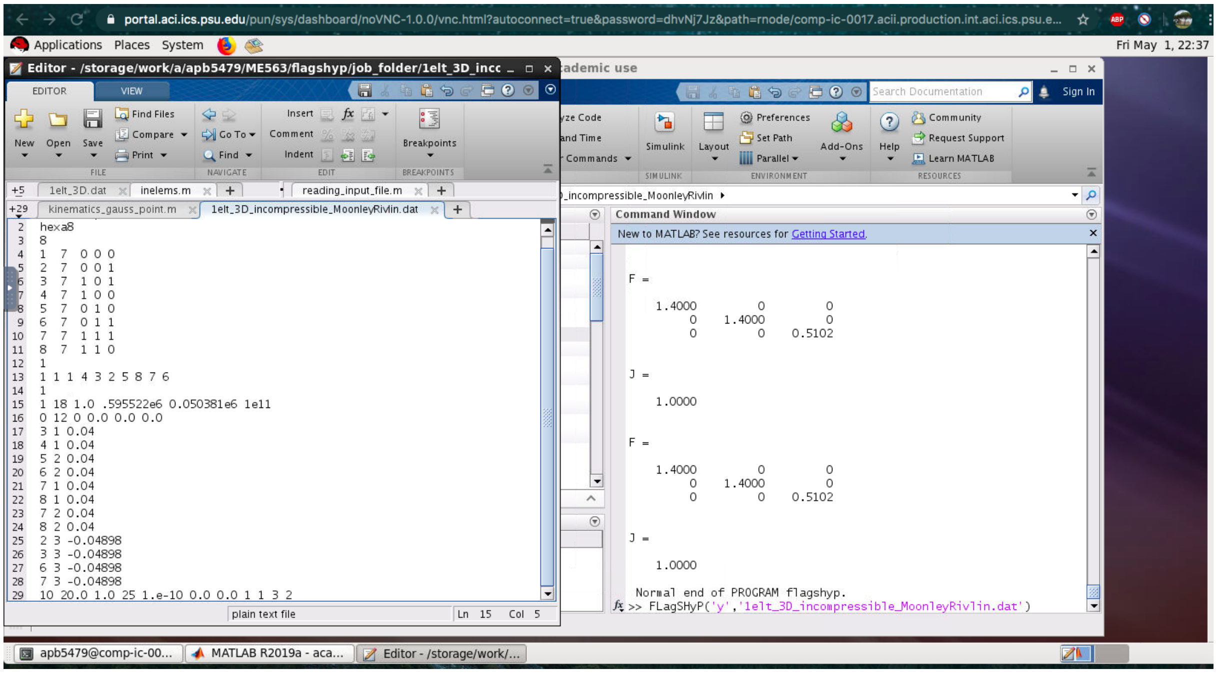

Figure 3: 1 Element Test with Moonley-Rivlin elasticity model for nearly incompressible material

paper, perhaps the authors were imposing a traction force on the faces and not a displacements. Qualitatively, the Neo-Hookean material shows a lower level of stresses, which is consistent with the paper. I also tried to create traction forces (of magnitude of the nominal stress in current configuration) on the faces where BC's were imposed before, but it did not converge, as shown on figure fig. 5. For this case I only use the homogeneous BC's of the previous case, with all other dofs free. The BC codes are then as follows:

- Node 1 $\in$ Faces 1,2,3 : x=y=z=0 BC7
- Node 2 $\in$ Faces 1,2: x=y=0, z is free BC3
- Node 3 $\in$ Faces 1,4: y=0, x is free, z is free: BC2
- Node 4 $\in$ Faces 1, 3, 4: y=z=0: BC6
- Node 5 $\in$ Faces 2, 3, 5: x=z=0 : BC5
- Node 6 $\in$ Faces 2,5: x=0: BC1
- Node 7 $\in$Faces 5,4: BC0
- Node 8 $\in$ Faces 3,4,5: z=0: BC4

