UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS: Part III Physics — NST3PHY
MASTER OF ADVANCED STUDY IN PHYSICS — MAPY

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MAJOR TOPICS
Paper 1/TQM (Theories of Quantum Matter)

Answer **two** *questions only.* $\hbar = 1$ ***throughout this paper***.

The approximate number of marks allocated to each part of a question is indicated in the right-hand margin where appropriate.

The paper contains **??** *sides including this one and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.*
You should use a **separate Answer Book** *for each question.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2 × 20-page answer books	Mathematical formulae handbook
Rough workpad	Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

Draft: 25th January 2023

1 The Hamiltonian of the spin-s Heisenberg ferromagnetic spin chain is

$$H = -J \sum_{j=1}^{N} \mathbf{s}_j \cdot \mathbf{s}_{j+1},$$

where $J > 0$ and $\mathbf{s}_j = (s_j^x, s_j^y, s_j^z)$ obey the usual angular momentum commutation relations and we assume periodic boundary conditions: $\mathbf{s}_{N+1} = \mathbf{s}_1$.

(a) The Holstein–Primakoff representation is

$$\begin{aligned} s_j^+ &= \sqrt{2s}\left(1 - \frac{a_j^\dagger a_j}{2s}\right)^{1/2} a_j \\ s_j^- &= \sqrt{2s}a_j^\dagger\left(1 - \frac{a_j^\dagger a_j}{2s}\right)^{1/2} \\ s_j^z &= \left(s - a_j^\dagger a_j\right). \end{aligned}$$

where $s_j^\pm = s_j^x \pm i s_j^y$ and $a_j^\dagger, a_j$ satisfy the standard bosonic commutation relations $\left[a_j, a_k^\dagger\right] = \delta_{jk}$. By approximating H by a Hamiltonian quadratic in the $a_j^\dagger, a_j$, show that the operators

$$a_\eta = \frac{1}{\sqrt{N}} \sum_j a_j \exp\left(-i\eta_n j\right),$$

diagonalize this Hamiltonian, where $\eta_n = 2\pi n/N$ with $n \le (N-1)/2$ assuming N odd. Find the dispersion $\omega(\eta)$ of the (magnon) excitations they describe. [8]

(b) Find the contribution to the Hamiltonian that is *quartic* in $a_j^\dagger, a_j$. [4]

(c) Show that in terms of $a_\eta^\dagger, a_\eta$ the quartic Hamiltonian is

$$H_4 = \sum_{\substack{\eta_1,\eta_2,\eta_3,\eta_4 \\ \eta_1+\eta_2=\eta_3+\eta_4}} U(\eta_1,\eta_2,\eta_3,\eta_4) a_{\eta_1}^\dagger a_{\eta_2}^\dagger a_{\eta_3} a_{\eta_4}.$$

Give the form of $U(\eta_1,\eta_2,\eta_3,\eta_4)$. [8]

(d) At finite temperature the occupation number of magnons has a Bose distribution

$$\langle a_\eta^\dagger a_\eta \rangle = \frac{1}{\exp(\beta\omega(\eta)) - 1}, \qquad \eta \neq 0$$

where $\beta = 1/k_B T$ is the inverse temperature (the $\eta = 0$ mode is unoccupied). Find an expression for $\langle H_4 \rangle$ in the $N \to \infty$ limit for the Heisenberg chain expressed as a multiple integral over the Brillouin zone. Find the temperature dependence of this contribution to the energy in the limit $\beta J \ll 1$. [10]

Solution 1

(a) **[Bookwork]** The lowest order approximation (valid at large s) coming from the HP representation is

$$s_j^+ = \sqrt{2s}a \qquad s_j^- = \sqrt{2s}a^\dagger$$
$$s_j^z = \left(s - a^\dagger a\right).$$

Write the Hamiltonian in the form

$$H = J\sum_j \left[s_j^z s_{j+1}^z + \frac{1}{2}\left(s_j^+ s_{j+1}^- + s_{j+1}^+ s_j^-\right)\right].$$

