Common Inequalities in Computer Science

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Contents

- Why inequalities?
- Used in: almost everywhere..

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  - (Can almost be called the backbone of mathematics..)

See it in real life

- So you have an algorithm - how do you prove it to be optimal (or close to optimal)?

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- So you have an algorithm - how do you prove it to be optimal (or close to optimal)?
- You have to show that there is a *lower bound* for the resources that the algorithm takes.
- Essentially - prove inequalities!

A simple inequality

- Setting: We are given n positive real numbers $x_1, x_2, \cdots x_n$, such that: $x_1 + x_2 + \cdots + x_n \geqslant n$.
- Prove: $\sum_{i:x_i > 1/2} x_i \geqslant n/2$.

A simple inequality

- Break up the sum $x_1 + x_2 + \cdots + x_n$ into two parts, collecting all terms i such that $x_i > 1/2$ and terms i such that $x_i \leqslant 1/2$.

$$\sum_i x_i = \sum_{i: x_i \leqslant 1/2} x_i + \sum_{i: x_i > 1/2} x_i$$

- Let k denote the number of terms such that $x_i \leqslant 1/2$, so that:

$$n \leqslant \sum_i x_i \leqslant k/2 + \sum_{i: x_i > 1/2} x_i$$

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- So that: $\sum_{i:x_i > 1/2} x_i \geqslant (n - k/2) \geqslant n/2$.

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- Given the n numbers, the mean is $\geqslant 1$.
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- A concentration of measure result.

Inequality 2

- Setting:
  - We are given a *digraph* $D = (V, A)$.
  - Given a vertex v, let $i(v)$ denote the indegree of the vertex, and let $d(v)$ denote the total degree (in + out).
  - Let $V_1 = \{v : i(v) \geqslant d(v)/3\}$
- Required to prove that:

$$\sum_{v \in V_1} d(v) \geqslant |A|/3.$$

Inequality 2

- Another application of concentration of measure.
- Collect facts:
  - $\sum_{v \in V} d(v) = 2|A|$
  - $\sum_{v \in V} i(v) = |A|$
- Rewrite the blue equality above as $\sum_{v \in V} d(v) \cdot \frac{i(v)}{d(v)} = |A|$.
- From here, we want to get: $\sum_{v \in V_1} d(v) \geqslant |A|/3$.

Inequality 2: Use blue fact

Facts

- $\sum_{v \in V} d(v) = 2|A|$
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- Have: $\sum_{v \in V} d(v) \cdot \frac{i(v)}{d(v)} = |A|$
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- Want to have: $\sum_{v \in V_1} d(v) \geqslant |A|/3$.
- Separate out the vertices as $v \in V_1$ and $v \in V \setminus V_1$.
- For $v \in V \setminus V_1$, $i(v)/d(v) < 1/3$. For $v \in V_1$, $i(v)/d(v) \leqslant 1$.
- $|A| = \sum_{v \in V} d(v) \cdot \frac{i(v)}{d(v)} \leqslant \sum_{v \in V_1} d(v) + \sum_{v \in V \setminus V_1} d(v)/3$

Inequality 2: Use red fact

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- $\sum_{v \in V} d(v) = 2|A|$
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- $|A| = \sum_{v \in V} d(v) \cdot \frac{i(v)}{d(v)} \leqslant \sum_{v \in V_1} d(v) + \sum_{v \in V \setminus V_1} d(v)/3$
- i.e. $|A| \leqslant \sum_{v \in V_1} d(v) + \frac{2|A|}{3}$, which gives
- $\sum_{v \in V_1} d(v) \geqslant \frac{|A|}{3}$.

Inequality 2: Quiz Problem 1 (Extra Credit)

- We showed:

$$\sum_{v \in V_1} d(v) \geqslant |A|/3.$$

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- Instead show that:

$$\sum_{v \in V_1} d(v) \geqslant |A|/2.$$

Inequality 2: Quiz Problem 1 (Bonus Credit)

- Show that:

$$\sum_{v \in V_1} d(v) \geqslant |A|$$

Comments

- Why is this a "concentration of measure" result?
- Note that $\sum_{v \in V} d(v) = 2 \sum_{v \in V} i(v)$, so *on average* $i(v)$ is roughly half of $d(v)$.
- As before, we move slightly away from the mean and ask: how many vertices v will satisfy $i(v) \geqslant d(v)/3$?
- While we don't get a *count*, we get a result for the *mass*.

Concentration of measure

- Chebyshev Inequalities
- Markov Inequalities
- Chernoff-Hoeffding bounds.
- Talagrand Inequality

Example: Chebyshev's Inequality

Recall Chebyshev's inequality:

- Setting: let X be a random variable with $E(X) = \mu$ and variance $\sigma^2 = var(X)$.
- Then: $P(|X - \mu| \geqslant t) \leqslant \frac{\sigma^2}{t^2}$

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How do we prove it?

