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SUBSYSTEM-BASED ROBUSTNESS ASSESSMENT OF (n, k)-STAR GRAPHS

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Abstract. *The reliability of a generic system is one robustness index to measure the probability that the system can work under a given suite of operational and environmental conditions in a specified time interval. In 2016, Li et al. proposed a combinatorial formulation for computing both upper and lower bounds on the subsystem reliability of (n, k)-star graph. However, their derived lower bound turns out to be incorrect almost surely. As the lower-bounded reliability plays a more informative role in evaluating a system's degree of sustainability, this paper establishes the correct formula for the lower bound on the subsystem reliability of (n, k)-star graphs. Numerical simulations are run under the probability fault model for validating the combinatorial formulation.*

Keywords: Reliability, Subsystem reliability, Combinatorics, Probability fault model, (n, k)-star graph

1. **Introduction.** In recent years, the Internet of Things (IoT) prevails in a wide range of life applications. Within the IoT environment, everything (including physical devices, buildings and individual users) is equipped with one or more sensors, software, and Internet access to collect and exchange numerous kinds of data in a just-in-time way. The IoT is an application-oriented network system that relies heavily upon the realization of high-performance parallel and distributed computing. To achieve this goal, it is usually the most fundamental to design a suitable underlying topology for the network system, which dominates the layout of all objects and communication links.

The (n, k)-star graph is an interconnection network based on the coset of symmetry group, and it was first proposed by Chiang and Chen [1, 2] and has been investigated by many researchers [3, 4, 5, 6, 7, 8]. In particular, the (n, k)-star graph includes the n-star graph [9] and the alternating group graph [10] as special cases. Previously, Li et al. [11] developed a combinatorial formulation for computing both upper and lower bounds on the subsystem reliability of (n, k)-star graph under the probability fault model. Unfortunately, the lower bound derived in [11] proved incorrect. The reason will be explained later in Section 3. The major contributions and innovation of this paper are summarized below.

- A correct lower bound formula is established for the subsystem reliability of (n, k)-star graphs.
- Based on the corrected formulation, numerical simulations are done to validate the proposed bounds.
- Compared with state-of-the-art studies, many researchers considered only upper bounds in the reliability estimation. Instead, lower bounds can avoid the risk of overestimating a system's reliability and are valuable to squeeze the true subsystem reliability.

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The rest of this paper is structured as follows. Section 2 introduces the fundamentals of the probability fault model and topological properties of the (n,k)-star graph. Section 3 derives the correct lower and upper bounds on the subsystem reliability of the (n,k)-star graph, together with its simple approximation. Section 4 presents numerical results based on the probabilistic model of the exponential distribution. Finally, our concluding remarks are drawn in Section 5.

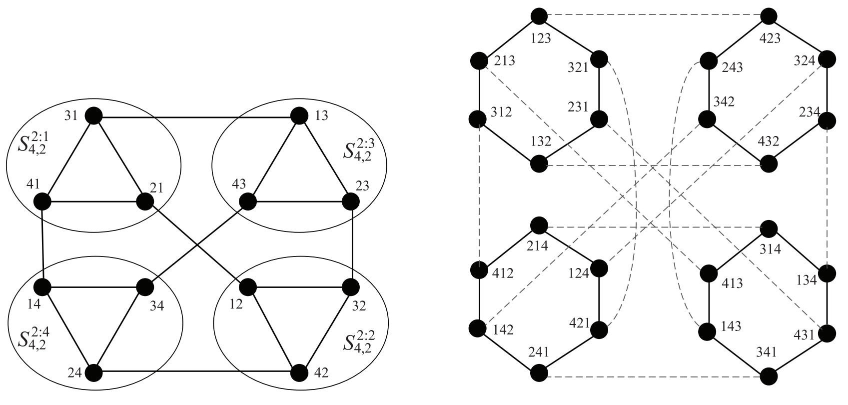
2. **Preliminaries.** A simple, loopless undirected graph G consists of a vertex set $V(G)$ and an edge set $E(G)$, where $V(G)$ is a finite set, and $E(G)$ is a subset of the set of all unordered pairs of two distinct elements in $V(G)$. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For any nonempty subset $S \subseteq V(G)$, the subgraph of G induced by S is a graph whose vertex set is S and whose edge set consists of all the edges of G joining any two vertices in S.

2.1. **The (n,k)-star graph.** Let n be any positive integer. For convenience, let $\langle n\rangle$ denote the set of all positive integers from 1 to n; i.e., $\langle n\rangle = \{1,2,\ldots,n\}$.

