



MAX-PLANCK-GESELLSCHAFT



Exactly-solvable many-body quantum dynamics in biunitary circuits

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Quantum 6, 738 (2022)

Phys. Rev. Res. 4, 043212 (2022)

arXiv:2302.07280 (2023)

In collaboration with



Austen Lamacraft

University of Cambridge



Jamie Vicary

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Quantum 6, 738 (2022)

Phys. Rev. Res. 4, 043212 (2022)

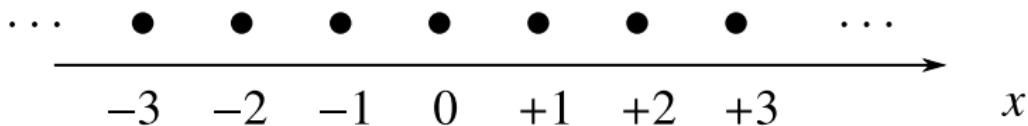
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1. Dual-unitary circuits
2. Shaded calculus
3. Biunitary circuits
4. Solvable states

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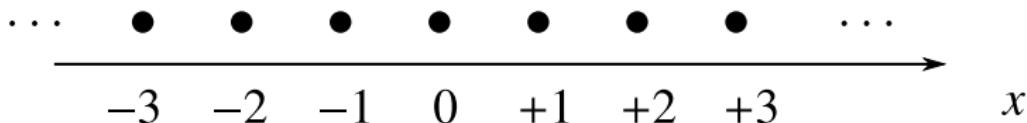
Local unitary dynamics

- Setting: 1+1D lattice models



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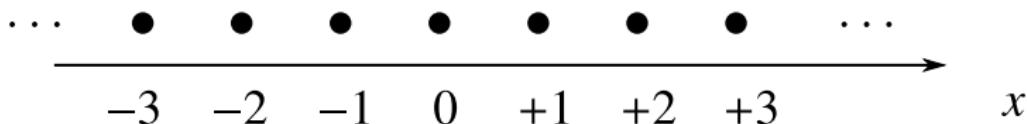


- Hamiltonian evolution with **local** interactions

$$U(t) = \exp [-iHt] \quad \text{with} \quad H = \sum_i h_{i,i+1}$$

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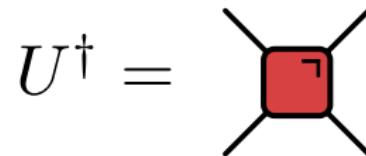
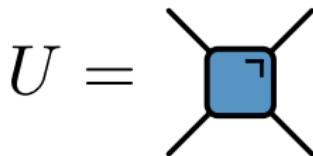
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⇒ Can we find **minimal models** for many-body dynamics?

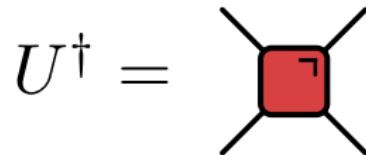
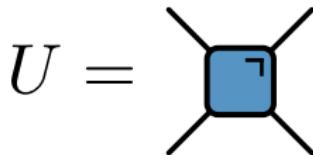
Unitary circuits

- Consider time evolution generated by two-site **unitary circuits**



Unitary circuits

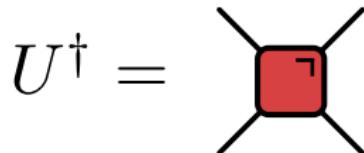
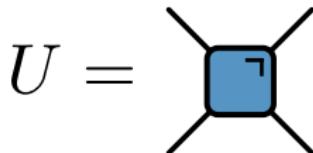
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E.g. $U_{n,n+1} = \exp[-ih_{n,n+1}\Delta t]$

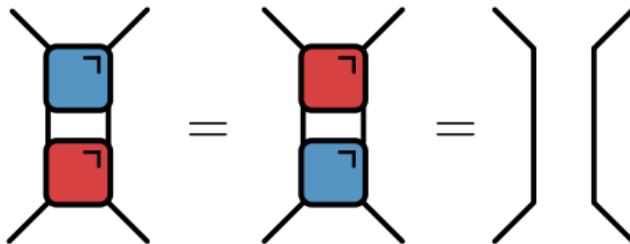
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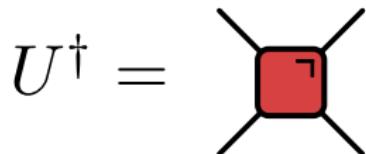
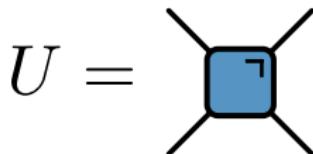
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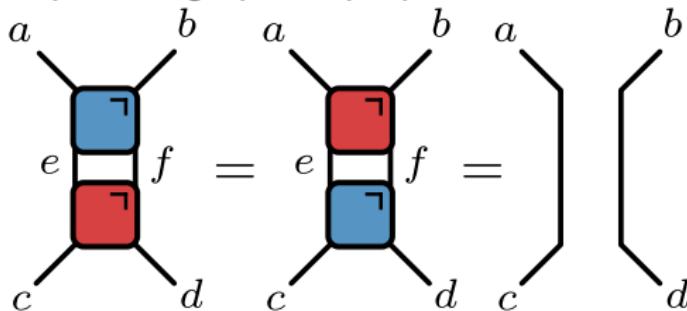
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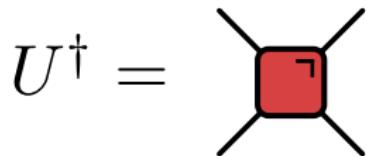
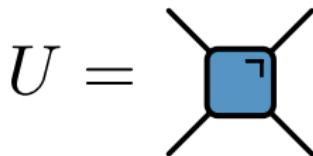
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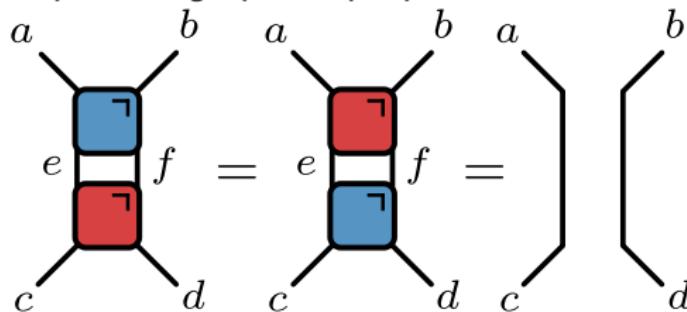
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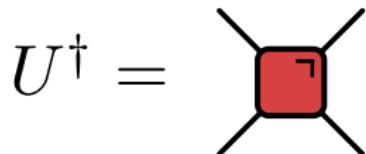
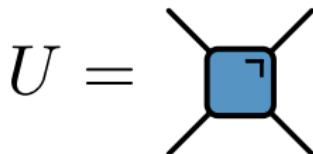
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$$\sum_{ef} U_{ab,ef} U_{ef,cd}^\dagger = \sum_{ef} U_{ab,ef}^\dagger U_{ef,cd} = \delta_{ac} \delta_{bd}$$

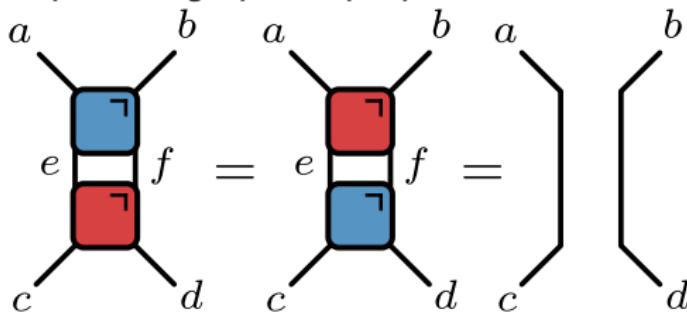
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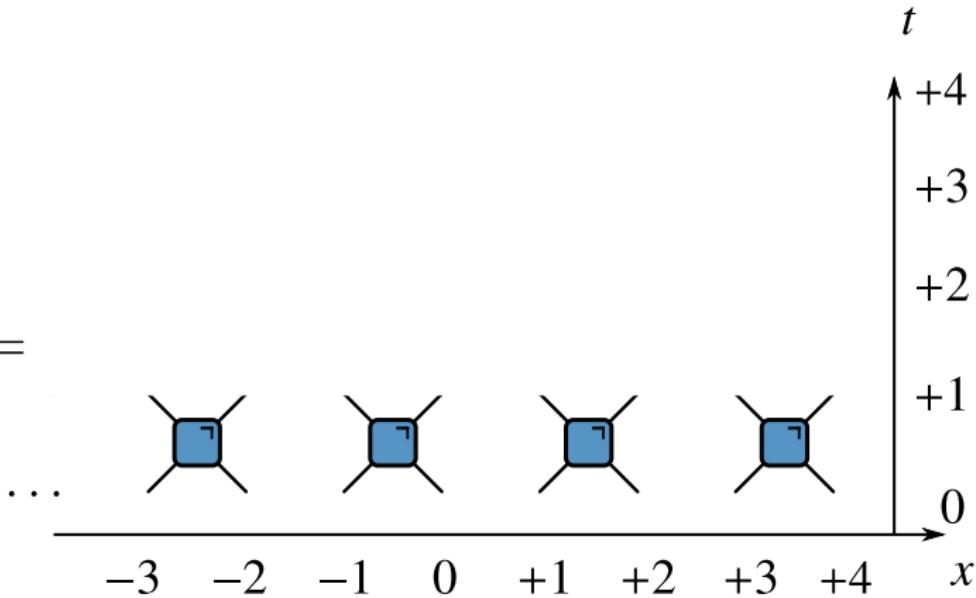


$$UU^\dagger = U^\dagger U = \mathbb{1}$$

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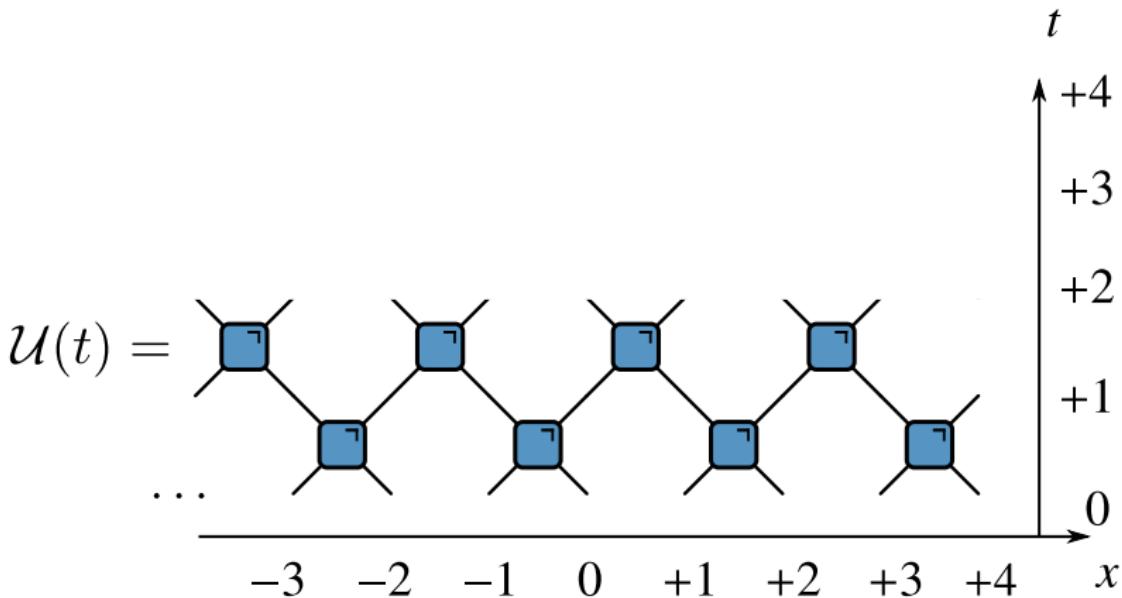
$$\mathcal{U}(t) =$$