Substituting in the HP representation gives

$$H = J\sum_j \left[(s - a_j^\dagger a_j)(s - a_{j+1}^\dagger a_{j+1})\right.$$
$$\left. +s\left(1 - \frac{a_j^\dagger a_j}{2s}\right)^{1/2} a_j a_{j+1}^\dagger \left(1 - \frac{a_{j+1}^\dagger a_{j+1}}{2s}\right)^{1/2} + s\left(1 - \frac{a_{j+1}^\dagger a_{j+1}}{2s}\right)^{1/2} a_{j+1} a_j^\dagger \left(1 - \frac{a_j^\dagger a_j}{2s}\right)^{1/2}\right] \tag{1}$$

The lowest order approximation corresonds to taking the square roots to be 1. This yields the zeroth order term $-NJs^2$ (not required) and the quadratic Hamiltonian

$$H_2 = 2Js\sum_j \left[a_j^\dagger a_j^\dagger - \frac{1}{2}\left(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j\right)\right].$$

Substitute in the given mode expansion and sum over j, giving

$$H_2 = \sum_\eta \omega(\eta) a_\eta^\dagger a_\eta$$

where $\omega(\eta) = 4Js\sin^2(\eta/2)$.

(b) Continuing the expansion of the HP representation by expanding the square roots

$$\left(1 - \frac{a_j^\dagger a_j}{2s}\right)^{1/2} \sim 1 - \frac{a_j^\dagger a_j}{4s}$$

gives the quartic Hamiltonian

$$H_4 = J\sum_j \left[-a_j^\dagger a_j a_{j+1}^\dagger a_{j+1} + \frac{1}{4}\left(a_j^\dagger a_j a_j a_{j+1}^\dagger + a_j a_{j+1}^\dagger a_{j+1}^\dagger a_{j+1} + \text{h.c.}\right)\right].$$

(c) Substituting in the mode expansion

$$H_4 = \frac{J}{N^2} \sum_{j,\eta_1,\eta_2,\eta_3,\eta_4} a^\dagger_{\eta_1} a^\dagger_{\eta_2} a_{\eta_3} a_{\eta_4} \left[-e^{i(\eta_1-\eta_3)} + \frac{1}{4}\left(e^{i\eta_1} + e^{i\eta_3} + \text{h.c.}\right)\right] e^{i(\eta_1+\eta_2-\eta_3-\eta_4)j}.$$

Summing over sites gives

$$H_4 = \sum_{\substack{\eta_1,\eta_2,\eta_3,\eta_4 \\ \eta_1+\eta_2=\eta_3+\eta_4}} U(\eta_1,\eta_2,\eta_3,\eta_4) a^\dagger_{\eta_1} a^\dagger_{\eta_2} a_{\eta_3} a_{\eta_4}$$

with

$$U(\eta_1,\eta_2,\eta_3,\eta_4) = \frac{J}{N}\left[-\cos(\eta_1-\eta_3) + \frac{1}{4}\left(\cos(\eta_1)+\cos(\eta_2)+\cos(\eta_3)+\cos(\eta_4)\right)\right].$$

[There are many equivalent ways to write this because of the symmetry of the operators]

(d) Take the expectation value using

$$\langle a^\dagger_{\eta_1} a^\dagger_{\eta_2} a_{\eta_3} a_{\eta_4}\rangle = \begin{cases} \langle a^\dagger_{\eta_2} a_{\eta_2}\rangle \langle a^\dagger_{\eta_3} a_{\eta_3}\rangle & \text{if } \eta_1 = \eta_3, \eta_2 = \eta_4 \\ \langle a^\dagger_{\eta_2} a_{\eta_2}\rangle \langle a^\dagger_{\eta_1} a_{\eta_1}\rangle & \text{if } \eta_2 = \eta_3, \eta_1 = \eta_4. \end{cases}$$

This gives the expression

$$\langle H_4\rangle = \sum_{\eta_1,\eta_2} \left[U(\eta_1,\eta_2,\eta_1,\eta_2) + U(\eta_1,\eta_2,\eta_2,\eta_1)\right] N(\eta_1)N(\eta_2).$$

In the continuum limit the resulting integral is

$$\langle H_4\rangle = \left(\frac{N}{2\pi}\right)^2 \int_{\eta_1,\eta_2\in \text{BZ}} \left[U(\eta_1,\eta_2,\eta_1,\eta_2) + U(\eta_1,\eta_2,\eta_2,\eta_1)\right] N(\eta_1)N(\eta_2).$$

The question asked for the $\beta J \ll 1$ limit, from which it's simple to get $\langle H_4\rangle \propto T^2$. However, I actually meant to ask about the *low temperature* limit, in which we will only have significant contribution from small η, so we can expand U. The lowest nonvanishing order gives $[-Js\eta_1\eta_2]$, but leads to a separable integral that vanishes. So the first order that survives (by virtue of being *even* in η_1 and η_2) is *quartic* in η.