Definition 2.1. [1] *The (n,k)-star graph, denoted by $S_{n,k}$, is specified by two positive parameters n and k, where $1 \le k \le n-1$. The vertex set of $S_{n,k}$ is defined by $V(S_{n,k}) = \{c_1c_2\cdots c_k | c_i \in \langle n\rangle$ for $1 \le i \le k$, and $c_i \ne c_j$ if $i \ne j\}$. The edge set of $S_{n,k}$ is regularized as follows.*

- *A vertex $c_1c_2\cdots c_i\cdots c_k$ is adjacent to the vertex $c_ic_2\cdots c_1\cdots c_k$ through an i-edge by a transposition between c_1 and c_i, where $2 \le i \le k$.*
- *A vertex $c_1c_2\cdots c_i\cdots c_k$ is adjacent to the vertex $xc_2\cdots c_i\cdots c_k$ through a 1-edge by replacing c_1 with x for each $x \in \langle n\rangle \backslash \{c_1,c_2,\ldots,c_k\}$.*

According to Definition 2.1, $S_{n,k}$ is vertex-symmetric and $(n-1)$-regular [1]. For two integers $i \in \langle k\rangle$ and $x \in \langle n\rangle$, let $V_{n,k}^{i:x}$ be the set of all vertices whose ith digits have the same identifying code x in $S_{n,k}$. Then, $\left\{V_{n,k}^{i:x} \middle| 1 \le x \le n\right\}$ forms a partition of $V(S_{n,k})$. Let $S_{n,k}^{i:x}$ denote the subgraph of $S_{n,k}$ induced by $V_{n,k}^{i:x}$. Clearly, $S_{n,k}^{i:x}$ is isomorphic to $S_{n-1,k-1}$ if $i \ne 1$. For instance, $S_{4,2}$ is partitioned into four $S_{3,1}$'s. Figure 1(a) illustrates $S_{4,2}$ and its $S_{3,1}$-subgraphs.



(a) (b)

FIGURE 1. (a) The $(4,2)$-star graph $S_{4,2}$ and its $S_{3,1}$-subgraphs; (b) $S_{4,3} - E_{4,3}^1$ consists of four components isomorphic to $S_{3,2}$, in which 1-edges are represented by dash lines

For any vertex v of $S_{n,k}$, its ith digit is denoted by $(v)_i$, and for $1 \le m \le k-1$, let $S_{n,k}^{i_1:x_1,\ldots,i_m:x_m}$ be the subgraph induced by $\{v \in V(S_{n,k}) | (v)_{i_1} = x_1, (v)_{i_2} = x_2, \ldots, (v)_{i_m} = x_m\}$, where $\{i_1,i_2,\ldots,i_m\}$ and $\{x_1,x_2,\ldots,x_m\}$ are m-element subsets of $\{2,3,\ldots,k\}$ and $\langle n\rangle$, respectively. Then, $S_{n,k}^{i_1:x_1,i_2:x_2,\ldots,i_m:x_m}$ is isomorphic to the $(n-m,k-m)$-star graph.

In this way, $S_{n,k}$ can be partitioned into $\binom{n}{m}m!$ disjoint $S_{n-m,k-m}$-subgraphs [2]. Denote by $E^1_{n,k}$ the set of all 1-edges in $S_{n,k}$. Then, $S_{n,n-1} - E^1_{n,n-1}$ consists of n components, every of which is isomorphic to an $(n-1, n-2)$-star graph. For instance, Figure 1(b) illustrates that $S_{4,3} - E^1_{4,3}$ consists of four components that are isomorphic to $S_{3,2}$. However, in [11], this kind of components is not seen as any $S_{n-1,n-2}$-subsystem of $S_{n,n-1}$, because it does not strictly meet the formal definition of (n, k)-star graphs. Thus, the total number of distinct $S_{n-m,k-m}$-subgraphs in $S_{n,k}$ is $\binom{k-1}{m}\binom{n}{m}m!$.

2.2. **The subsystem reliability under the probability fault model.** The *probability fault model* originated from the pioneer work of Chang and Bhuyan [12] to evaluate the subcube reliability of the hypercube architecture by assuming a mutually independent, homogeneous vertex reliability. However, the exact formulation of the subsystem reliability becomes much more complicated for a variety of interconnection networks. Therefore, many researchers turned their attention to establishing the reliability's upper and lower bounds as alternative approaches [13, 14, 15, 16, 17, 18, 19, 20].

Let $R^{n-1,k-1}_{n,k}(p)$ denote the probability that there exists at least one fault-free $S_{n-1,k-1}$-subgraph in the (n, k)-star graph $S_{n,k}$, where p is a given homogeneous vertex reliability. Let $A^{(i:x)}_{n,k}$ denote the probabilistic event that every vertex of $S^{i:x}_{n,k}$ is fault-free for $i \in \{2, 3, \ldots, k\}$ and $x \in \langle n\rangle$. Under the probability fault model, $P\left(A^{(i:x)}_{n,k}\right) = p^{\frac{(n-1)!}{(n-k)!}}$ and $R^{n-1,k-1}_{n,k}(p) = P\left(\bigcup_{i=2}^{k}\bigcup_{x=1}^{n} A^{(i:x)}_{n,k}\right)$.

