Common Inequalities in Computer Science

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Contents

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- Used in: almost everywhere..

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  - (Can almost be called the backbone of mathematics..)

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- So you have an algorithm - how do you prove it to be optimal (or close to optimal)?
- You have to show that there is a *lower bound* for the resources that the algorithm takes.
- Essentially - prove inequalities!

A simple inequality

- Setting: We are given n positive real numbers $x_1, x_2, \cdots x_n$, such that: $x_1 + x_2 + \cdots + x_n \geqslant n$.
- Prove: $\sum_{i: x_i > 1/2} x_i \geqslant n/2$.

A simple inequality

- Break up the sum $x_1 + x_2 + \cdots + x_n$ into two parts, collecting all terms i such that $x_i > 1/2$ and terms i such that $x_i \leqslant 1/2$.

$$\sum_i x_i = \sum_{i : x_i \leqslant 1/2} x_i + \sum_{i : x_i > 1/2} x_i$$

- Let k denote the number of terms such that $x_i \leqslant 1/2$, so that:

$$n \leqslant \sum_i x_i \leqslant k/2 + \sum_{i : x_i > 1/2} x_i$$

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- Let k denote the number of terms such that $x_i \leqslant 1/2$, so that:
$$n \leqslant \sum_i x_i \leqslant k/2 + \sum_{i:x_i > 1/2} x_i$$
- So that: $\sum_{i:x_i > 1/2} x_i \geqslant (n - k/2) \geqslant n/2$.

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- A concentration of measure result.

Inequality 2

- Setting:
  - We are given a *digraph* $D = (V, A)$.
  - Given a vertex v, let $i(v)$ denote the indegree of the vertex, and let $d(v)$ denote the total degree (in + out).
  - Let $V_1 = \{v : i(v) \geqslant d(v)/3\}$
- Required to prove that:

$$\sum_{v \in V_1} d(v) \geqslant |A|/2.$$

Inequality 2

- Another application of concentration of measure.
- Collect facts:
  - $\sum_{v \in V} d(v) = 2|A|$
  - $\sum_{v \in V} i(v) = |A|$
- Rewrite the blue inequality above as $\sum_{v \in V} d(v) \cdot \frac{i(v)}{d(v)} = |A|$.
- From here, we want to get: $\sum_{v \in V_1} d(v) \geqslant |A|/2$.

Inequality 2: Use blue fact

Facts

- $\sum_{v \in V} d(v) = 2|A|$
- $\sum_{v \in V} i(v) = |A|$

- Have: $\sum_{v \in V} d(v) \cdot \frac{i(v)}{d(v)} = |A|$
- Want to have: $\sum_{v \in V_1} d(v) \geqslant |A|/2$.
- Separate out the vertices as $v \in V_1$ and $v \in V \setminus V_1$.
- For $v \in V \setminus V_1$, $i(v)/d(v) < 1/3$. For $v \in V_1$, $i(v)/d(v) \leqslant 1$.
- $|A| = \sum_{v \in V} d(v) \cdot \frac{i(v)}{d(v)} \leqslant \sum_{v \in V_1} d(v) + \sum_{v \in V \setminus V_1} d(v)/3$