- Infinite lattice, discrete time

Unitary circuits

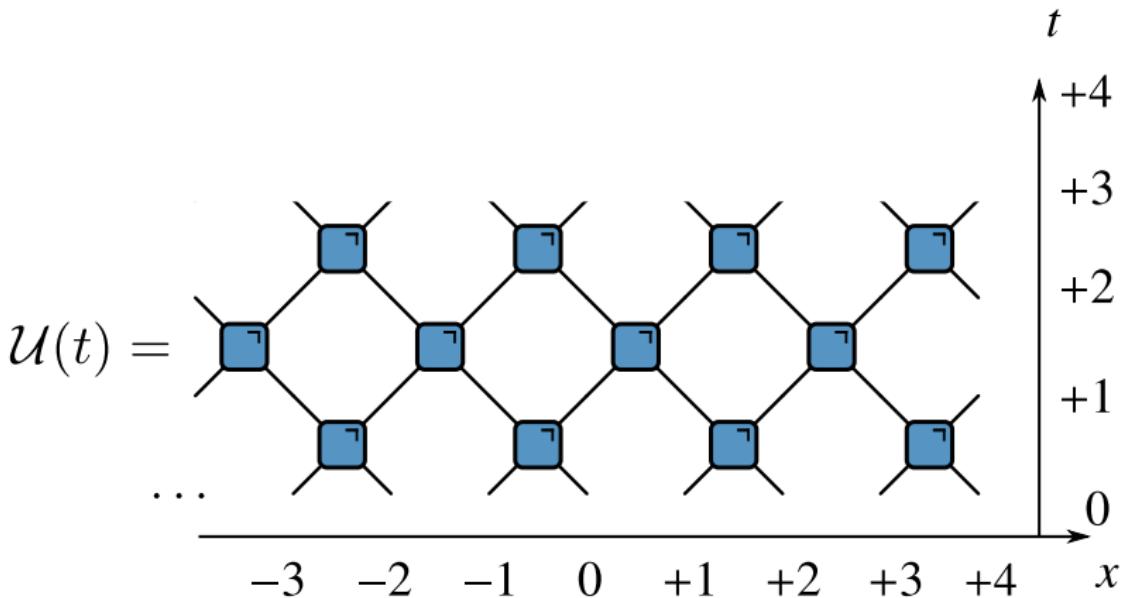
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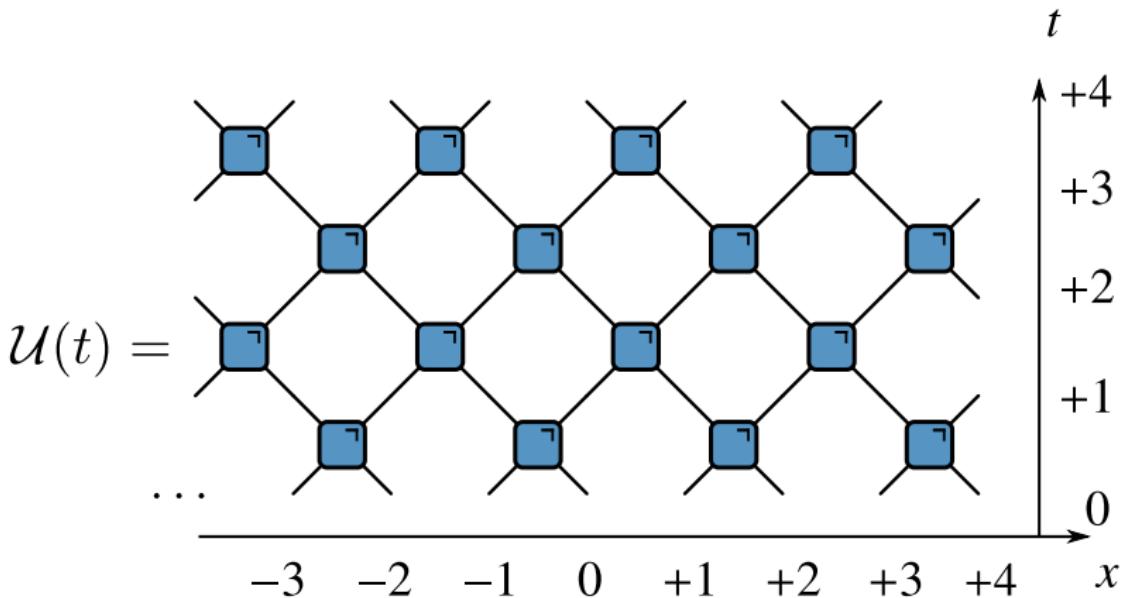
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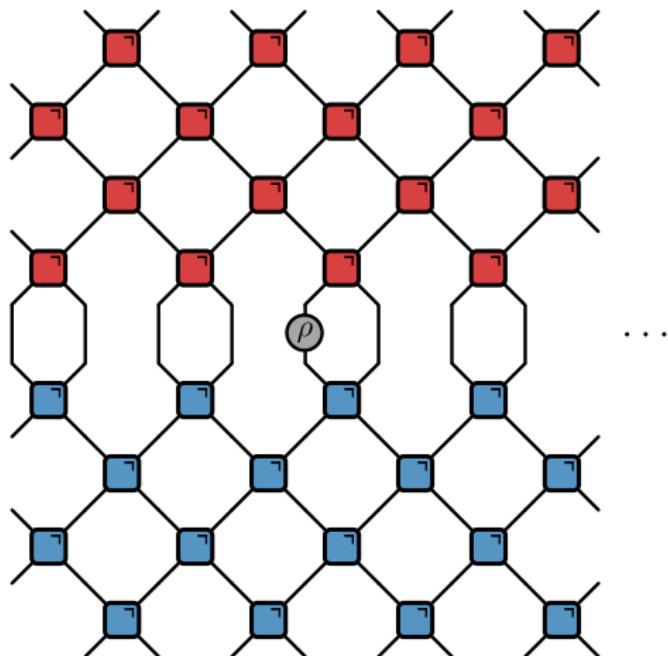


- Infinite lattice, discrete time

Causal light cone

- Light cone baked in!

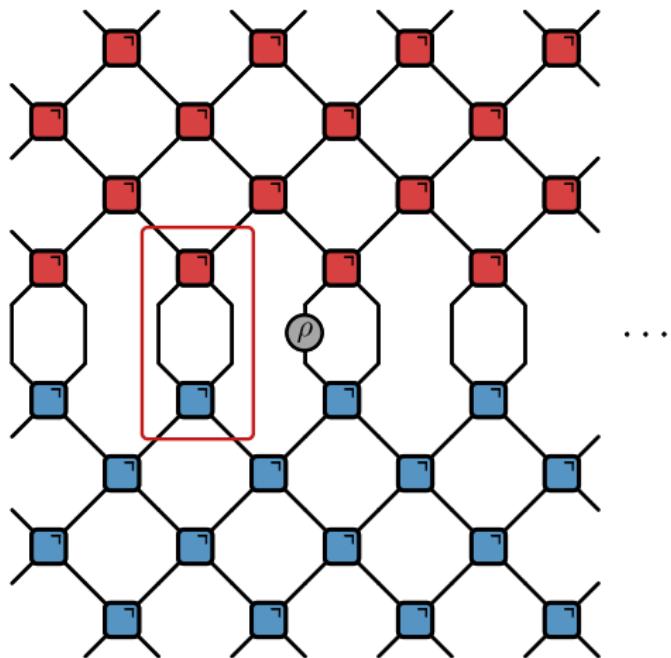
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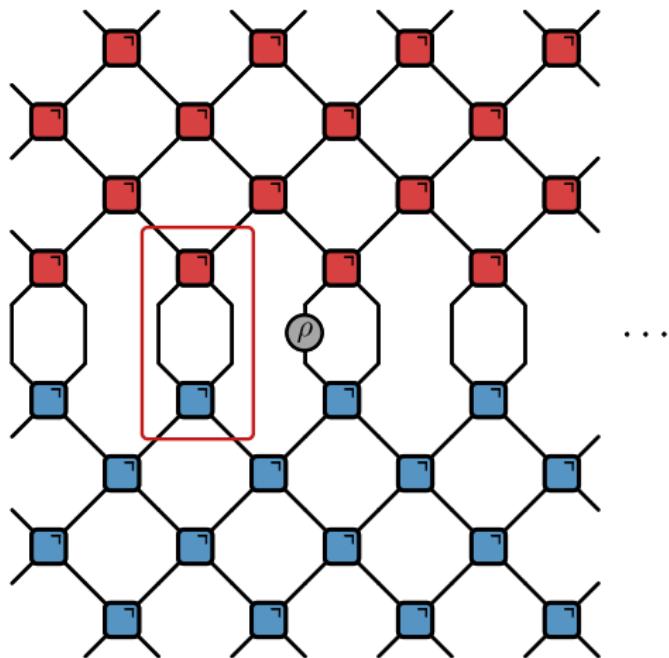
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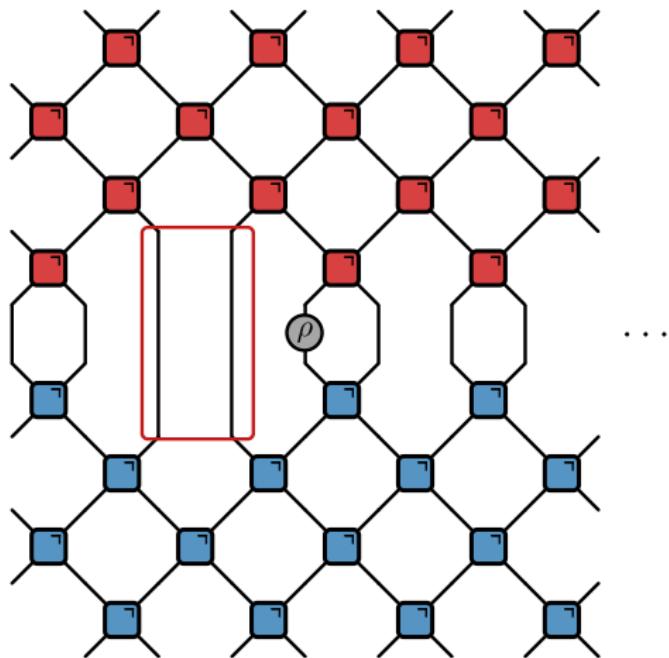
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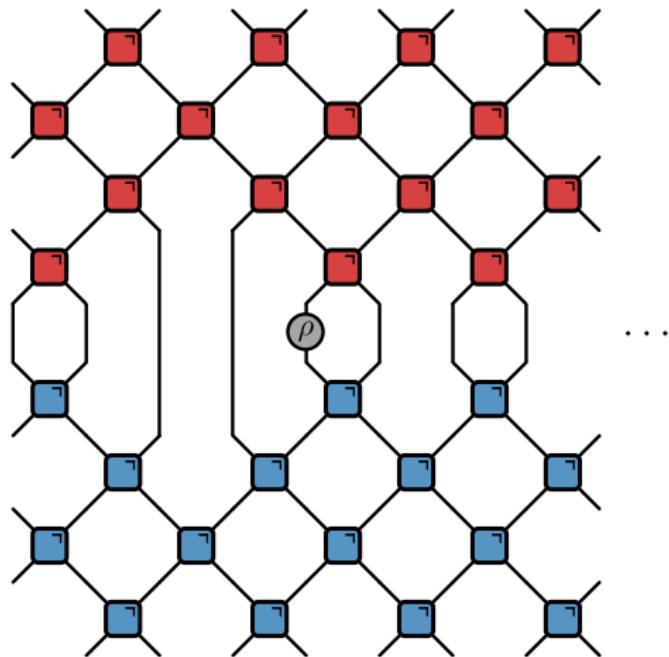
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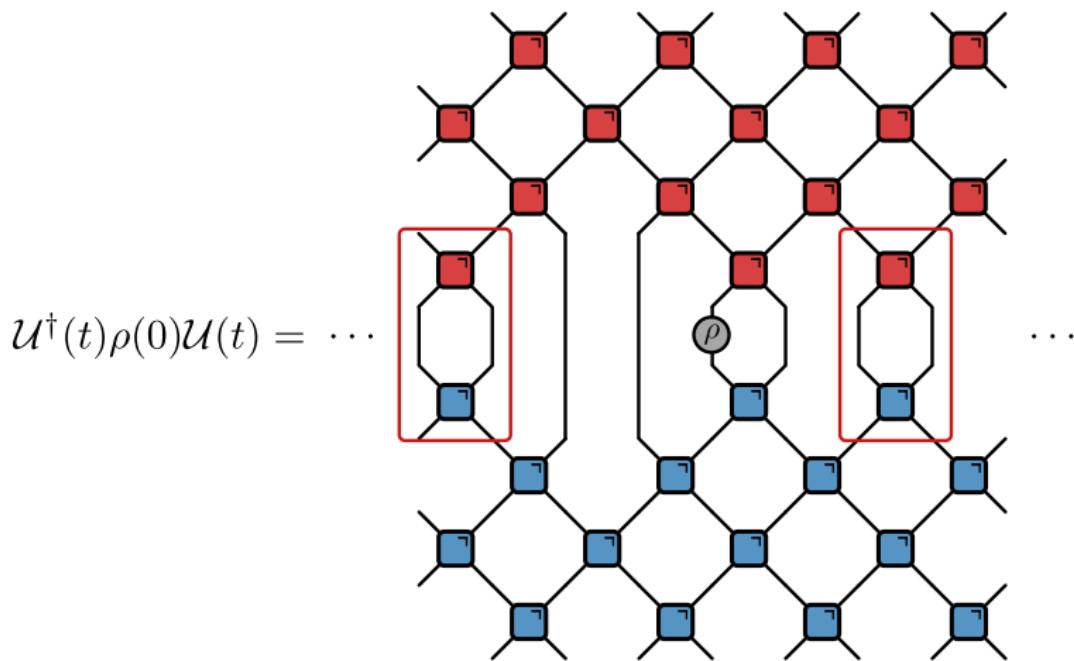
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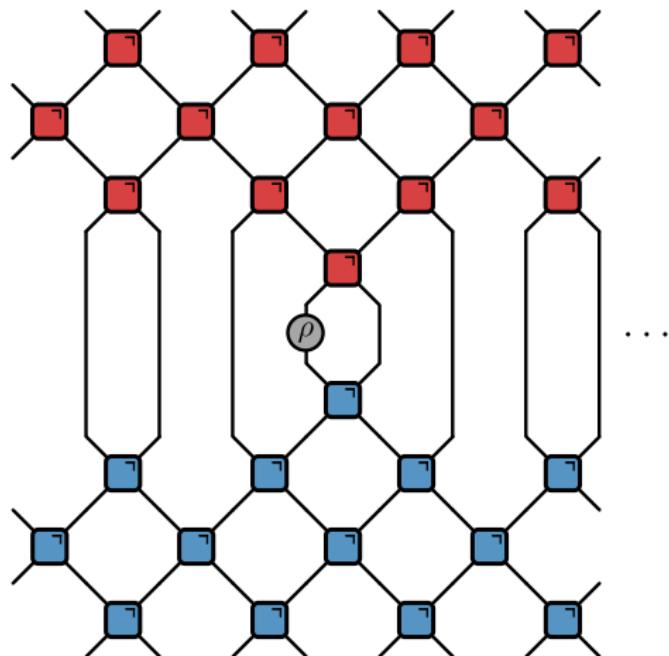
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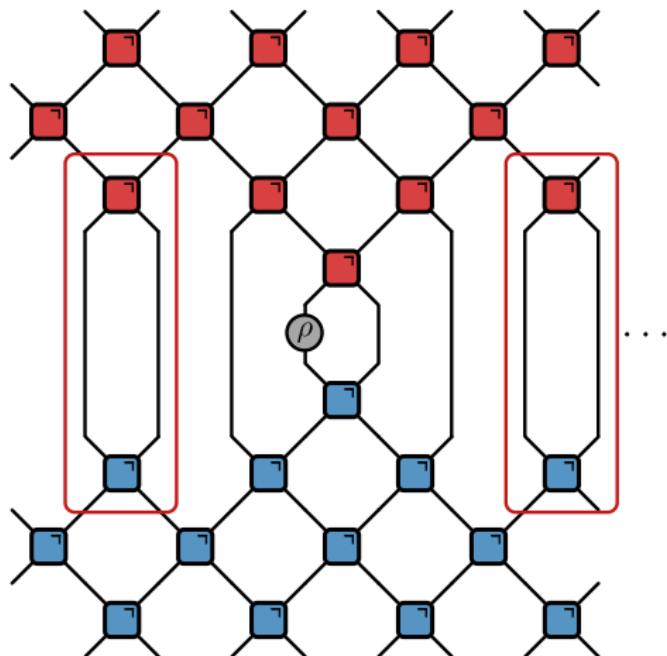
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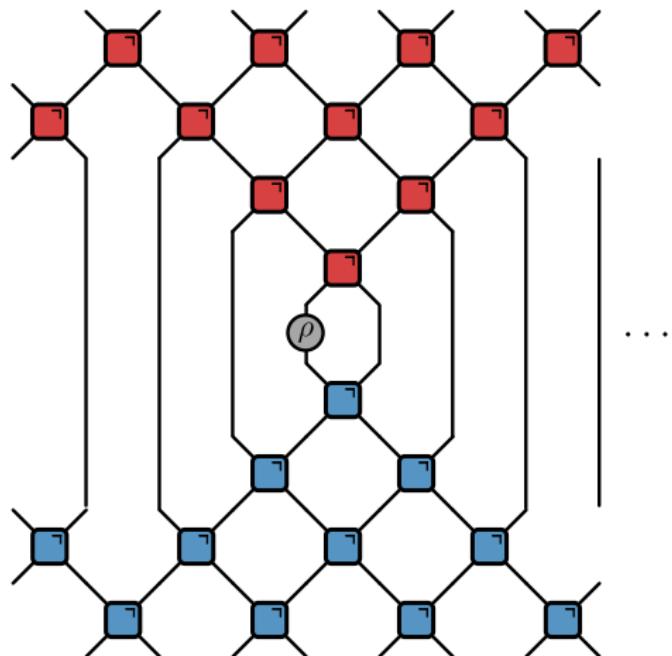
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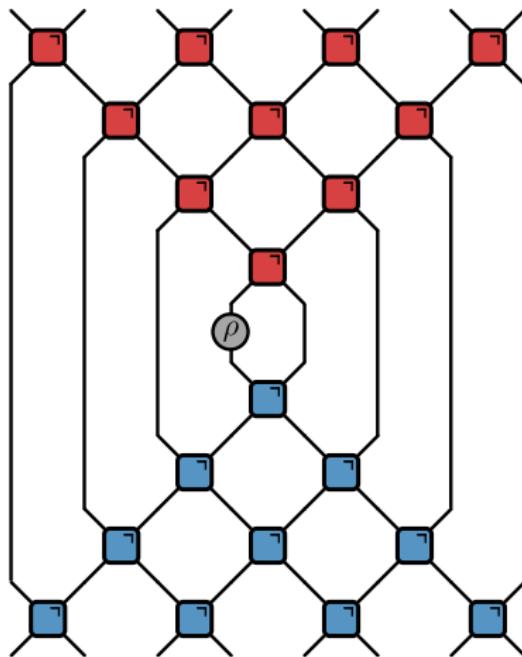
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Dual-unitary gates