3. **Evaluation of $\boldsymbol{R^{n-1,k-1}_{n,k}(p)}$.** There are $n(k-1)$ distinct $S_{n-1,k-1}$-subgraphs in $S_{n,k}$. For convenience, denote these $S_{n-1,k-1}$-subgraphs by $S^1_{n-1,k-1}, S^2_{n-1,k-1}, \ldots, S^{n(k-1)}_{n-1,k-1}$. For $1 \le i \le n(k-1)$, let O_i denote the event that $S^i_{n-1,k-1}$ is fault-free. According to the principle of inclusion and exclusion, $R^{n-1,k-1}_{n,k}(p)$ is expressed as follows:

$$\begin{aligned} R^{n-1,k-1}_{n,k}(p) = & \sum_{i=1}^{n(k-1)} P(O_i) - \sum_{i<j} P(O_i \cap O_j) + \sum_{i<j<l} P(O_i \cap O_j \cap O_l) \\ & - \sum_{i<j<l<q} P(O_i \cap O_j \cap O_l \cap O_q) + \cdots - (-1)^{nk-n} P\left(\bigcap_{i=1}^{nk-n} O_i\right). \end{aligned} \tag{1}$$

Li et al. [11] evaluated both upper and lower bounds on $R^{n-1,k-1}_{n,k}(p)$ under the probability fault model. They approximated $R^{n-1,k-1}_{n,k}(p)$ by omitting those after the 4th (respectively, 3rd) term of Equation (1) to form a lower (respectively, an upper) bound. All

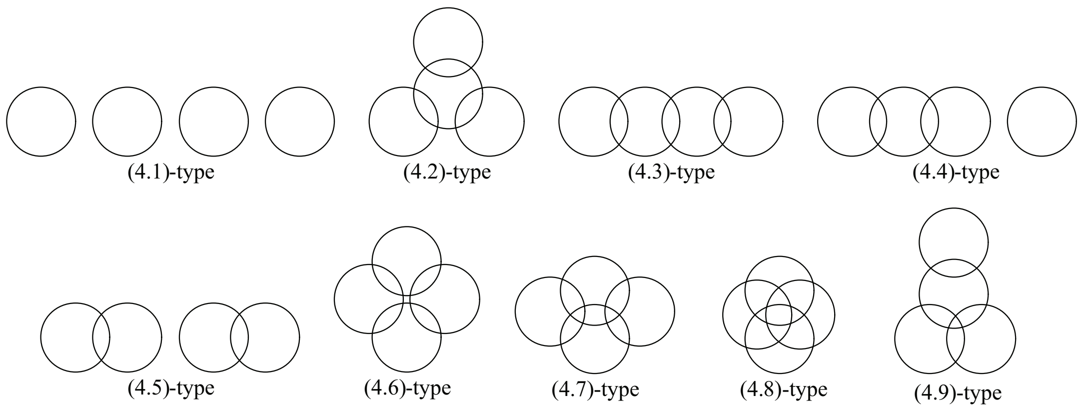


Figure 2. The nine intersection types of four $S_{n-1,k-1}$-subgraphs in $S_{n,k}$

Table 1. Combinatorial formulas in [11] for all intersection types of four $S_{n-1,k-1}$-subgraphs, in contrast to the corrected ones

Intersection	Incorrrect # of groups [11]	$n=8$ $k=6$	$n=9$ $k=7$	Corrrect # of groups	$n=8$ $k=6$	$n=9$ $k=7$
(4.1)-type	$\binom{k-1}{4}\left(\binom{4}{1}\binom{n}{4}+\binom{n}{1}\right)$	1,440	7,695	$\binom{k-1}{1}\binom{n}{4}+\binom{k-1}{4}\binom{n}{1}$	390	891
(4.2)-type	$2\binom{k-1}{4}\binom{4}{2}\binom{n}{3}\binom{n-3}{1}^{*}$ $+\binom{k-1}{4}\left[\binom{n}{2}\left(\binom{4}{1}+\binom{4}{3}\right)\right]$	17,920	95,040	$2\binom{k-1}{2}\binom{n}{3}\binom{n-3}{1}$ $+8\binom{k-1}{4}\binom{n}{2}$	6,720	19,440
(4.3)-type	$2\binom{k-1}{4}\binom{4}{2}\binom{n}{3}\binom{3}{2}$ $+3\binom{k-1}{4}\binom{4}{3}\binom{n}{2}\times 2$	13,440	58,320	$6\binom{k-1}{2}\binom{n}{3}+6\binom{k-1}{3}\binom{n}{2}$	5,040	11,880
(4.4)-type	$2\binom{k-1}{4}\binom{4}{2}\binom{n}{3}\binom{3}{1}$ $+3\binom{k-1}{4}\binom{4}{3}\binom{n}{2}\times 2$	13,440	58,320	$6\binom{k-1}{2}\binom{n}{3}+6\binom{k-1}{3}\binom{n}{2}$	5,040	11,880
(4.5)-type	$\binom{k-1}{4}\binom{4}{2}\binom{n}{2}$	840	3,240	$\binom{k-1}{2}\binom{n}{2}$	280	540
(4.6)-type	$\binom{k-1}{4}\binom{4}{2}\binom{n}{4}\binom{4}{2}$ $+3\binom{k-1}{4}\binom{4}{3}\binom{n}{3}\binom{3}{2}$ $+\binom{k-1}{4}\binom{n}{2}\binom{4}{2}$	23,520	116,640	$\binom{k-1}{2}\binom{n}{2}\binom{n-2}{2}$ $+9\binom{k-1}{3}\binom{n}{3}$ $+6\binom{k-1}{4}\binom{n}{2}$	10,080	29,700
(4.7)-type	$3\binom{k-1}{4}\binom{4}{3}\binom{n}{4}\binom{4}{2}\times 2$ $+2\binom{k-1}{4}\binom{n}{3}\binom{4}{1}\binom{3}{1}$ $+\binom{k-1}{4}\binom{n}{3}\binom{4}{2}\times 2$	60,480	317,520	$36\binom{k-1}{3}\binom{n}{4}$ $+36\binom{k-1}{4}\binom{n}{3}$	35,280	136,080
(4.8)-type	$\binom{k-1}{4}\binom{n}{1}\binom{n-1}{1}\binom{n-2}{1}\binom{n-3}{1}$	8,400	45,360	$24\binom{k-1}{4}\binom{n}{4}$	8,400	45,360
(4.9)-type	$3\binom{k-1}{4}\binom{4}{3}\binom{n}{3}\binom{3}{2}\times 4$	40,320	181,440	$36\binom{k-1}{3}\binom{n}{3}$	20,160	60,480
Total	$\neq\binom{n(k-1)}{4}$	179,800	883,575	$=\binom{n(k-1)}{4}$	91,390	316,251