After making the substitution $u_{1,2} = \sqrt{\beta J s}\eta_{1,2}$ to scale the temperature out of the Bose distribution, we arrive at the overall dependence

$$\langle H_4\rangle = JN(\beta J s)^{-3} \times \text{numerical factor} \propto \frac{N(k_B T)^3}{J^2 s^3}$$

2 A gas of spin 1 bosons of mass M is described by the Hamiltonian

$$H = \int d\mathbf{r}\left[\sum_{m=-1}^{+1}\left(\frac{\nabla\psi_m^\dagger\cdot\nabla\psi_m}{2M} + (q|m|-\mu)\psi_m^\dagger\psi_m\right) + \frac{c_0}{2} : \rho(\mathbf{r})^2 : + \frac{c_2}{2} : \mathbf{S}(\mathbf{r})^2 :\right].$$

The three fields $\psi_m(\mathbf{r})$ ($m = -1, 0, +1$) satisfy the usual bosonic commutation relations. q is an energy shift of the $m = \pm 1$ bosons relative to the $m = 0$ boson, and μ is the chemical potential. c_0 and c_2 are interaction constants. For operators A, B we use the notation $: AB :$ to indicate that the operator product AB is normal ordered. The density $\rho(\mathbf{r})$ and spin density $\mathbf{S}(\mathbf{r})$ are defined as

$$\rho(\mathbf{r}) = \sum_{m=-1}^{1}\psi_m^\dagger(\mathbf{r})\psi_m(\mathbf{r}), \qquad \mathbf{S}(\mathbf{r}) = \sum_{m,m'=-1}^{1}\psi_m^\dagger(\mathbf{r})\mathbf{s}_{mm'}\psi_{m'}(\mathbf{r})$$

where the spin 1 matrices $\mathbf{s} = (s_x, s_y, s_z)$ are

$$s^x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad s^y = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad s^z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(a) In a cubic box of side L with periodic boundary conditions the field operators have the expansion

$$\psi_m(\mathbf{r}) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}}\psi_{\mathbf{k},m}\exp\left(-i\mathbf{k}\cdot\mathbf{r}\right), \qquad \mathbf{k} = 2\pi\left(\frac{n_x}{L}, \frac{n_y}{L}, \frac{n_z}{L}\right), \quad n_{x,y,z} \text{ integer.}$$

Assume that all N bosons start in the $m = 0$ state. Show that in the Bogoliubov approximation the $m = \pm 1$ bosons are described by the Hamiltonian

$$H_{\mathrm{B}} = \sum_{\mathbf{k}}\left[\sum_{m=\pm 1}\psi_{\mathbf{k},m}^\dagger\left[\varepsilon_k + q + c_2 n\right]\psi_{\mathbf{k},m} + c_2 n\left(\psi_{\mathbf{k},1}^\dagger\psi_{-\mathbf{k},-1}^\dagger + \psi_{\mathbf{k},1}\psi_{-\mathbf{k},-1}\right)\right],$$

where $n = N/L^3$ is the density, and $\varepsilon_{\mathbf{k}} = \frac{\mathbf{k}^2}{2M}$ is the free particle dispersion. [10]

(b) Show that H_{B} can be expressed as a system of free bosons by the transformation

$$a_k = \cosh\kappa_{\mathbf{k}}\,\psi_{k,1} - \sinh\kappa_{\mathbf{k}}\,\psi_{-k,-1}^\dagger$$
$$b_k = \cosh\kappa_{\mathbf{k}}\,\psi_{k,-1} - \sinh\kappa_{\mathbf{k}}\,\psi_{-k,1}^\dagger$$

where you should find $\kappa_{\mathbf{k}}$ and show that the dispersion relation of the bosons is

$$\omega(\mathbf{k}) = \sqrt{(\varepsilon_{\mathbf{k}} + q)^2 + 2c_2 n(\varepsilon_{\mathbf{k}} + q)}.$$