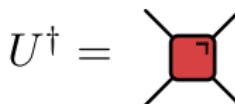
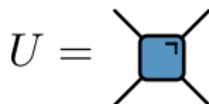
- Time evolution generated by two-site **unitary gates**

$$U = \begin{array}{c} \diagup \quad \diagdown \\ \textcolor{blue}{\square} \\ \diagdown \quad \diagup \end{array}$$

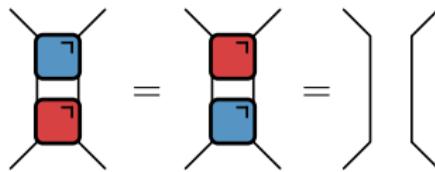
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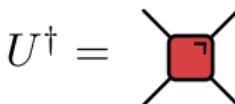
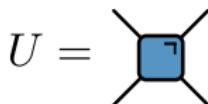


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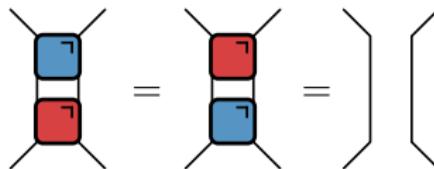


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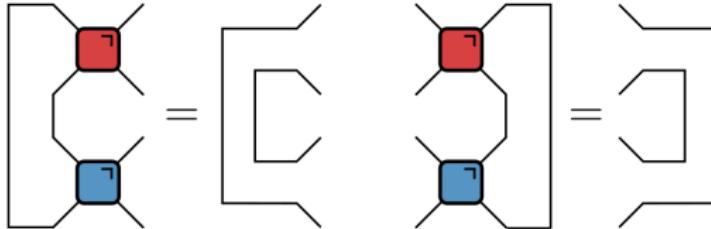
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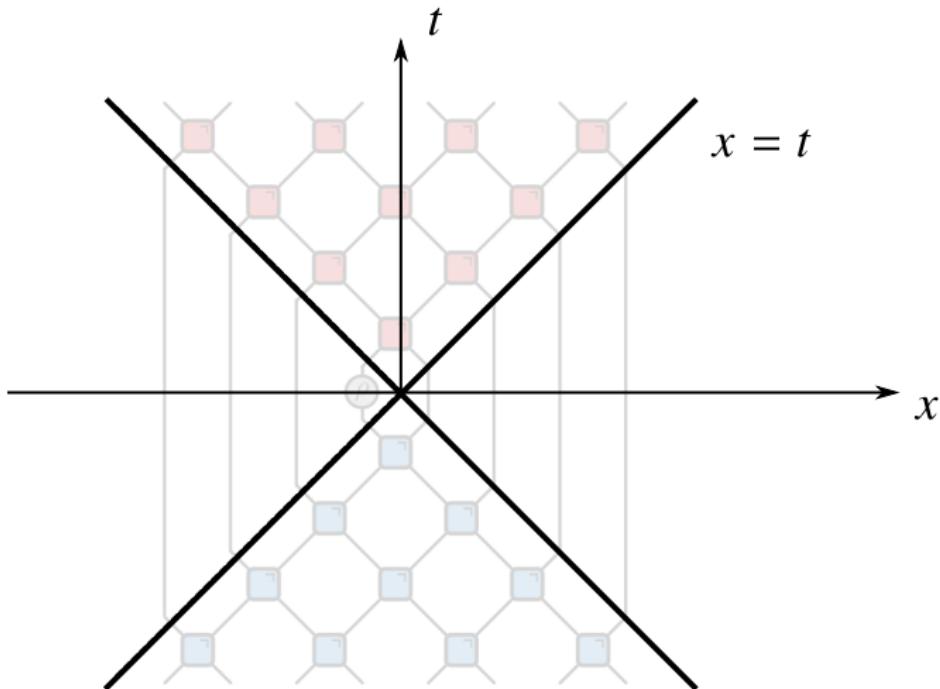


- Dual-unitary gates additionally satisfy



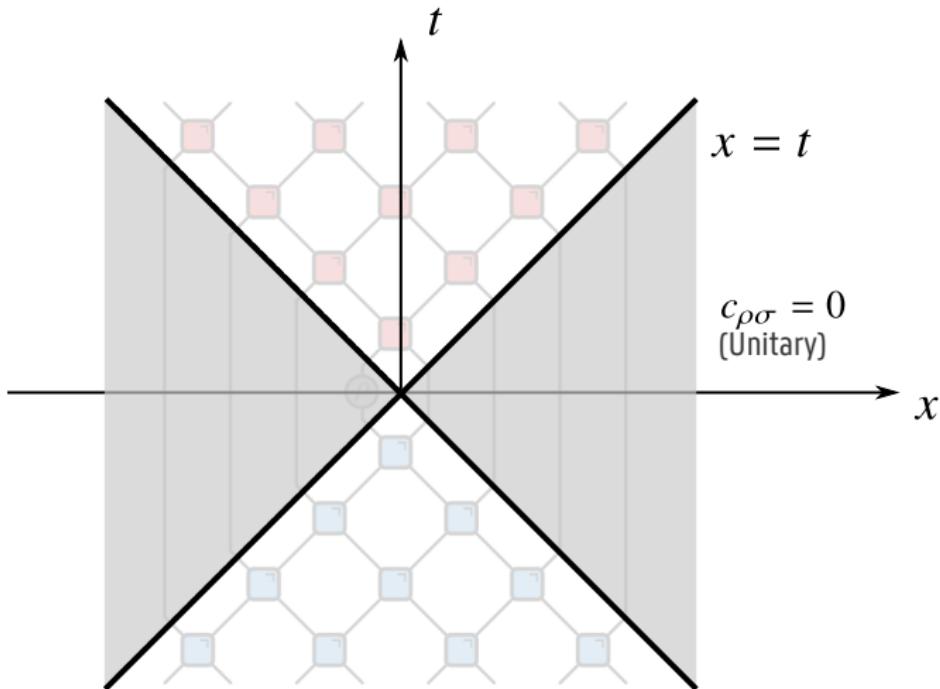
Effect of dual-unitarity

- Correlation functions $c_{\rho\sigma}(x, t) \sim \text{Tr} [\rho(0, t)\sigma(x)]$