* In [11], this term appears to be $2\binom{k-1}{4}\binom{4}{2}\binom{n}{3}\binom{n-1}{3}$, which should be a typo.

possible intersection ways of four $S_{n-1,k-1}$-subgraphs are grouped into various types, as shown in Figure 2, and incorrect number of combinations corresponding to each intersection type was presented in [11], as summarized in the left part of Table 1. By contrast, the correct results are listed in the right part of Table 1.

The following lemma verifies that the newly established formulation is really correct. For convenience, denote by $\phi_{n,k}(p)$ the exact combinatorial formula of $\sum_{i<j<l<q} P(O_i \cap O_j \cap O_l \cap O_q)$.

Lemma 3.1. *Given a homogeneous vertex reliability p of $S_{n,k}$, $\phi_{n,k}(p)$ is given below:*

$$\begin{aligned}
&\phi_{n,k}(p)\\
\equiv &\sum_{1\le i<j<l<q\le n(k-1)} P(O_i\cap O_j\cap O_l\cap O_q)\\
= &\left[(k-1)\binom{n}{4}+n\binom{k-1}{4}\right]p^{\frac{4(n-1)!}{(n-k)!}}+\left[2(n-3)\binom{k-1}{2}\binom{n}{3}+8\binom{k-1}{4}\binom{n}{2}\right.\\
&\left.+6\binom{k-1}{2}\binom{n}{3}+6\binom{k-1}{3}\binom{n}{2}\right]p^{\frac{4(n-1)!-3(n-2)!}{(n-k)!}}
\end{aligned}$$

$$+\left[6\binom{k-1}{2}\binom{n}{3}+6\binom{k-1}{3}\binom{n}{2}+\binom{k-1}{2}\binom{n}{2}\right]p^{\frac{4(n-1)!-2(n-2)!}{(n-k)!}}$$
$$+\left[\binom{k-1}{2}\binom{n}{2}\binom{n-2}{2}+9\binom{k-1}{3}\binom{n}{3}+6\binom{k-1}{4}\binom{n}{2}\right]p^{\frac{4(n-1)!-4(n-2)!}{(n-k)!}}$$
$$+36\binom{k-1}{3}\binom{n}{4}p^{\frac{4(n-1)!-5(n-2)!+2(n-3)!}{(n-k)!}}+36\binom{k-1}{4}\binom{n}{3}p^{\frac{4(n-1)!-5(n-2)!+2(n-3)!}{(n-k)!}}$$
$$+24\binom{k-1}{4}\binom{n}{4}p^{\frac{4(n-1)!-6(n-2)!+4(n-3)!-(n-4)!}{(n-k)!}}+36\binom{k-1}{3}\binom{n}{3}p^{\frac{4(n-1)!-4(n-2)!+(n-1)!}{(n-k)!}}. \quad (2)$$

Proof: The combinatorial derivation of $\phi_{n,k}(p)$ is described below. Without loss of generality, suppose that probabilistic events O_i, O_j, O_l, and O_q correspond to $A_{n,k}^{(i_1:x_1)}$, $A_{n,k}^{(i_2:x_2)}$, $A_{n,k}^{(i_3:x_3)}$, and $A_{n,k}^{(i_4:x_4)}$, respectively. That is, $P(O_i \cap O_j \cap O_l \cap O_q) = P\left(\bigcap_{j=1}^4 A_{n,k}^{(i_j:x_j)}\right)$.

Case 1: $|\{i_1,i_2,i_3,i_4\}| = 1$. Obviously, we have $|\{x_1,x_2,x_3,x_4\}| = 4$, so $S_{n,k}^{i_1:x_1}$, $S_{n,k}^{i_2:x_2}$, $S_{n,k}^{i_3:x_3}$, and $S_{n,k}^{i_4:x_4}$ are mutually disjoint. Figure 3(a) illustrates the union of these four subgraphs ((4.1)-type). Then, $P\left(\bigcap_{j=1}^4 A_{n,k}^{(i_j:x_j)}\right) = p^{\frac{4(n-1)!}{(n-k)!}}$. Accordingly, this case corresponds to $\binom{k-1}{1}\binom{n}{4}$ 4-subgraph combinations.