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How do we prove it? How do we prove Markov's Inequality?

Quiz Problem 2

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  - Question: A one-sided version of Chebyshev

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  - Setting: let X be a random variable with $E(X) = \mu$ and variance $\sigma^2 = var(X)$.
  - Then: $P(|X - \mu| \geqslant t) \leqslant \frac{\sigma^2}{t^2}$
  - Question: A one-sided version of Chebyshev
  - Prove that:

$$P(X - \mu \geqslant t) \leqslant \frac{\sigma^2}{\sigma^2 + t^2}.$$

Shifting Gears: Lagrangian Relaxations

- Suppose we are given an optimization problem.
- What is an optimization problem?
  - Has an objective
  - Has constraints

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Shifting Gears: Lagrangian Relaxations

- Suppose we are given an optimization problem.
- What is an optimization problem?
  - Has an objective
  - Has constraints
- Some constraints are easy to satisfy (called “soft” constraints)
- while some others are harder - hard constraints.

An example

An optimization problem:

$$\min \sum_{i \in V} c_i x_i$$

$$\begin{aligned} \forall (i,j) \in E \quad & x_i + x_j \geqslant 1 \\ \forall i \in V \quad & x_i \geqslant 0 \end{aligned}$$

Looks familiar?

Example modified

Modified Vertex Cover problem where we have to cover at least k edges (the so-called “partial vertex cover” problem):

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$$\min \sum_{i \in V} c_i x_i$$

$$\begin{array}{rc} \forall (i,j) \in E & x_i + x_j \geqslant y_e \\ & \sum_{e \in E} y_e \geqslant k \\ \forall i \in V & 1 \geqslant x_i \geqslant 0 \end{array}$$

Here, the $x_i + x_j \geqslant y_e$ are the "soft" constraints, and $\sum_{e \in E} y_e \geqslant k$ is the "hard" constraint.

Lagrangian Relaxations

- The big idea behind Lagrangian Relaxations is to “relax” the hard constraints by *pulling them* into the objective.

General Instance

Optimization problem O_1:

$$\min f(x)$$

s.t.

$$s(x) \leqslant 0$$
$$h(x) \leqslant 0$$

where, $s(x)$ stands for the soft constraints, and $h(x)$ for the hard constraints.

Lagrangian Relaxation

Relaxation $O_2 = \mathrm{LR}(\lambda)$ of O_1:

$$\min f(x) + \lambda \cdot h(x)$$

$$\text{s.t.}$$

$$s(x) \leqslant 0$$

$$\lambda \geqslant 0$$

Here, $\lambda \in [0, \infty)$ is called a Lagrange Multiplier.

Major Impact: $\mathrm{LR}(\lambda) \leqslant \mathbf{O_1}$!!!

Hölder's Inequality

Setting:

- Vectors $\mathbf{x} = (x_1, \cdots, x_n), \mathbf{y} = (y_1, \cdots, y_n)$, where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_{\geqslant 0}$.
- Reals $p, q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Prove that:

$$(x_1^p + x_2^p + \cdots + x_n^p)^{1/p} \cdot (y_1^q + y_2^q + \cdots y_n^q)^{1/q} \geqslant \mathbf{x} \cdot \mathbf{y}$$

Hölder's Inequality

Have we seen (any special case of) this before? With $p = q = 2$:

$$(x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2} \cdot (y_1^2 + y_2^2 + \cdots y_n^2)^{1/2} \geqslant \mathbf{x} \cdot \mathbf{y}$$

which is the Cauchy Schwarz inequality!

Another Hölder's Inequality

The following is also often called Hölder's Inequality (same conditions on p, q, non-negative real numbers x, y):

$$\frac{x^p}{p} + \frac{y^q}{q} \geqslant xy$$

Proof is easy by $\mathrm{AM} - \mathrm{GM}$. (Quiz Problem 3). Call this the "little" Hölder's Inequality. Condition for equality: $x^p = y^q$. Now, from here to the first Hölder's Inequality? We will only discuss $n = 2$.

$n = 2$

We want:

$$(x_1^p + x_2^p)^{1/p} \cdot (y_1^q + y_2^q)^{1/q} \geqslant \mathbf{x} \cdot \mathbf{y}$$

First attempt: We can write down 2 “little” Hölder Inequalities (one for x_1, y_1 another for x_2, y_2) but that gives us:

$$\frac{x_1^p + x_2^p}{p} + \frac{y_1^q + y_2^q}{q} \geqslant \mathbf{x} \cdot \mathbf{y}$$

This is weaker than what we want; why?

Weaker?