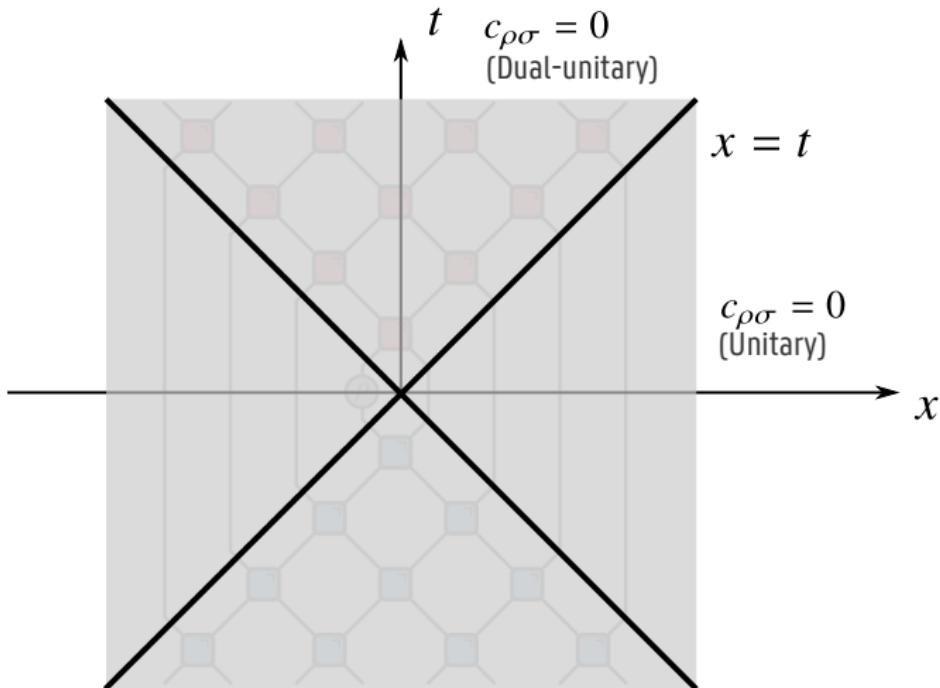
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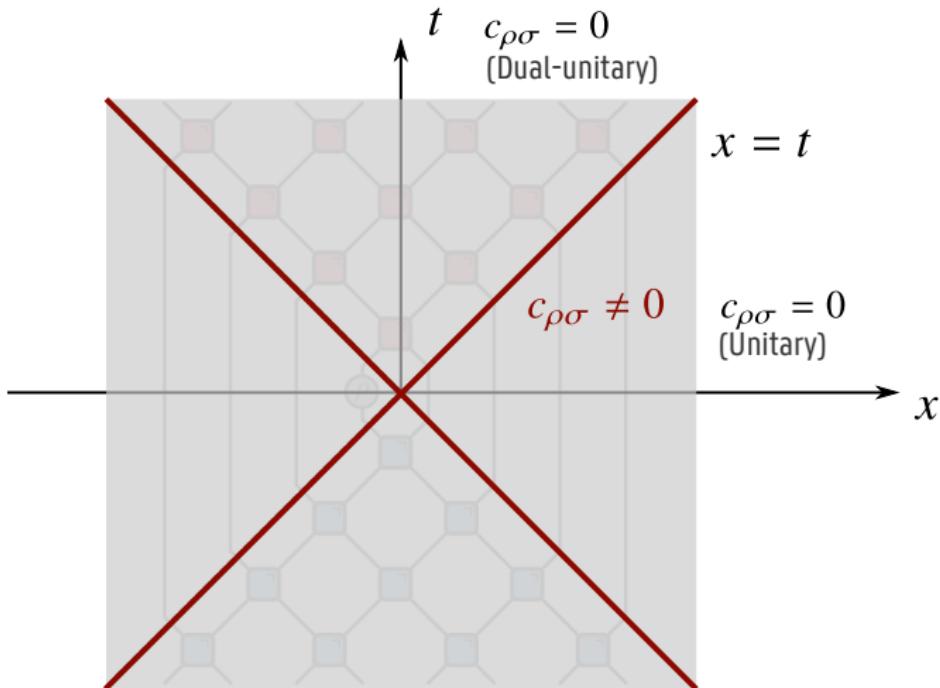
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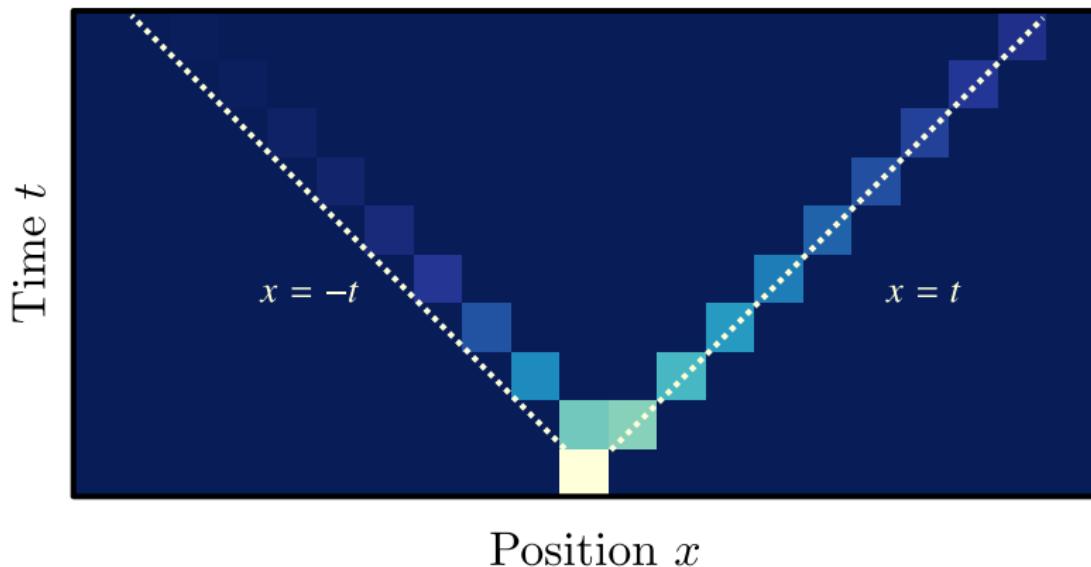
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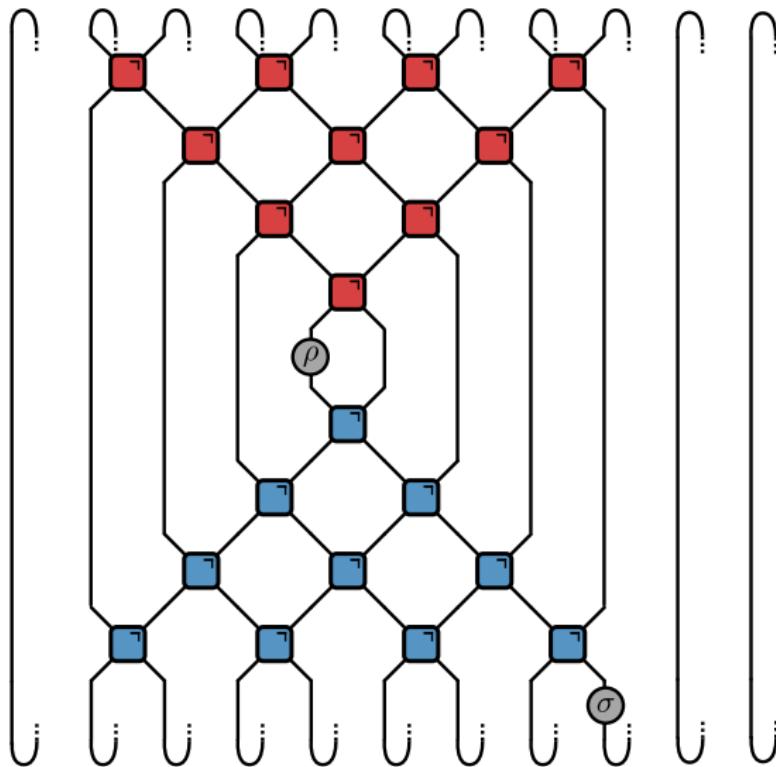
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$$c_{\rho\sigma}(x, t)$$



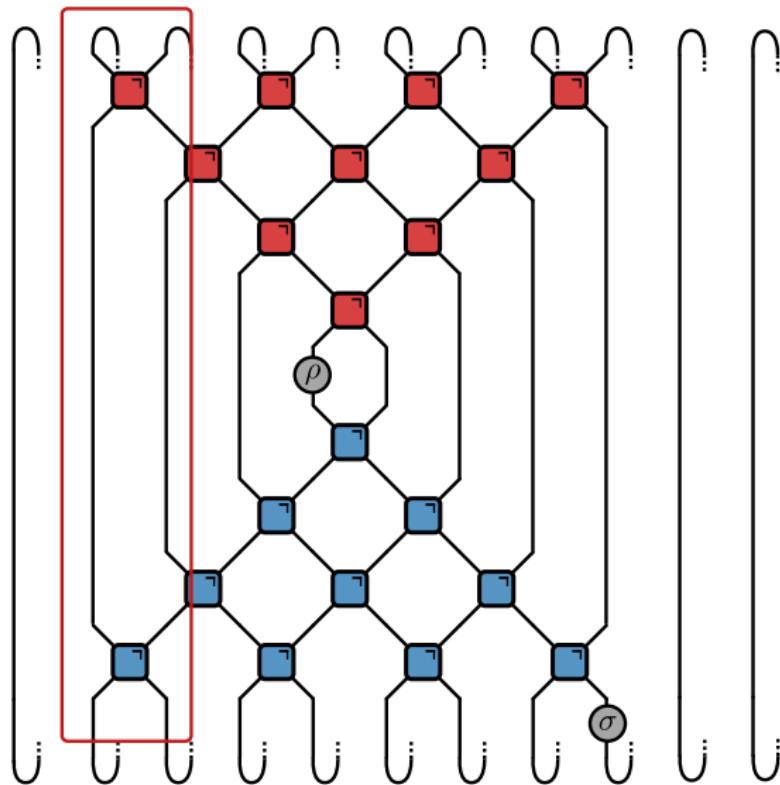
Light cone correlations

- Correlations on light cone $\text{Tr} [\mathcal{U}^\dagger(t)\rho(0)\mathcal{U}(t)\sigma(x = t)]$