Case 2: $|\{i_1,i_2,i_3,i_4\}| = 2$. We assume that $\{i_1,i_2,i_3,i_4\} = \{i_1,i_3\}$ and consider the following subcases.

Subcase 2.1: $|\{x_1,x_2,x_3,x_4\}| = 4$. If digit i_1 or i_3 corresponds to three identifying codes, say $i_2 = i_3 = i_4$, then there are $\binom{k-1}{2}\binom{2}{1}\binom{n}{3}\binom{n-3}{1} = 2\binom{k-1}{2}\binom{n}{3}\binom{n-3}{1}$ 4-subgraph combinations with $P\left(\bigcap_{j=1}^4 A_{n,k}^{(i_j:x_j)}\right) = p^{\frac{4(n-1)!-3(n-2)!}{(n-k)!}}$ (see Figure 3(b): (4.2)-type). If both digits i_1 and i_3 correspond to two identifying codes, say $i_1 = i_2$ and $i_3 = i_4$, then there are $\binom{k-1}{2}\binom{n}{2}\binom{n-2}{2}$ 4-subgraph combinations with $P\left(\bigcap_{j=1}^4 A_{n,k}^{(i_j:x_j)}\right) = p^{\frac{4(n-1)!-4(n-2)!}{(n-k)!}}$ (see Figure 3(c): (4.6)-type).

Subcase 2.2: $|\{x_1,x_2,x_3,x_4\}| = 3$. If digit i_1 or i_3 corresponds to three identifying codes, say $i_2 = i_3 = i_4$, then there are $\binom{k-1}{2}\binom{2}{1}\binom{n}{3}\binom{3}{1} = 6\binom{k-1}{2}\binom{n}{3}$ 4-subgraph combinations with $P\left(\bigcap_{j=1}^4 A_{n,k}^{(i_j:x_j)}\right) = p^{\frac{4(n-1)!-2(n-2)!}{(n-k)!}}$ (see Figure 3(d): (4.4)-type). If both digits i_1 and i_3 correspond to two identifying codes, say $i_1 = i_2$ and $i_3 = i_4$, then there are $\binom{k-1}{2}\binom{n}{3}\binom{3}{1} \times 2! = 6\binom{k-1}{2}\binom{n}{3}$ 4-subgraph combinations with $P\left(\bigcap_{j=1}^4 A_{n,k}^{(i_j:x_j)}\right) = p^{\frac{4(n-1)!-3(n-2)!}{(n-k)!}}$ (see Figure 3(e): (4.3)-type).

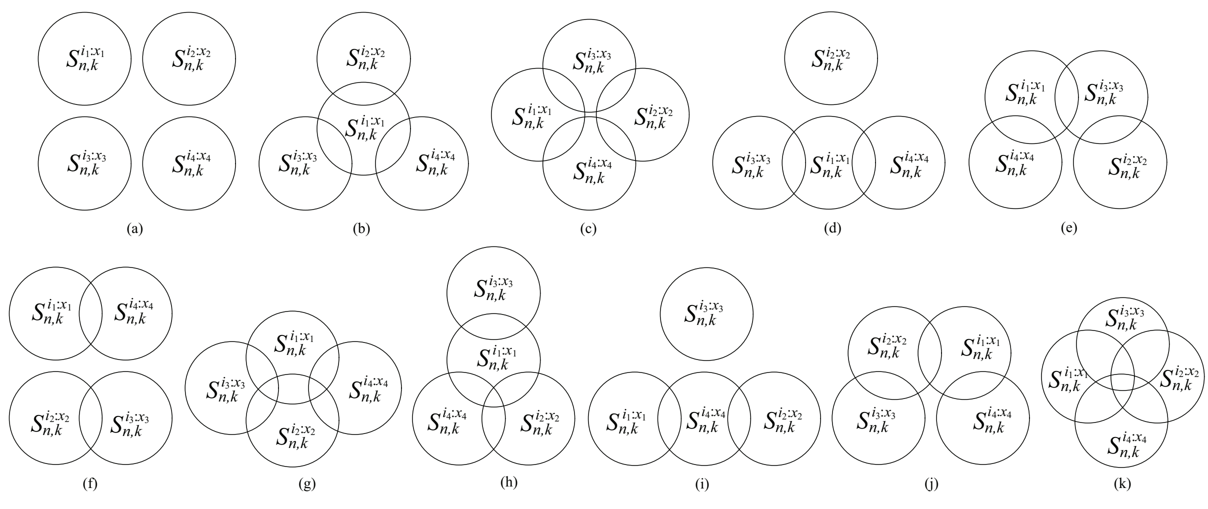


FIGURE 3. Union of $S_{n-1,k-1}$-subgraphs $S_{n,k}^{i_1:x_1}$, $S_{n,k}^{i_2:x_2}$, $S_{n,k}^{i_3:x_3}$, and $S_{n,k}^{i_4:x_4}$

Subcase 2.3: $|\{x_1,x_2,x_3,x_4\}|=2$. Without loss of generality, we assume $i_1=i_2$ and $i_3=i_4$ and $x_1=x_3$ and $x_2=x_4$. Thus, there are $\binom{k-1}{2}\binom{n}{2}$ 4-subgraph combinations with $P\left(\bigcap_{j=1}^{4}A_{n,k}^{(i_j:x_j)}\right)=p^{\frac{4(n-1)!-2(n-2)!}{(n-k)!}}$ (see Figure 3(f): (4.5)-type).

