

Common Inequalities in Computer Science

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- Why inequalities?
- Used in: almost everywhere..

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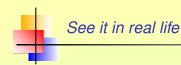
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 - **...**
 - (Can almost be called the backbone of mathematics..)



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- You have to show that there is a *lower bound* for the resources that the algorithm takes.
- Essentially prove inequalities!

A simple inequality

- Setting: We are given n positive real numbers $x_1, x_2, \dots x_n$, such that: $x_1 + x_2 + \dots + x_n \ge n$.
- ▶ Prove: $\sum_{i:x_i>1/2} x_i \ge n/2$.



▶ Break up the sum $x_1 + x_2 + \cdots + x_n$ into two parts, collecting all terms i such that $x_i \le 1/2$.

$$\sum_{i} x_i = \sum_{i:x_i \leqslant 1/2} x_i + \sum_{i:x_i > 1/2} x_i$$

Let k denote the number of terms such that $x_i \le 1/2$, so that:

$$n \leqslant \sum_i x_i \leqslant k/2 + \sum_{i:x_i>1/2} x_i$$



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• So that: $\sum_{i:x_i>1/2} x_i \ge (n-k/2) \ge n/2$.



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A concentration of measure result.



Setting:

- We are given a digraph D = (V, A).
- Given a vertex v, let i(v) denote the indegree of the vertex, and let d(v) denote the total degree (in + out).
- ▶ Let $V_1 = \{v : i(v) \ge d(v)/3\}$
- Required to prove that:

$$\sum_{v\in V_1} d(v) \geqslant |A|/2.$$

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- Another application of concentration of measure.
- Collect facts:

$$\sum_{v \in V} i(v) = |A|$$

► Rewrite the blue inequality above as

$$\sum_{v \in V} d(v) \cdot \frac{i(v)}{d(v)} = |A|.$$

▶ From here, we want to get: $\sum_{v \in V_1} d(v) \ge |A|/2$.



Inequality 2: Use blue fact

Facts

- $\blacktriangleright \sum_{v \in V} i(v) = |A|$
- ► Have: $\sum_{v \in V} d(v) \cdot \frac{i(v)}{d(v)} = |A|$
- ▶ Want to have: $\sum_{v \in V_1} d(v) \ge |A|/2$.
- ▶ Separate out the vertices as $v \in V_1$ and $v \in V \setminus V_1$.
- ▶ For $v \in V \setminus V_1$, i(v)/d(v) < 1/3. For $v \in V_1$, $i(v)/d(v) \leq 1$.
- ► $|A| = \sum_{v \in V} d(v) \cdot \frac{i(v)}{d(v)} \le \sum_{v \in V_1} d(v) + \sum_{v \in V \setminus V_1} d(v)/3$