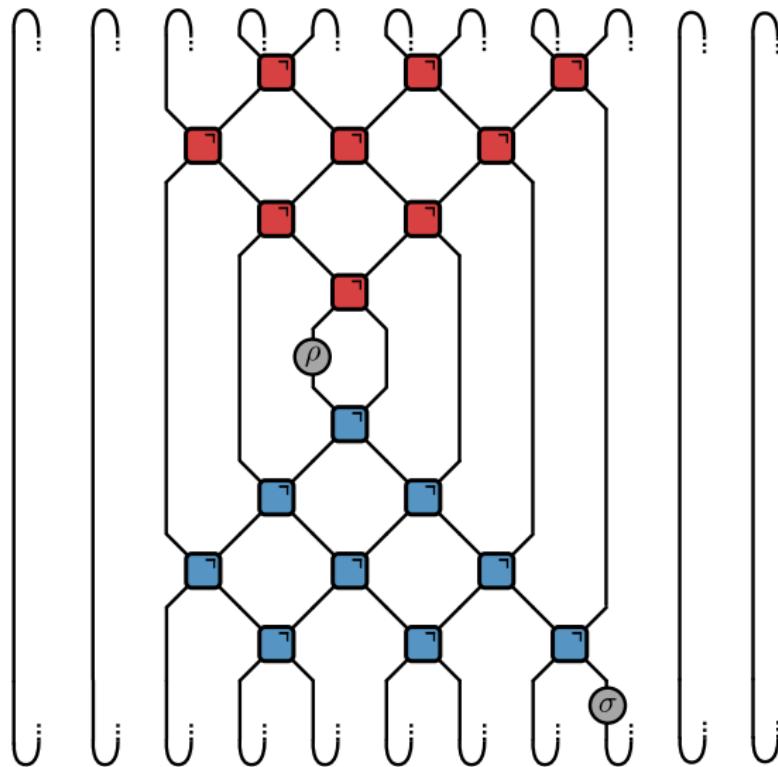
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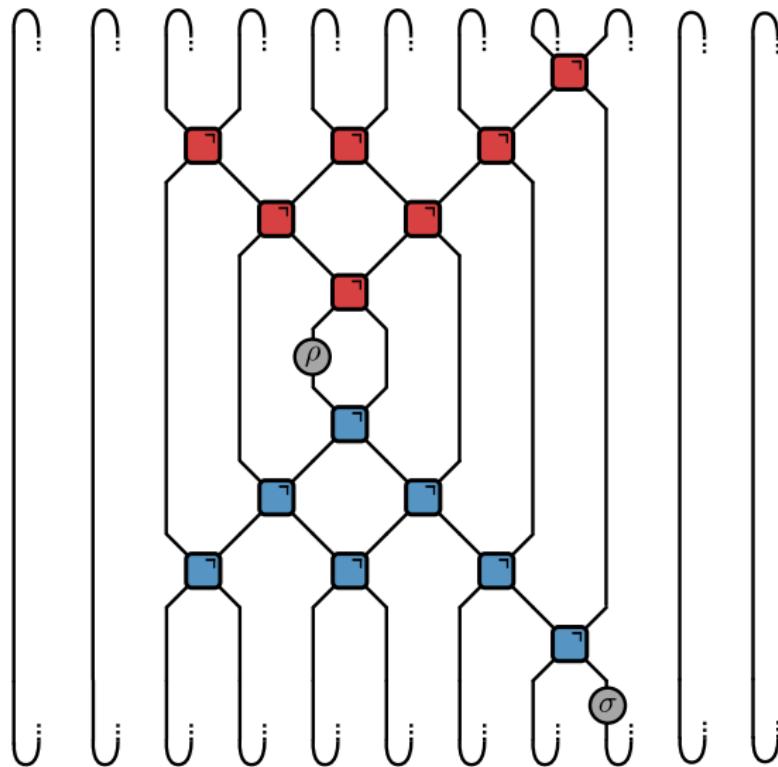
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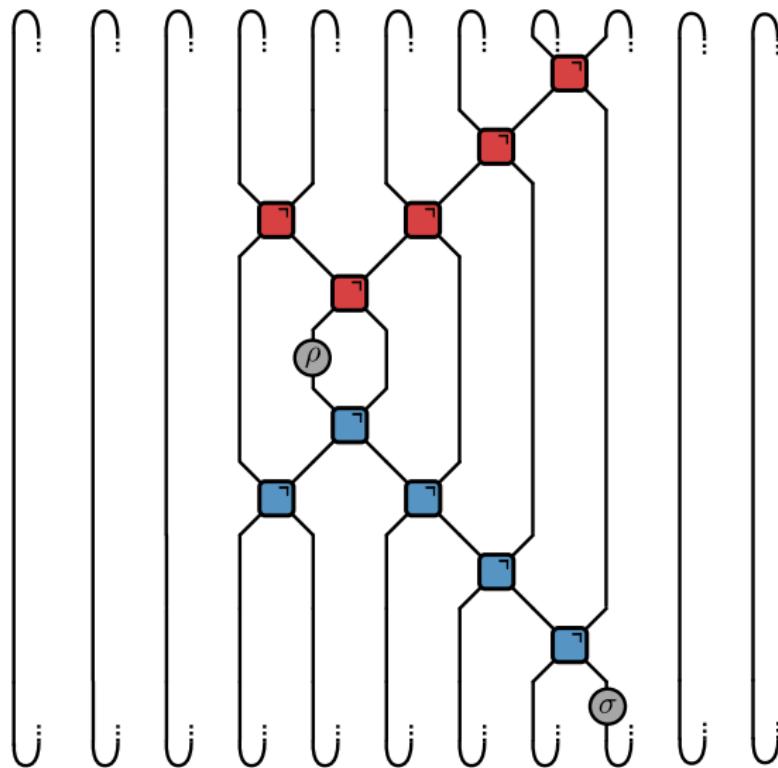
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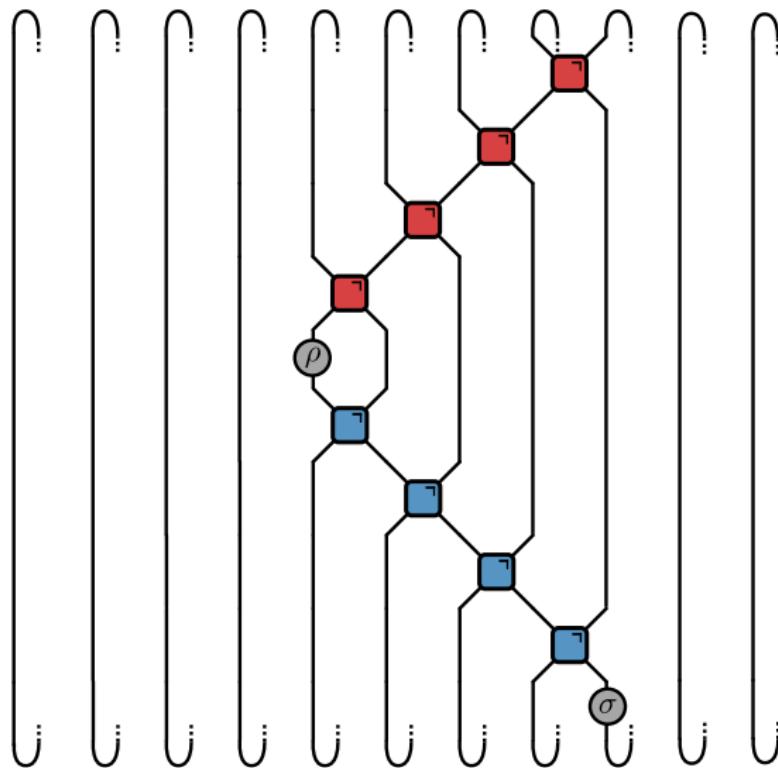
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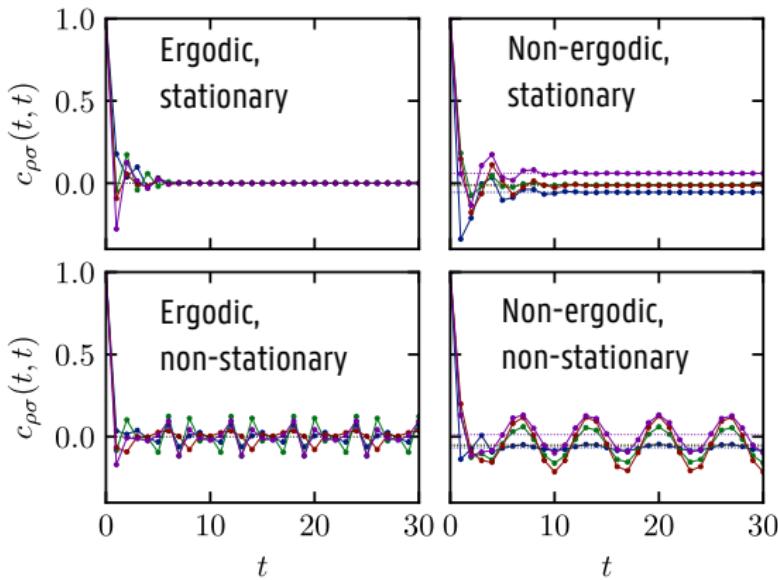
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Light-cone correlations

- Can be analytically calculated for arbitrary long times
- Different (dual-unitary) circuits \Rightarrow different dynamics



Motivation

Dual-unitary circuits are a minimal model of **local** and **unitary** many-body dynamics with many interesting properties

- S. Gopalakrishnan and A. Lamacraft, Phys. Rev. B 100, 064309 (2019)
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- ... and many more

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- **Exact solvability.** Supporting exact calculations of thermalization, scrambling, entanglement growth
- **Maximally chaotic.** Allow for analytic connections with random matrix theory
- **Maximal entanglement velocity.** Entanglement spreads at fastest possible rate.

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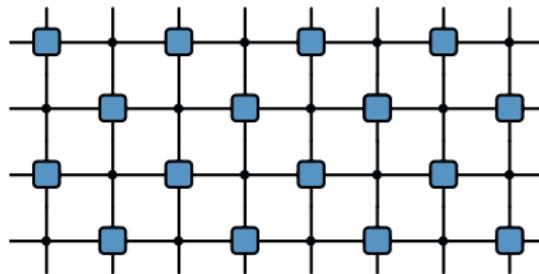
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Dual-unitarity round-a-face

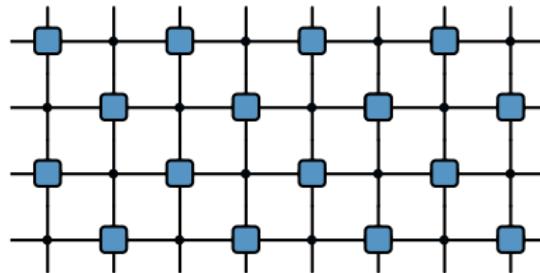
- Circuits introduced in Prosen, Chaos 31, 093101 (2021) with similar properties, built from 1-site 2-controlled gates



$$(U_{a,c})_{b,d} = \begin{array}{c} |a \\ \bullet \\ |b \\ \square \\ |d \\ \bullet \\ |c \end{array}$$

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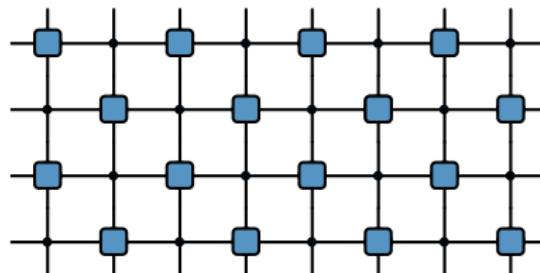
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- **Dual-unitarity:** enforce unitarity when exchanging indices

$$\Rightarrow \tilde{U}_{a,c} \text{ unitary with } (\tilde{U}_{a,c})_{b,d} = (U_{b,d})_{a,c}$$

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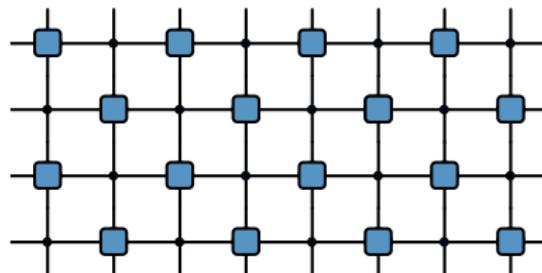


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- Similar **light-cone dynamics** as in dual-unitary brickwork circuits

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- Similar **light-cone dynamics** as in dual-unitary brickwork circuits
- Dual-unitary interactions round-a-face (DUIRF or 'clockwork')

1. Dual-unitary circuits
- 2. Shaded calculus**
3. Biunitary circuits
4. Solvable states

Shaded tensor networks

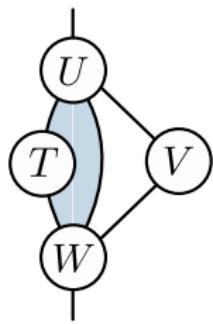
- Planar string diagram where some of the regions are shaded

Shaded tensor networks

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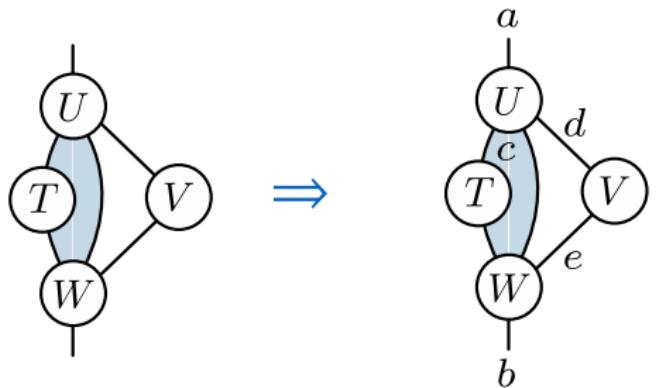


Shaded region = index

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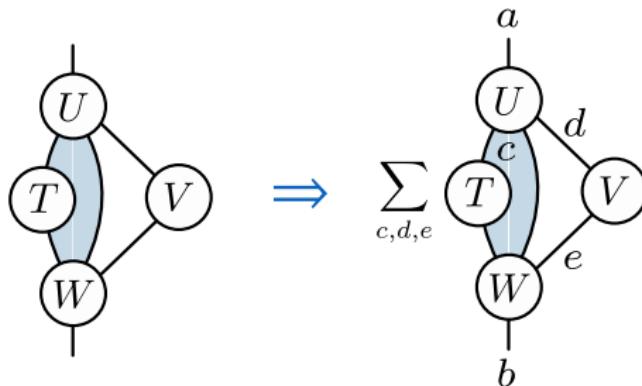


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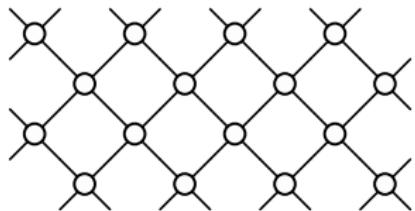
$$\sum_{c,d,e} T_c U_{acd} V_{de} W_{ceb}$$