Case 3: $|\{i_1,i_2,i_3,i_4\}|=3$. Without loss of generality, we assume $i_3=i_4$. Then, x_3 is different from x_4 so that $S_{n,k}^{i_3:x_3}$ and $S_{n,k}^{i_4:x_4}$ are disjoint. We distinguish the following subcases.

Subcase 3.1: $|\{x_1,x_2,x_3,x_4\}|=4$. Obviously, both $S_{n,k}^{i_1:x_1}$ and $S_{n,k}^{i_2:x_2}$ overlap with the others (see Figure 3(g): (4.7)-type). Thus, $P\left(\bigcap_{j=1}^{4}A_{n,k}^{(i_j:x_j)}\right)=p^{\frac{4(n-1)!-5(n-2)!+2(n-3)!}{(n-k)!}}$, and this subcase corresponds to $\binom{k-1}{3}\binom{3}{1}\binom{n}{4}\binom{4}{2}\times 2!=36\binom{k-1}{3}\binom{n}{4}$ 4-subgraph combinations.

Subcase 3.2: $|\{x_1,x_2,x_3,x_4\}|=3$. If $\{x_1,x_2\}\cap\{x_3,x_4\}=\emptyset$, then there are $9\binom{k-1}{3}\binom{n}{3}$ $=\binom{k-1}{3}\binom{3}{1}\binom{n}{3}\binom{3}{2}$ 4-subgraph combinations with $P\left(\bigcap_{j=1}^{4}A_{n,k}^{(i_j:x_j)}\right)=p^{\frac{4(n-1)!-4(n-2)!}{(n-k)!}}$ (see Figure 3(c): (4.6)-type). On the other hand, if $\{x_1,x_2\}\cap\{x_3,x_4\}\neq\emptyset$, say $x_2=x_3$, then there are $\binom{k-1}{3}\binom{3}{1}\binom{n}{3}\binom{3}{2}\binom{2}{1}\times 2!=36\binom{k-1}{3}\binom{n}{3}$ 4-subgraph combinations with $P\left(\bigcap_{j=1}^{4}A_{n,k}^{(i_j:x_j)}\right)=p^{\frac{4(n-1)!-4(n-2)!+(n-3)!}{(n-k)!}}$ (see Figure 3(h): (4.9)-type).

Subcase 3.3: $|\{x_1,x_2,x_3,x_4\}|=2$. If $x_1=x_2$ (supposedly $x_1=x_2=x_3$), then there are $\binom{k-1}{3}\binom{3}{1}\binom{n}{2}\binom{2}{1}=6\binom{k-1}{3}\binom{n}{2}$ 4-subgraph combinations with $P\left(\bigcap_{j=1}^{4}A_{n,k}^{(i_j:x_j)}\right)=$ $p^{\frac{4(n-1)!-2(n-2)!}{(n-k)!}}$ (see Figure 3(i): (4.4)-type). On the other hand, if $x_1\neq x_2$, supposedly $x_1=x_3$ and $x_2=x_4$, then there are $\binom{k-1}{3}\binom{3}{1}\binom{n}{2}\times 2!=6\binom{k-1}{3}\binom{n}{2}$ 4-subgraph combinations with $P\left(\bigcap_{j=1}^{4}A_{n,k}^{(i_j:x_j)}\right)=p^{\frac{4(n-1)!-3(n-2)!}{(n-k)!}}$ (see Figure 3(j): (4.3)-type).

Case 4: $|\{i_1,i_2,i_3,i_4\}|=4$.

Subcase 4.1: $|\{x_1,x_2,x_3,x_4\}|=4$. Obviously, there are $\binom{k-1}{4}\binom{n}{4}\times 4!=24\binom{k-1}{4}\binom{n}{4}$ 4-subgraph combinations with $P\left(\bigcap_{j=1}^{4}A_{n,k}^{(i_j:x_j)}\right)=p^{\frac{4(n-1)!-6(n-2)!+4(n-3)!-(n-4)!}{(n-k)!}}$ (see Figure 3(k): (4.8)-type).

Subcase 4.2: $|\{x_1,x_2,x_3,x_4\}|=3$. Without loss of generality, suppose that $x_3=x_4$. Thus, there are $\binom{k-1}{4}\binom{n}{3}\binom{3}{1}\binom{4}{2}\times 2!=36\binom{k-1}{4}\binom{n}{3}$ 4-subgraph combinations with $P\left(\bigcap_{j=1}^{4}A_{n,k}^{(i_j:x_j)}\right)=p^{\frac{4(n-1)!-5(n-2)!+2(n-3)!}{(n-k)!}}$ (see Figure 3(g): (4.7)-type).