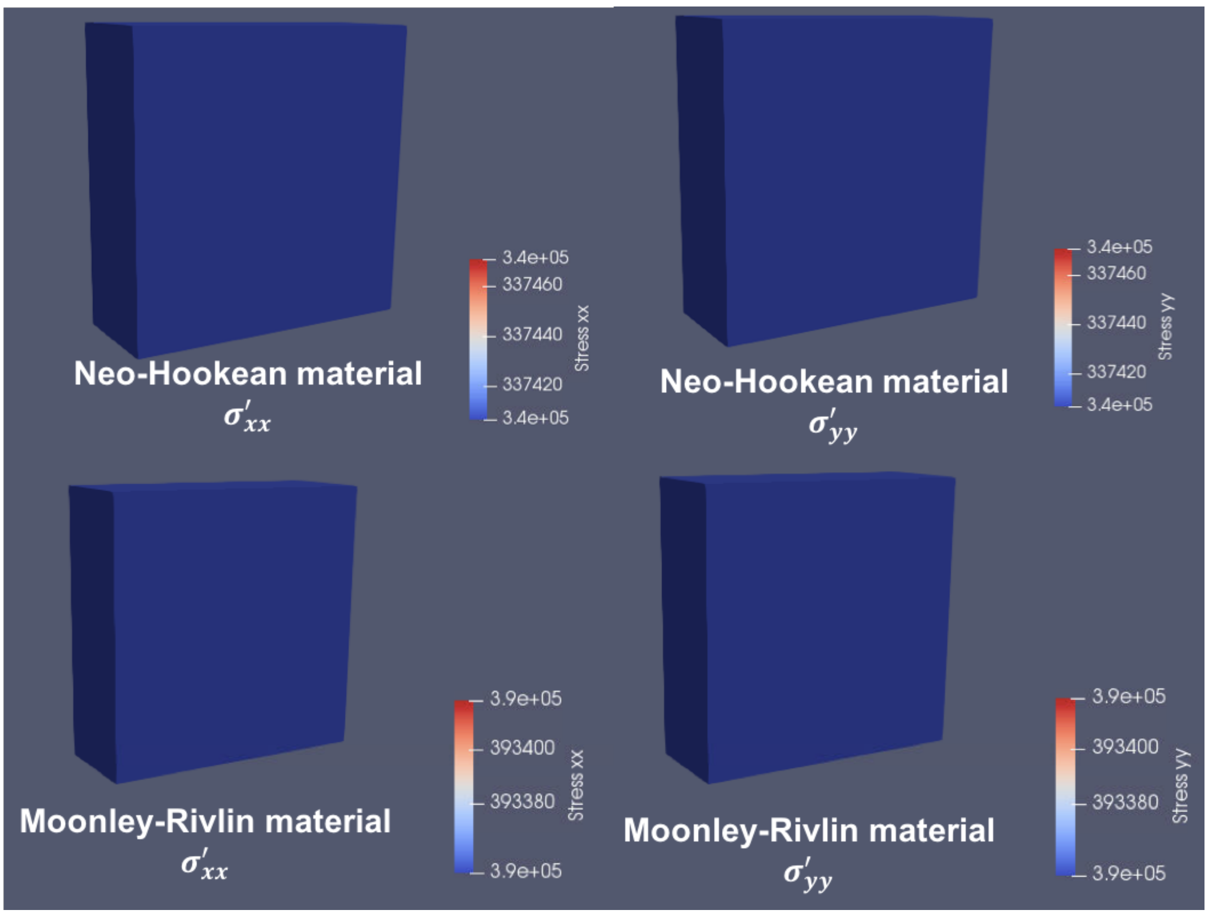

Figure 4: 1 Element Test with Moonley-Rivlin and Neo-Hookean elasticity model for nearly incompressible materials

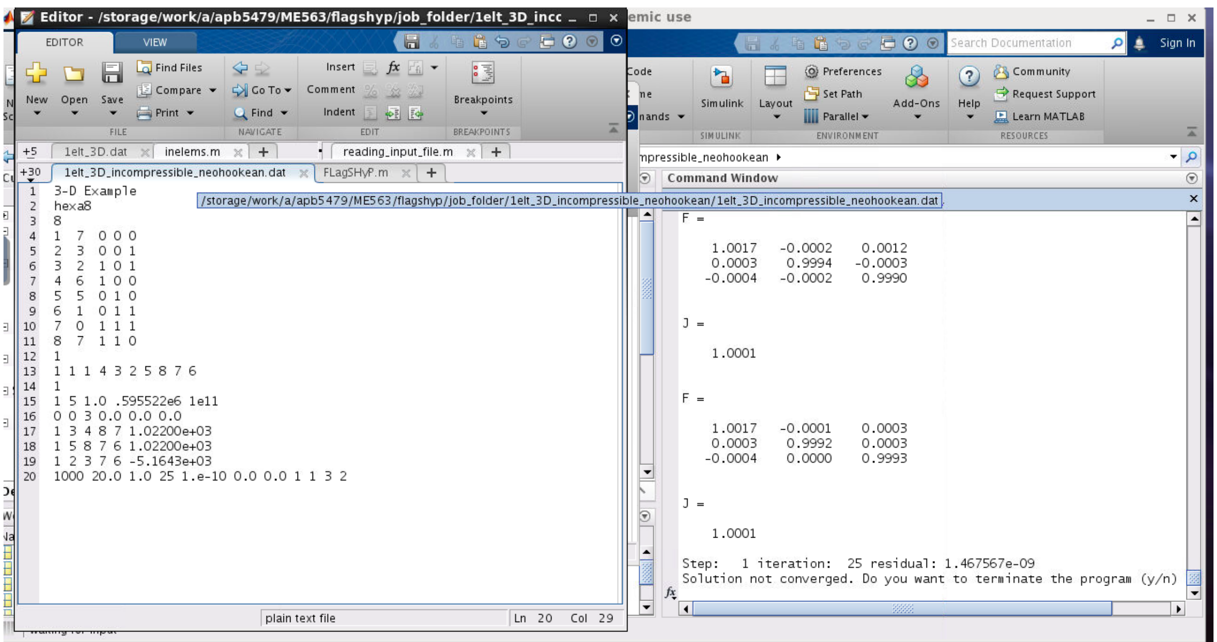

Figure 5: 1 Element Test with Moonley-Rivlin and Neo-Hookean elasticity model for nearly incompressible materials with traction forces on the faces