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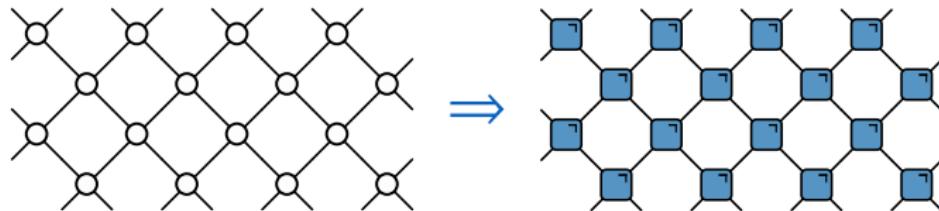
Shaded calculus representation of circuits

No shading corresponds to **brickwork** circuits



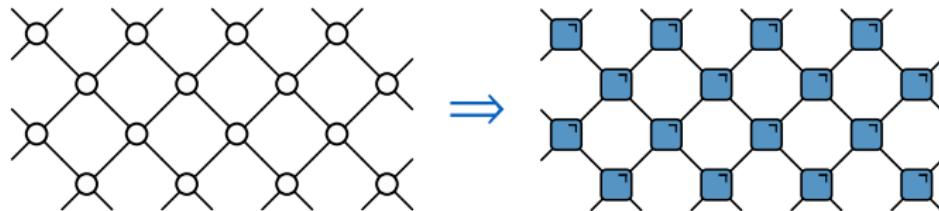
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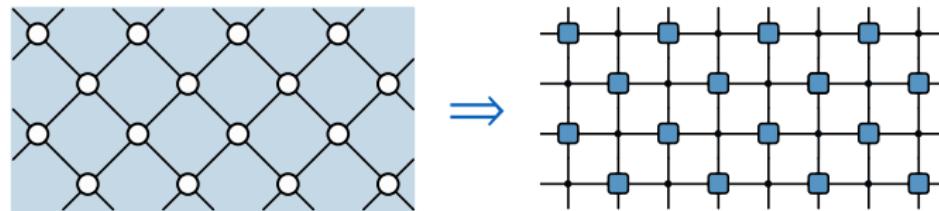


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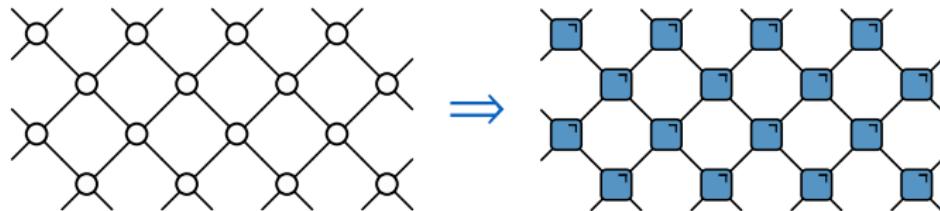


Fully shaded corresponds to **clockwork** circuits

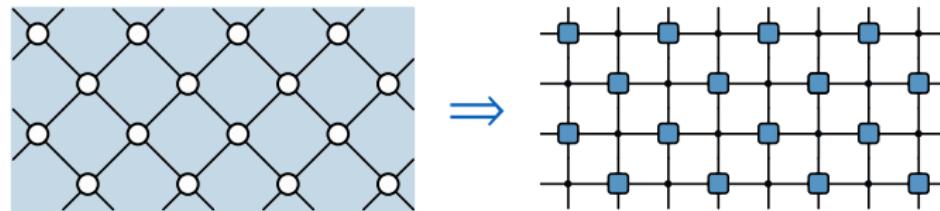


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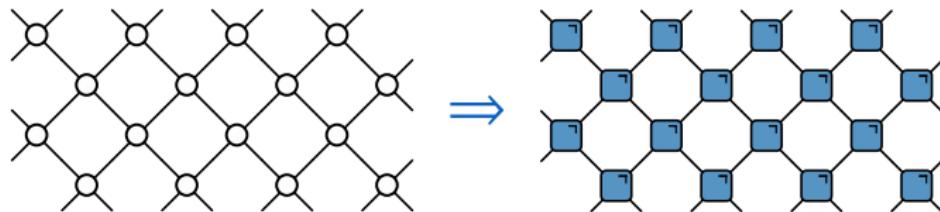


...writing

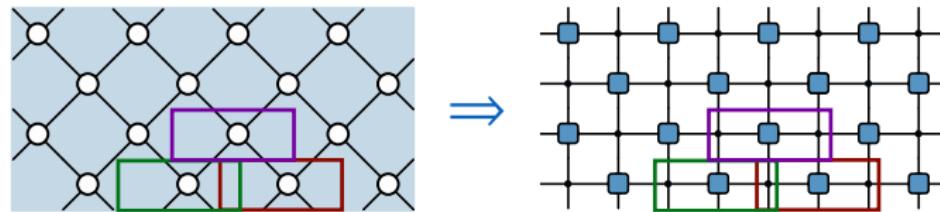
$$\text{...writing} \quad \begin{array}{c} b \\ \backslash \quad / \\ \text{---} \\ a \quad U \quad c \\ \backslash \quad / \\ d \end{array} = \begin{array}{c} b \\ | \quad | \\ a \quad \text{---} \quad c \\ | \quad | \\ d \end{array}$$

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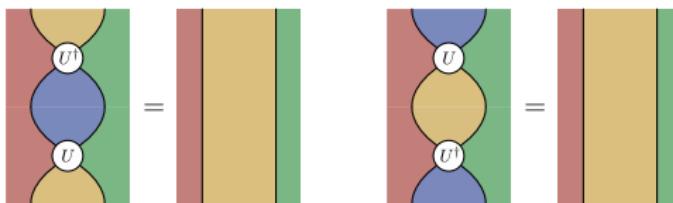
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1. Dual-unitary circuits
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- 3. Biunitary circuits**
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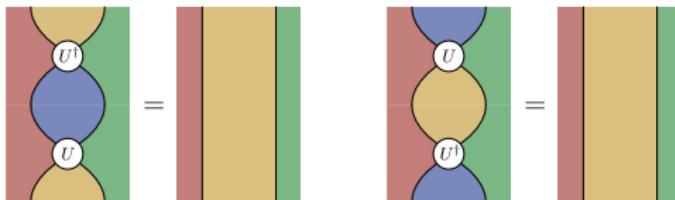
Biunitarity

- In shaded tensor networks, a 4-valent vertex U can be **vertically unitary**:

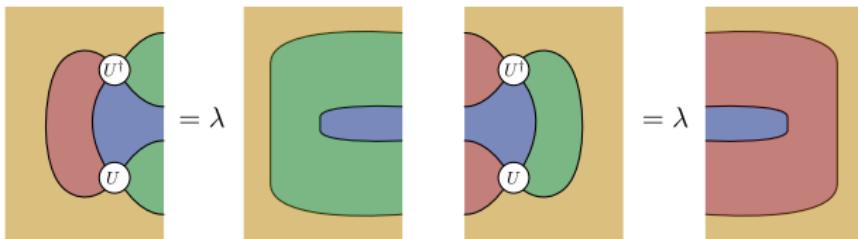


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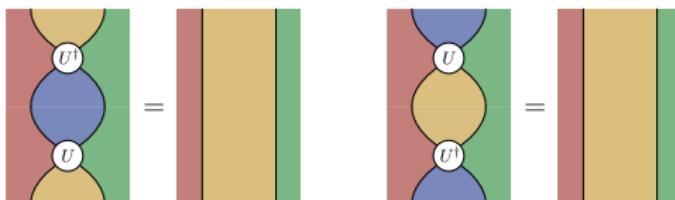


- It can also be **horizontally unitary**:

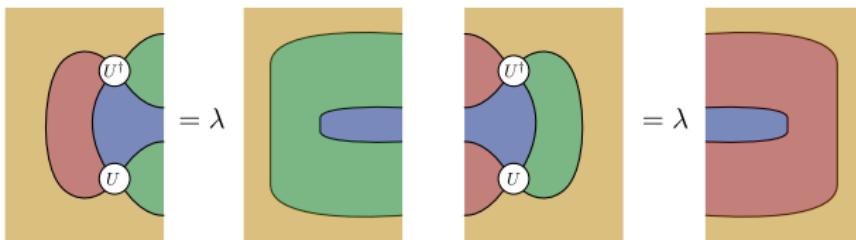


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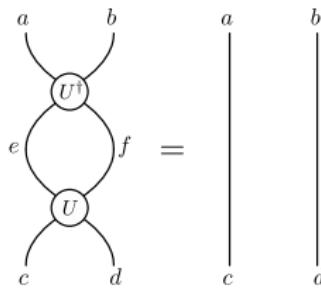


A 4-valent map is **biunitary** when it is **vertically and horizontally unitary**.

Dual-unitarity from biunitarity

- A biunitary without any shading corresponds to a **dual-unitary gate**

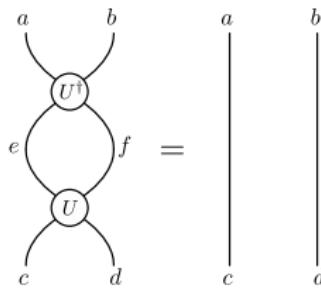
Vertical unitarity:



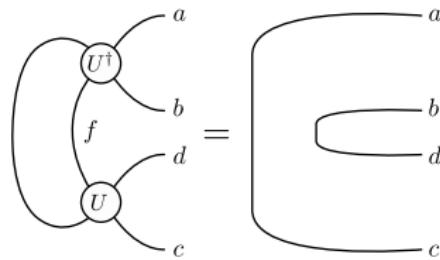
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$$\begin{array}{c} \text{Diagram: } \begin{array}{c} a \\ \backslash \\ e \\ \diagup \\ U^\dagger \\ \diagdown \\ f \\ \diagup \\ U \\ \diagdown \\ c \\ / \\ d \end{array} = \begin{array}{c} a \\ | \\ b \\ | \\ c \\ | \\ d \end{array} \end{array}$$
$$\sum_{e,f=1}^q U_{ab,ef}^\dagger U_{ef,cd} = \delta_{ac}\delta_{bd}$$

Horizontal unitarity:

$$\begin{array}{c} \text{Diagram: } \begin{array}{c} a \\ \backslash \\ e \\ \diagup \\ U^\dagger \\ \diagdown \\ f \\ \diagup \\ U \\ \diagdown \\ c \\ / \\ b \\ / \\ d \end{array} = \begin{array}{c} a \\ \backslash \\ b \\ \backslash \\ d \\ \backslash \\ c \end{array} \end{array}$$

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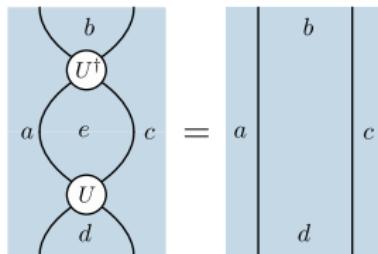
Vertical unitarity:

$$\begin{array}{c} \text{Diagram:} \\ \text{Two circles connected by a horizontal line. The top circle is labeled } U^\dagger \text{ and the bottom circle is labeled } U. \text{ Inputs } a \text{ and } c \text{ enter the left side, and outputs } b \text{ and } d \text{ exit the right side. Input } e \text{ enters the horizontal line between the two circles.} \\ = \\ \text{Matrix:} \\ \begin{array}{c|c|c} & b & \\ a & & & c \\ & d & \end{array} \end{array}$$

Dual-unitarity from biunitarity

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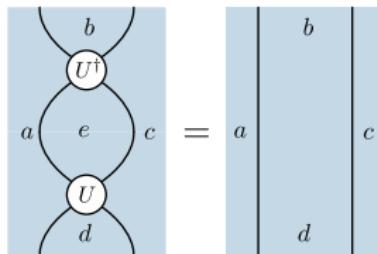


$$\sum_e (U_{a,c})_{b,e}^\dagger (U_{a,c})_{e,d} = \delta_{bd}, \quad \forall a, c$$

Dual-unitarity from biunitarity

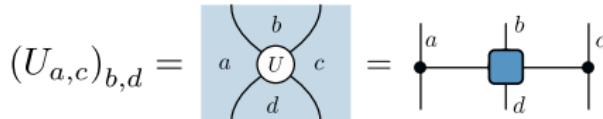
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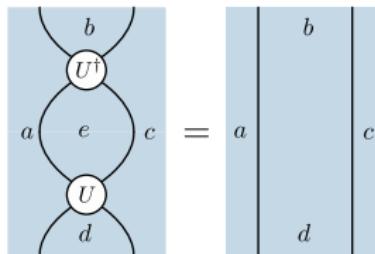
- Can express fully shaded biunitary as **2-controlled 1-site unitary**