Subcase 4.3: $|\{x_1,x_2,x_3,x_4\}|=2$. Suppose that $\{x_1,x_2,x_3,x_4\}=\{x_1,x_3\}$. If x_1 or x_3 is associated with three digits, supposedly $x_2=x_3=x_4$, then there are $\binom{k-1}{4}\binom{n}{2}\left[\binom{4}{1}+\binom{4}{3}\right]$ $=8\binom{k-1}{4}\binom{n}{2}$ 4-subgraph combinations with $P\left(\bigcap_{j=1}^{4}A_{n,k}^{(i_j:x_j)}\right)=p^{\frac{4(n-1)!-3(n-2)!}{(n-k)!}}$ (see Figure 3(b): (4.2)-type). If both x_1 and x_3 are associated with two digits, supposedly $x_1=x_2$ and $x_3=x_4$, then there are $\binom{k-1}{4}\binom{n}{2}\frac{4!}{2!2!}=6\binom{k-1}{4}\binom{n}{2}$ 4-subgraph combinations with $P\left(\bigcap_{j=1}^{4}A_{n,k}^{(i_j:x_j)}\right)=p^{\frac{4(n-1)!-4(n-2)!}{(n-k)!}}$ (see Figure 3(c): (4.6)-type).

Subcase 4.4: $|\{x_1,x_2,x_3,x_4\}|=1$. Obviously, $S_{n,k}^{i_1:x_1}$, $S_{n,k}^{i_2:x_2}$, $S_{n,k}^{i_3:x_3}$, and $S_{n,k}^{i_4:x_4}$ are mutually disjoint (see Figure 3(a): (4.1)-type). Thus, there are $\binom{k-1}{4}\binom{n}{1}$ 4-subgraph combinations with $P\left(\bigcap_{j=1}^{4}A_{n,k}^{(i_j:x_j)}\right)=p^{\frac{4(n-1)!}{(n-k)!}}$.

To conclude, the nine intersection types of all 4-subgraph combinations have been completely addressed in the above analytic formulation. □

3.1. **Upper and lower bounds on $R_{n,k}^{n-1,k-1}(p)$.** According to Equation (1), an upper bound on the $R_{n,k}^{n-1,k-1}(p)$ can be formulated by taking account of only the joint probability of at most three probabilistic events. That is,

$$R_{n,k}^{n-1,k-1}(p) \leq \sum_{i=1}^{n(k-1)} P(O_i) - \sum_{1\leq i<j\leq n(k-1)} P(O_i \cap O_j) + \sum_{1\leq i<j<l\leq n(k-1)} P(O_i \cap O_j \cap O_l) \tag{3}$$

Denote by $\Omega_{n,k}(p)$ the right-hand side of the above Inequality (3). As expressed in Theorem 3.1, Li et al. [11] gave an explicit formula for $\Omega_{n,k}(p)$.

Theorem 3.1. [11] *Given a homogeneous vertex reliability* p *of* $S_{n,k}$*, the analytic formula of the upper bound* $\Omega_{n,k}(p)$ *on* $R_{n,k}^{n-1,k-1}(p)$ *is given as follows*[1]*:*

$$\begin{aligned}
&\Omega_{n,k}(p) \\
&= n(k-1)p^{\frac{(n-1)!}{(n-k)!}} - \left[(k-1)\binom{n}{2} + n\binom{k-1}{2}\right] p^{\frac{2(n-1)!}{(n-k)!}} - 2\binom{n}{2}\binom{k-1}{2} p^{\frac{2(n-1)!-(n-2)!}{(n-k)!}} \\
&\quad + \left[(k-1)\binom{n}{3} + n\binom{k-1}{3}\right] p^{\frac{3(n-1)!}{(n-k)!}} + 6\binom{n}{3}\binom{k-1}{3} p^{\frac{3(n-1)!-3(n-2)!+(n-3)!}{(n-k)!}} \\
&\quad + 4\binom{n}{2}\binom{k-1}{2} p^{\frac{3(n-1)!-(n-2)!}{(n-k)!}} + (2n+2k-10)\binom{n}{2}\binom{k-1}{2} p^{\frac{3(n-1)!-2(n-2)!}{(n-k)!}}
\end{aligned} \tag{4}$$

A lower bound on $R_{n,k}^{n-1,k-1}(p)$ follows directly from Theorem 3.1 and Lemma 3.1.