[8]

(c) Discuss the stability of the Bogoliubov excitations for different parameter regimes of q and c_2. [4]

(d) Find the Fourier component of transverse spin density $S^{\perp}(\mathbf{r}) = S^{x}(\mathbf{r}) + iS^{y}(\mathbf{r})$ in the Bogoliubov approximation and evaluate the correlation function $\langle S^{\perp}_{\mathbf{q}} \left(S^{\perp}_{\mathbf{q}}\right)^{\dagger} \rangle$ in the Bogoliubov ground state. [8]

Solution 2

(a) This is a variant on lecture material. First express the Hamiltonian in terms of plane wave states

$$H = \sum_{\mathbf{k},m} \left[\varepsilon(\mathbf{k}) + (q|m| - \mu)\right] a^\dagger_{\mathbf{k},m} a_{\mathbf{k},m}$$
$$+ \frac{1}{2V} \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4} \left[c_0 \sum_{m,m'} a^\dagger_{\mathbf{k}_1,m} a^\dagger_{\mathbf{k}_2,m'} a_{\mathbf{k}_3,m'} a_{\mathbf{k}_4,m} + c_2 \sum_{m_1,m_2,m_3,m_4} \mathbf{s}_{m_2 m_3} \cdot \mathbf{s}_{m_1 m_4} a^\dagger_{\mathbf{k}_1,m_1} a^\dagger_{\mathbf{k}_2,m_2} a_{\mathbf{k}_3,m_3} a_{\mathbf{k}_4,m_4} \right],$$

After the substitution $a^\dagger_{0,0} = a_{0,0} = \sqrt{N}$ the key realizations are:

1. Only the transverse part of the spin-spin interaction contributes because the $m = 0$ state is occupied
2. The chemical potential cancels the contribution of the density-density interaction c_0

(b) After substitution of the transformation into the Hamiltonian the anomalous terms are cancelled by the choice

$$\tanh 2\kappa_k = -\frac{c_2 n}{\varepsilon_k + q + c_2 n}. \tag{2}$$

For future reference

$$\sinh 2\kappa_k = \frac{|c_2|n}{\omega(k)}, \qquad \cosh 2\kappa_k = \frac{\varepsilon_k + q - |c_2|n}{\omega(k)}. \tag{3}$$

The transformed Hamiltonian is

$$H_\mathrm{B} = \sum_k \omega(\mathbf{k}) \left[a^\dagger_\mathbf{k} a_\mathbf{k} + b^\dagger_\mathbf{k} b_\mathbf{k}\right], \tag{4}$$

with

$$\omega(\mathbf{k}) = \sqrt{(\varepsilon_\mathbf{k} + q)^2 + 2c_2 n(\varepsilon_\mathbf{k} + q)},$$

(c) Unstable modes correspond to imaginary $\omega(\mathbf{k})$. This occurs when only one of the factors $(\varepsilon_\mathbf{k} + q)$ and $(\varepsilon_\mathbf{k} + q + 2c_2 n)$ is negative. Thus we can have

1. $q < 0$, $2c_2 n > 0$, $|q| - 2c_2 n < \varepsilon_\mathbf{k} < |q|$ or
2. $q > 0$, $2c_2 n < -q$, $\varepsilon_\mathbf{k} < q + 2c_2 n$

(d) The Fourier components of the spin density are

$$\mathbf{S}_\mathbf{q} = \sum_{\mathbf{p},m,m'} \psi^\dagger_{\mathbf{p}-\mathbf{q}/2,m} \mathbf{s}_{mm'} \psi_{\mathbf{p}+\mathbf{q}/2,m} \tag{5}$$