Dual-unitarity from biunitarity

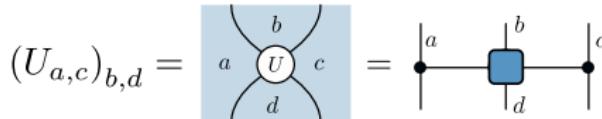
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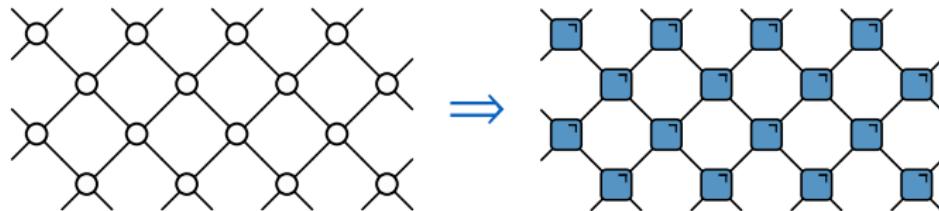
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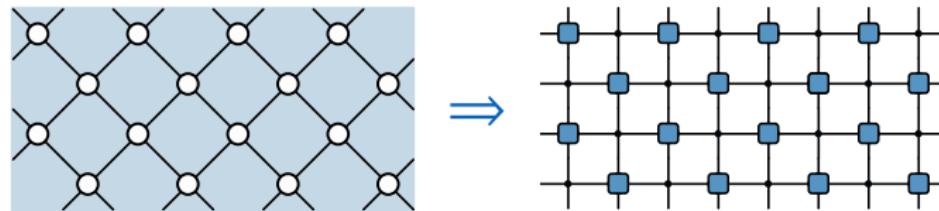
- Horizontal unitarity** enforces unitarity of $\tilde{U}_{a,c}$ with
 $(\tilde{U}_{a,c})_{b,d} = (U_{b,d})_{a,c}$

Shaded calculus representation of circuits

No shading corresponds to **brickwork** circuits

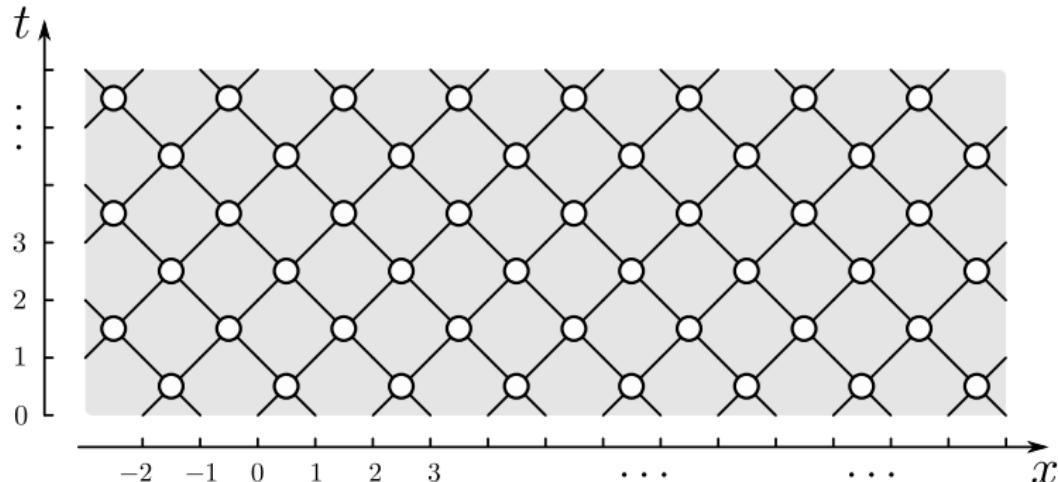


Fully shaded corresponds to **clockwork** circuits



Biunitary circuits

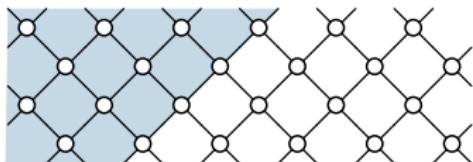
- We consider lattices where **every vertex is biunitary**



Gray background = region can be **shaded or not**

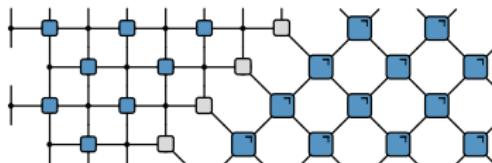
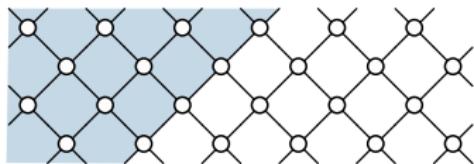
Heterogeneous biunitary circuits

- We can vary the shading pattern...



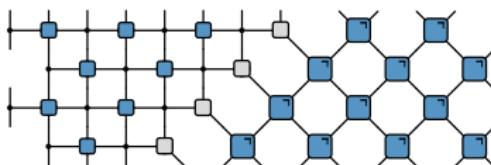
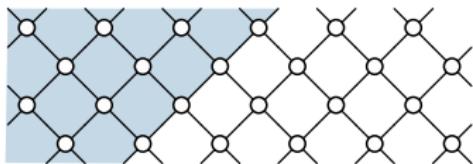
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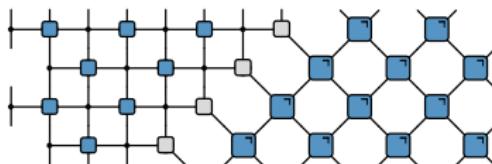
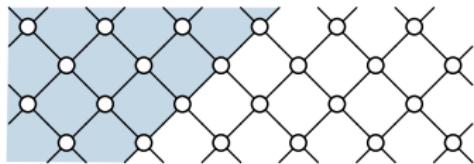


- ... by introducing new elements

$$(U_{a,b})_c = \begin{array}{c} b \\ \diagup \quad \diagdown \\ a & U & c \\ \diagdown \quad \diagup \end{array} = \begin{array}{c} |a \\ \bullet \quad \square \\ |b \\ c \end{array}$$

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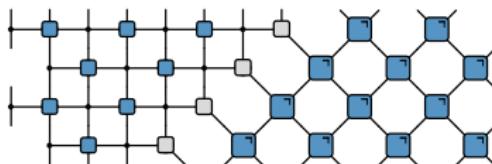
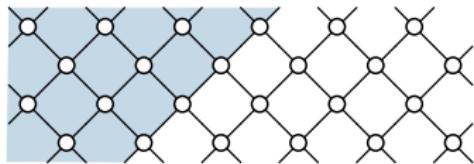
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⇒ Biunitarity implies that these vertices correspond to **quantum Latin squares**

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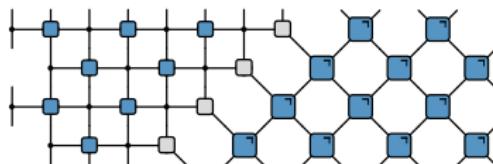
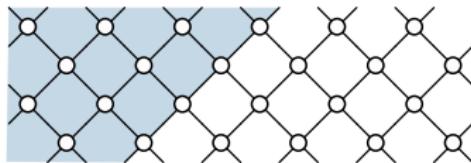
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⇒ Matrix of vectors $U_{a,b}$ for which every row and column forms an orthonormal basis

Heterogeneous biunitary circuits

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Latin square

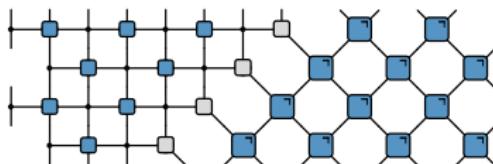
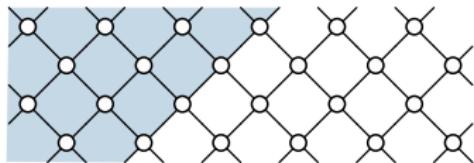
$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$$

Quantum Latin square

$$U = \begin{pmatrix} |1\rangle & |2\rangle & |3\rangle \\ |3\rangle & |1\rangle & |2\rangle \\ |2\rangle & |3\rangle & |1\rangle \end{pmatrix}$$

Heterogeneous biunitary circuits

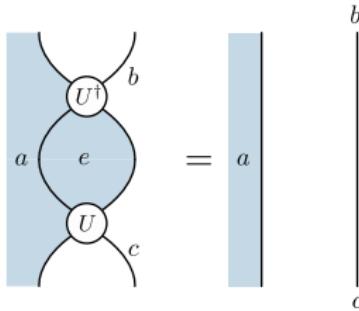
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$ 0\rangle$	$ 1\rangle$	$ 2\rangle$	$ 3\rangle$
$\frac{1}{\sqrt{2}}(1\rangle - 2\rangle)$	$\frac{1}{\sqrt{5}}(i 0\rangle + 2 3\rangle)$	$\frac{1}{\sqrt{5}}(2 0\rangle + i 3\rangle)$	$\frac{1}{\sqrt{2}}(1\rangle + 2\rangle)$
$\frac{1}{\sqrt{2}}(1\rangle + 2\rangle)$	$\frac{1}{\sqrt{5}}(2 0\rangle + i 3\rangle)$	$\frac{1}{\sqrt{5}}(i 0\rangle + 2 3\rangle)$	$\frac{1}{\sqrt{2}}(1\rangle - 2\rangle)$
$ 3\rangle$	$ 2\rangle$	$ 1\rangle$	$ 0\rangle$

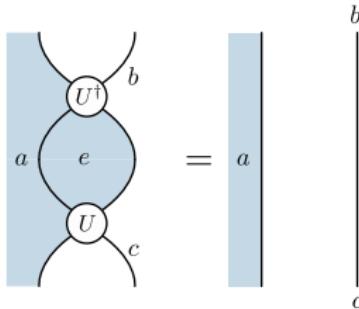
Quantum Latin squares

- Vertical unitarity determines the properties of the rows



Quantum Latin squares

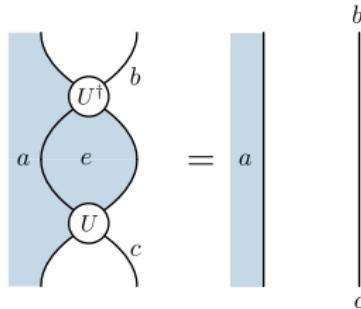
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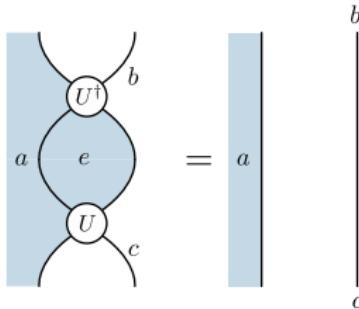


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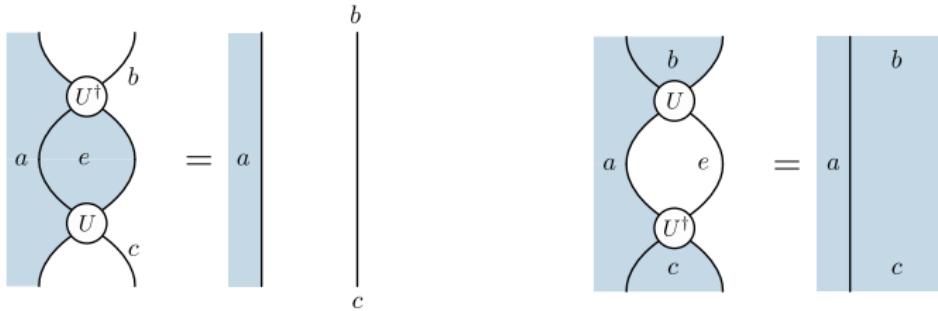
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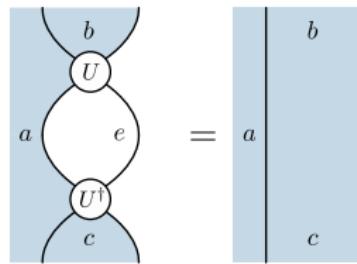
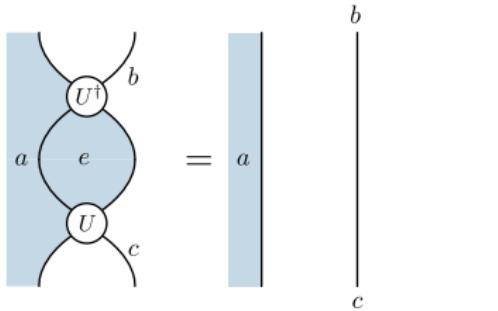
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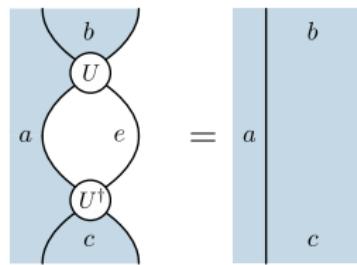
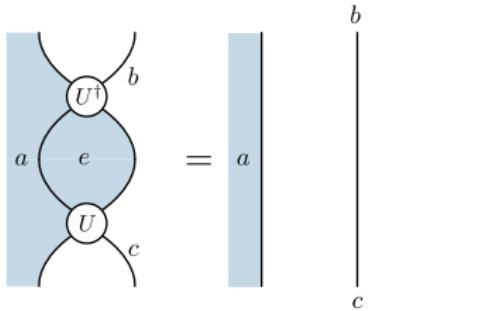
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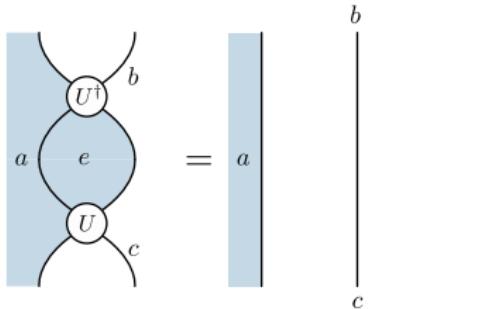
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Completeness