So far, we have:

$$\frac{x_1^p + x_2^p}{p} + \frac{y_1^q + y_2^q}{q} \geqslant \mathbf{x} \cdot \mathbf{y}$$

Let $z_1^p = x_1^p + x_2^p$ and $z_2^q = y_1^q + y_2^q$, then one application of the "little" Hölder Inequality gives:

$$\frac{x_1^p + x_2^p}{p} + \frac{y_1^q + y_2^q}{q} \geqslant z_1 \cdot z_2 = (x_1^p + x_2^p)^{1/p} \cdot (y_1^q + y_2^q)^{1/q}$$

Weaker to Stronger

Terry Tao's suggestion: in order to milk most out of an inequality, apply it to a scenario where equality can even conceivably hold.

Also called the "principle of maximum effectiveness".

We realize that for the application of the "little" Hölder Inequality above, the equality condition of $x_1^p + x_2^p = y_1^q + y_2^q$ may be widely flouted.

Enter Lagrangian Multipliers

Taking the cue from there, let's apply a Lagrangian Multiplier λ:

$$\frac{x_1^p + x_2^p}{p} + \lambda^q \frac{y_1^q + y_2^q}{q} \geqslant \lambda \mathbf{x} \cdot \mathbf{y}$$

Now we can at least hope to achieve the case of equality by suitably choosing $\lambda = \lambda_0$.

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$$\lambda_0^q = \frac{x_1^p + x_2^p}{y_1^q + y_2^q}$$

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$$\lambda_0^q = \frac{x_1^p+x_2^p}{y_1^q+y_2^q}$$

Applying $1/p + 1/q = 1$, the LHS $\frac{x_1^p+x_2^p}{p} + \lambda_0^q \frac{y_1^q+y_2^q}{q}$ simplifies to $(x_1^p + x_2^p)$ and we have: $(x_1^p + x_2^p) \geqslant \lambda_0 \mathbf{x} \cdot \mathbf{y}$.

Enter Lagrangian Multipliers

The LHS $\frac{x_1^p + x_2^p}{p} + \lambda_0^q \frac{y_1^q + y_2^q}{q}$ simplifies to$(x_1^p + x_2^p)$ and we have:
$(x_1^p + x_2^p) \geqslant \lambda_0 \mathbf{x} \cdot \mathbf{y}$.
One last application of $1/p + 1/q = 1$ gives

$$(x_1^p + x_2^p)^{1/p} \cdot (y_1^q + y_2^q)^{1/q} \geqslant \mathbf{x} \cdot \mathbf{y}$$

and we are done!

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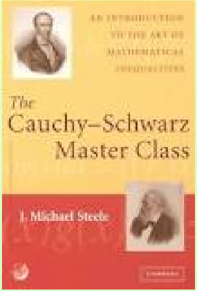
and we are done!
The inequality for n quantities is similar.

Ode to beautiful inequalities

The world of inequalities is mesmerizing:

- The mother of all inequalities: Cauchy Schwarz Inequality $(x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}(y_1^2 + y_2^2 + \cdots y_n^2)^{1/2} \geqslant \mathbf{x} \cdot \mathbf{y}$.



Ode to beautiful inequalities

Sample applications:

1. $(x^2 + y^2 + z^2) \geqslant (xy + yz + zx)$; restatement (AM-GM for 3 numbers): $x^3 + y^3 + z^3 \geqslant 3xyz$.
2. Loomis-Whitney Inequality: Compares the volume of a set in terms of the volumes of the projections of that set onto lower dimensional subspaces.
   For 3 dimensions: given a set $A \in \mathbb{R}^3$ and its projections onto the axes A_x, A_y, A_z it holds that:
   $$|A| \leqslant |A_x|^{1/2}|A_y|^{1/2}|A_z|^{1/2}$$

Ode to beautiful inequalities

1. Harker-Kasper Inequality in Crystallography

Ode to beautiful inequalities

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2. Cramér-Rao Lower Bound in Statistics

Ode to beautiful inequalities

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3. Jensen's Inequality in Convex Analysis, Expectation Maximization methods.

Ode to beautiful inequalities

1. Harker-Kasper Inequality in Crystallography
2. Cramér-Rao Lower Bound in Statistics
3. Jensen's Inequality in Convex Analysis, Expectation Maximization methods.
4. Many, many, many more...

Far reaching consequences

Question: Are all non-negative polynomials of two variables sums of squares of other polynomials?

Minkowski conjectured that this is not the case.
One can actually construct a counter-example (accompanied by a 1-page proof) motivated by the AM-GM inequality!

Final Challenge: Quiz Problem 4

Carleman's Inequality:

$$\sum_{k=1}^{\infty}(a_1 a_2 \cdots a_k)^{1/k} \leqslant e \sum_{k=1}^{\infty} a_k.$$

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Or...

Final Challenge: Quiz Problem 4

Carleman's Inequality:

$$\sum_{k=1}^{\infty}(a_1 a_2 \cdots a_k)^{1/k} \leqslant e \sum_{k=1}^{\infty} a_k.$$

Or... prove that:

$$3(a^2 + b^2 + c^2) \geqslant (a + b + c)^2$$

THANK YOU