The Bogoliubov approxmation amounts to retaining only terms with one $\psi^\dagger_{0,0}$ or $\psi_{0,0}$, with both set to $\sqrt{N}$. This gives

$$
\begin{aligned}
S_{x,q} &= \sqrt{\frac{N}{2}}\left[\psi_{1,-q}+\psi_{-1,-q}+\psi^{\dagger}_{1,q}+\psi^{\dagger}_{-1,q}\right] && (6)\\
&= \sqrt{\frac{N}{2}}e^{\kappa_{\mathbf{q}}}\left[\left(a+b^{\dagger}\right)_{-q}+\left(a^{\dagger}+b\right)_{q}\right]\\
S_{x,q} &= \sqrt{\frac{N}{2}}\left[i(\psi_{1,-q}-\psi_{-1,-q})-i(\psi^{\dagger}_{1,q}-\psi^{\dagger}_{-1,q})\right]\\
&= \sqrt{\frac{N}{2}}e^{\kappa_{\mathbf{q}}}\left[i\left(a+b^{\dagger}\right)_{-q}-i\left(a^{\dagger}+b\right)_{q}\right]\\
S_{\perp,q}\equiv\left(S_{x}+iS_{y}\right)_{q} &= \sqrt{2N}e^{\kappa_{\mathbf{q}}}\left(a^{\dagger}+b\right)_{q} && (7)
\end{aligned}
$$

Computing the transverse magnetization correlation function using the property that the ground state is a Bogoliubov vacuum gives

$$
\langle S_{\perp\mathbf{q}}S^{\dagger}_{\perp-\mathbf{q}}\rangle = 2Ne^{2\kappa_{\mathbf{q}}}\langle\left(a^{\dagger}+b\right)_{\mathbf{q}}\left(a+b^{\dagger}\right)_{-\mathbf{q}}\rangle = 2Ne^{2\kappa_{\mathbf{q}}} \tag{8}
$$

3 (a) In a one dimensional gas of fermions the Fourier components of the density operator have the second quantized representation

$$\rho_q = \sum_p a^\dagger_{p-q/2} a_{p+q/2},$$

where $a^\dagger_p\ a_p$ satisfy the canonical anticommutation relations $\{a_p, a^\dagger_{p'}\} = \delta_{pp'}$. Find the static structure factor $\langle \rho_q \rho_{-q} \rangle$ in the ground state of a system of noninteracting fermions. [8]

(b) To describe a one dimensional systems of fermions at low energies, the fermion field operator is written as the sum of contributions from the two Fermi points

$$\psi(x) = e^{ik_\mathrm{F}x}\psi_R(x) + e^{-ik_\mathrm{F}x}\psi_L(x),$$

where the modes contributing to $\psi_{L/R}(x)$ are restricted to a momentum shell of width 2Λ around the Fermi points, with $\Lambda \ll k_\mathrm{F}$:

$$\psi_{R/L}(x) = \sum_{k=-\Lambda}^{\Lambda} a_{k\pm k_\mathrm{F}} e^{ikx}.$$

The Fourier components of the density operator for *right moving* fermions are

$$\rho^R_q = \sum_{p=-\Lambda+|q|/2}^{\Lambda-|q|/2} a^\dagger_{p-q/2+k_\mathrm{F}} a_{p+q/2+k_\mathrm{F}}.$$

Show that

$$\left[\rho^R_q, \rho^R_{-q}\right] = \frac{qL}{2\pi}, \quad |q| \ll \Lambda,$$

assuming states near $k_\mathrm{F} - \Lambda$ are always occupied and those near $k_\mathrm{F} + \Lambda$ are always empty. [10]

(c) Find the commutation relation for the left moving density operators

$$\rho^L_q = \sum_{p=-\Lambda+|q|/2}^{\Lambda-|q|/2} a^\dagger_{p-q/2-k_\mathrm{F}} a_{p+q/2-k_\mathrm{F}},$$

[6]

(d) Use the above commutation relations to find the static structure factor of $\rho_q = \rho^R_q + \rho^L_q$ in a second way. [6]

[*Hint: If* $|0\rangle$ *is the ground state,* $\rho^R_q|0\rangle = 0$ *for* $q > 0$*, while* $\rho^L_q|0\rangle = 0$ *for* $q < 0$]

Solution 3

(a) **[Bookwork]** Substituting the expansion into the expression for the structure factor gives

$$\langle \rho_q \rho_{-q} \rangle = \sum_{p,p'} \langle a^\dagger_{p-q/2} a_{p+q/2} a^\dagger_{p'+q/2} a_{p'-q/2} \rangle \tag{9}$$