Theorem 3.2. *Given a homogeneous vertex reliability* p *of* $S_{n,k}$*, a lower bound on* $R_{n,k}^{n-1,k-1}(p)$ *is as follows:*

$$R_{n,k}^{n-1,k-1}(p) \geq \Omega_{n,k}(p) - \phi_{n,k}(p), \tag{5}$$

where $\phi_{n,k}(p)$ *has been derived in Lemma 3.1.*

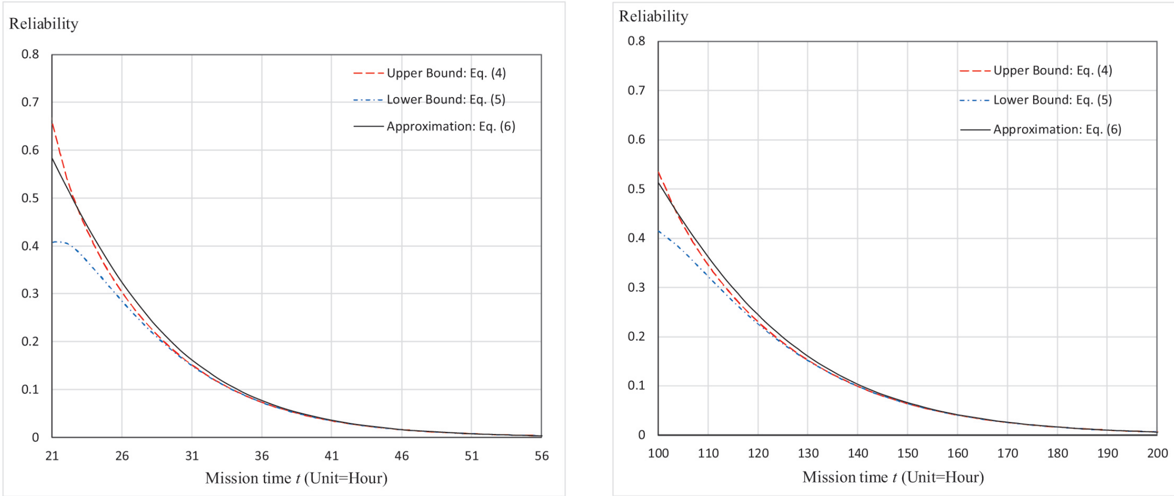
3.2. **Approximation of $\boldsymbol{R_{n,k}^{n-1,k-1}(p)}$.** Simply considering only the collection of disjoint $S_{n-1,k-1}$-subgraphs along each dimension, Li et al. [11] gave the following approximative expression of $R_{n,k}^{n-1,k-1}(p)$:

$$R_{n,k}^{n-1,k-1}(p) \approx 1 - \left(1 - p^{\frac{(n-1)!}{(n-k)!}}\right)^{n(k-1)} \tag{6}$$

4. **Numerical Comparison.** In this section, the comparison between the lower and upper bounds validates the numerical evaluation of $R_{n,k}^{n-1,k-1}(p)$, where p is a given homogeneous vertex reliability ranging from 0 to 1. In practice, the vertex reliability p may vary over time, and the number of faulty vertices accumulates as time passes by. Following the work of Wu and Latifi [21], the exponential failure model assumes that the expected number of faulty vertices at time t, denoted by $f(t)$, increases with a constant failure rate λ: $f(t) = N \times \left(1 - e^{-\lambda t}\right)$, where N is the total number of vertices. The vertex reliability function at time t, denoted by $p(t)$, is thus $p(t) = 1 - \frac{f(t)}{N} = e^{-\lambda t}$.

Figure 4 updates the numerical results in [11] according to the corrected formulation. The three estimates (upper bound: Equation (4), lower bound: Equation (5), and approximation: Equation (6)) of $R_{n,k}^{n-1,k-1}(p(t))$ are plotted with respect to different settings of parameters n, k, and λ. As you can see, for example, the subsystem reliability of $S_{6,5}$ drops to around 0.3 when the $S_{6,5}$ has been already operational for a period of 26 hours, provided that the homogeneous vertex reliability function is $p(t) = e^{-0.001321t}$. Analogously, the subsystem reliability of $S_{10,9}$ degrades to around 0.3 after $S_{10,9}$ has been operational for a period of 113 hours, provided that the homogeneous vertex reliability function is $p(t) = e^{-1.299\times 10^{-7}t}$. The value of upper bound is higher than the approximate estimate

[1]The second term of the right-hand side appears to be $\left[k\binom{n}{2} + n\binom{k-1}{2}\right] p^{\frac{2(n-1)!}{(n-k)!}}$ in [11], which should be a typo.



(a) $n = 6$, $k = 5$, $\lambda = 0.001321$ (b) $n = 10$, $k = 9$, $\lambda = 0.0000001299$

FIGURE 4. Approximation and lower/upper bounds on the values of $R_{n,k}^{n-1,k-1}(p(t))$

when p is at a higher level. However, the value of upper bound becomes lower than the approximate estimate as p drops off. Most of the time, it is evident that both lower and upper bounds are more accurate than Equation (6).

As time passes, the three formulas coincide with each other, and these three estimates eventually converge toward an identical level. It is implied that the estimation errors of both lower and upper bounds approach to 0. Apparently, the upper and lower bounds quickly approach the true reliability because the effect of those neglected terms in Equation (1) vanishes gradually when p goes smaller and smaller.

5. **Conclusions.** In this paper, the correct formula of lower bound on the subsystem reliability of (n, k)-star graphs is established, and numerical results are presented to validate the corrected analytical formulation. According to these results, the gap between lower and upper bounds remains big when vertex reliability p stays at a higher level. Thus, our future work will be devoted to a study on more precise estimations for reducing the difference between the lower and upper bounds of the subsystem reliability.

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