In a product state (of which the filled Fermi sea is an example) we get contributions for $q \neq 0$ only when $p = p'$ in which case the anticommutation relations give us

$$\langle \rho_q \rho_{-q} \rangle = \sum_p N_{p-q/2}(1 - N_{p+q/2}). \tag{10}$$

In the ground state the summand is only nonzero for $q > 0$ when $p - q/2$ is below the Fermi sea and $p + q/2$ is above, giving

$$\langle \rho_q \rho_{-q} \rangle = \frac{\min(q, 2k_F)L}{2\pi} \qquad q > 0 \tag{11}$$

Similar reasoning for $q < 0$ leads to the conclusion

$$\langle \rho_q \rho_{-q} \rangle = \frac{\min(|q|, 2k_F)L}{2\pi} \tag{12}$$

(b) We start by computing the following commutator of the density operator for right-movers ('right current')

$$\begin{aligned}
[\rho^R_q, \rho^R_{-q}] &= \sum_{k,k'=-\Lambda+|q/2|}^{\Lambda-|q/2|} [a^\dagger_{k_F+k'-q/2} a_{k_F+k'+q/2}, a^\dagger_{k_F+k+q/2} a_{k_F+k-q/2}] \\
&= \sum_{k=-\Lambda+|q|/2}^{\Lambda-|q/2|} a^\dagger_{k_F+k-q/2} a_{k_F+k-q/2} - a^\dagger_{k_F+k+q/2} a_{k_F+k+q/2} \\
&= \sum_{k=-\Lambda}^{\Lambda-q} a^\dagger_{k_F+k} a_{k_F+k} - \sum_{k=-\Lambda+q}^{\Lambda} a^\dagger_{k_F+k} a_{k_F+k},
\end{aligned}$$

where the second line follows from the canonical *anti*commutation relations, and the third is just a shift of the sum limits; had we not imposed momentum cutoffs, this line would now be seen vanish. Provided $\Lambda \gg q$, the edges of the domains where the sums differ are not too close to the Fermi points. At low energies, we can thus neglect fluctuations around the ground state and just replace the number operators by expectation values

$$\begin{aligned}
[\rho^R_q, \rho^R_{-q}] &= L \int_{-\Lambda}^{-\Lambda+q} \frac{dk}{2\pi}\, \Theta(-k) \;-\; L \int_{\Lambda-q}^{\Lambda} \frac{dk}{2\pi}\, \Theta(-k), \qquad (13) \\
&= L \int_{-\Lambda}^{-\Lambda+q} \frac{dk}{2\pi}\, \Theta(-k) \\
&= \frac{Lq}{2\pi} \qquad (14)
\end{aligned}$$

where we have used the momentum distribution $\langle n_{\mathrm{R}}(k)\rangle = \Theta(-k)$ appropriate for right-movers (i.e. the Fermi sea is empty for $k > k_{\mathrm{F}}$).

(c) The calculation for the left current ρ_q^L is essentially the same, the key difference being that the counterpart of Eq. (**??**) now contains $\Theta(k)$ (instead of $\Theta(-k)$, since the Fermi sea is empty for $k < -k_{\mathrm{F}}$), resulting in a contribution only from the second term in Eq. (**??**). Consequently, the left current commutator has the opposite sign and thus the total current ($\rho(x) = \rho^R(x) + \rho^L(x)$) commutes for all x, x'.

(d) The structure factor is

$$\langle \rho_q \rho_{-q}\rangle = \langle (\rho_q^L + \rho_q^R)(\rho_{-q}^L + \rho_{-q}^R)\rangle. \tag{15}$$

For $q > 0$ we have (using the hint)

$$\langle \rho_q \rho_{-q}\rangle = \langle \rho_q^R \rho_{-q}^R)\rangle = \langle [\rho_q^R, \rho_{-q}^R]\rangle = \frac{qL}{2\pi} \tag{16}$$

For $q < 0$ we get the opposite sign, leading to

$$\langle \rho_q \rho_{-q}\rangle = \frac{|q|L}{2\pi} \tag{17}$$

as before (the calculation applies at small $q \ll \Lambda \ll k_{\mathrm{F}}$).

END OF PAPER