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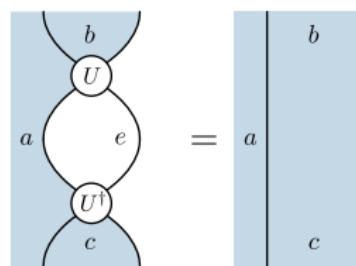
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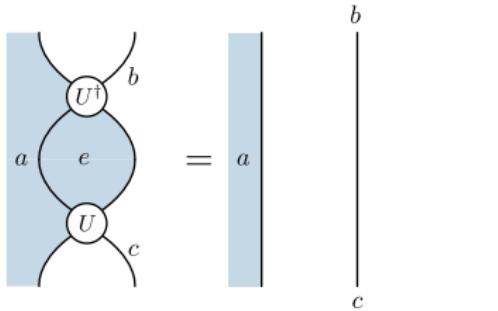
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Orthonormality

Quantum Latin squares

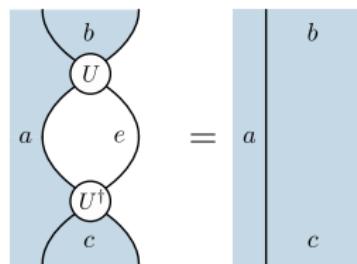
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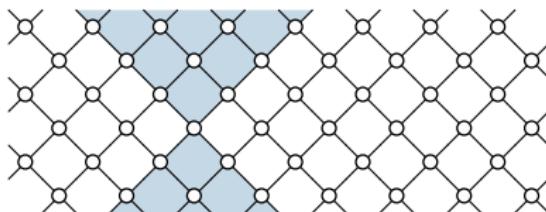
$$\langle U_{a,b} | U_{a,c} \rangle = \delta_{bc}$$

Orthonormality

- Horizontal unitarity fixes same property for the columns

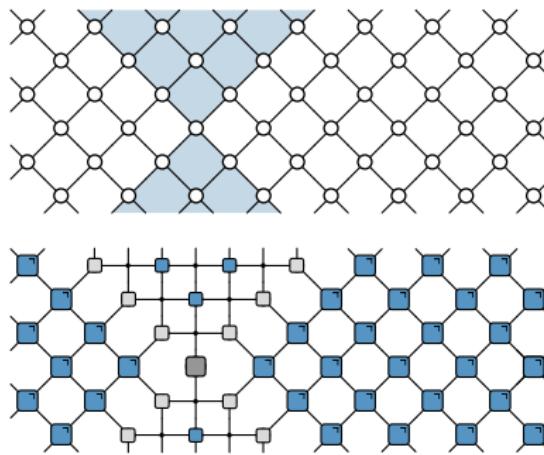
Heterogeneous biunitary circuits

- Introducing additional shading patterns



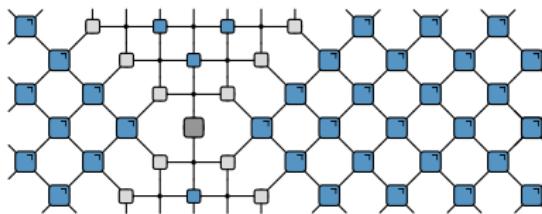
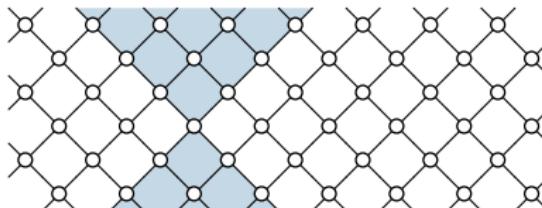
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Heterogeneous biunitary circuits

- Introducing additional shading patterns

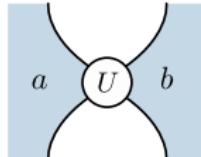


- ... by introducing new elements

$$U_{a,b} = \begin{array}{c} b \\ \text{\scriptsize } U \\ a \end{array} = \begin{array}{c} a \\ \text{\scriptsize } \square \\ b \end{array}$$

Complex Hadamard matrix

- A biunitary with two opposite shaded regions

$$U_{ab} = \begin{array}{c} \text{ } \\ \text{ } \\ \text{ } \\ U \\ \text{ } \\ \text{ } \\ \text{ } \end{array}$$


Complex Hadamard matrix

- A biunitary with two opposite shaded regions

$$U_{ab} = \begin{array}{c} \text{---} \\ | \quad \quad | \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ | \quad \quad | \\ \text{---} \end{array}$$

$a \quad U \quad b$

- Corresponds to a complex Hadamard matrix

$$U^\dagger U = U U^\dagger = q \mathbb{1} \quad \text{and} \quad |U_{ab}| = 1, \forall a, b$$

Complex Hadamard matrix

- A biunitary with two opposite shaded regions

$$U_{ab} = \begin{array}{c} \text{---} \\ | \quad | \\ a \quad b \\ \text{---} \end{array}$$

- Corresponds to a complex Hadamard matrix
- $U^\dagger U = UU^\dagger = q \mathbb{1}$ and $|U_{ab}| = 1, \forall a, b$
- Represents either **one-site unitary** or **two-site controlled phase**

$$\begin{array}{c} \text{---} \\ | \quad | \\ a \quad b \\ \text{---} \end{array} = \begin{array}{c} a \\ | \\ \text{---} \\ | \\ b \end{array}$$

Complex Hadamard matrix

- A biunitary with two opposite shaded regions

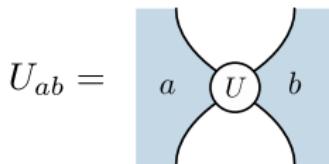
$$U_{ab} = \begin{array}{c} \text{---} \\ | \quad | \\ a \quad b \\ \text{---} \end{array}$$

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$$\begin{array}{c} \text{---} \\ | \quad | \\ a \quad b \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \quad | \\ \text{---} \end{array} \quad \text{or} \quad \begin{array}{c} \text{---} \\ | \quad | \\ a \quad b \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \quad | \\ \bullet \quad \text{---} \quad \text{---} \quad \bullet \\ | \quad | \\ a \quad b \end{array}$$

Complex Hadamard matrix

- A biunitary with two opposite shaded regions



- Corresponds to a complex Hadamard matrix
- $U^\dagger U = UU^\dagger = q \mathbb{1}$ and $|U_{ab}| = 1, \forall a, b$
- Represents either **one-site unitary** or **two-site controlled phase**

1-site unitary

$$\sqrt{\frac{1}{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

2-controlled phase

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Classification of biunitaries

Dual-unitary gates

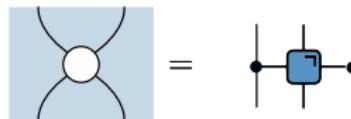


Classification of biunitaries

Dual-unitary gates



Quantum crosses



Classification of biunitaries

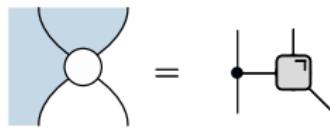
Dual-unitary gates



Quantum crosses



Quantum Latin squares

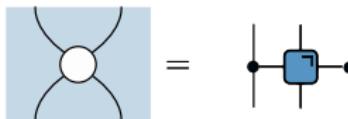


Classification of biunitaries

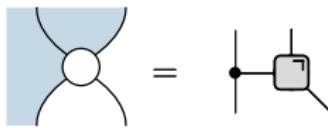
Dual-unitary gates



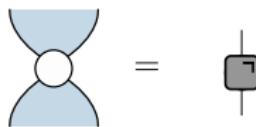
Quantum crosses



Quantum Latin squares



Complex Hadamard
matrices

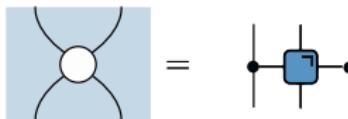


Classification of biunitaries

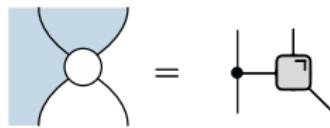
Dual-unitary gates



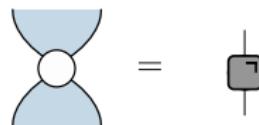
Quantum crosses



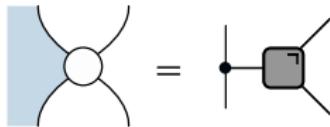
Quantum Latin squares



Complex Hadamard
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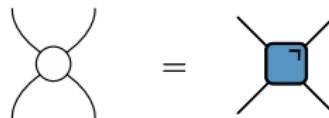


Unitary error bases

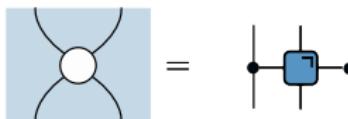


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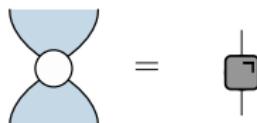
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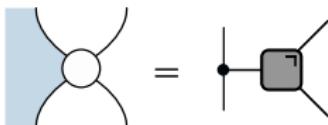


Complex Hadamard
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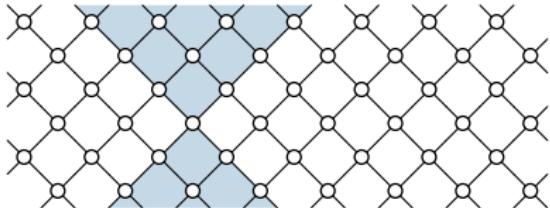


Unitary error bases

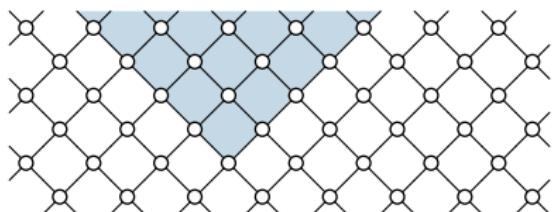
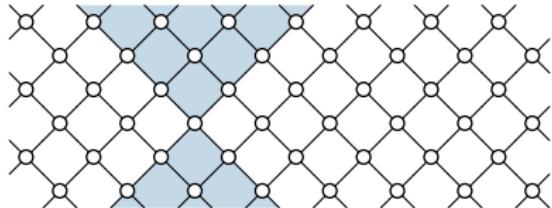
Orthogonal and complete set of unitaries,
e.g. Pauli matrices



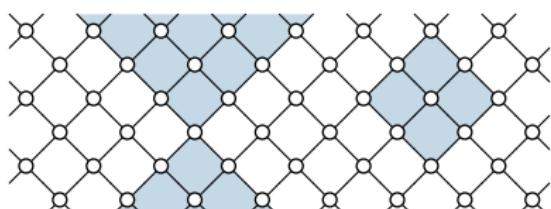
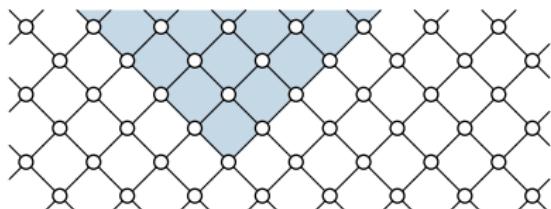
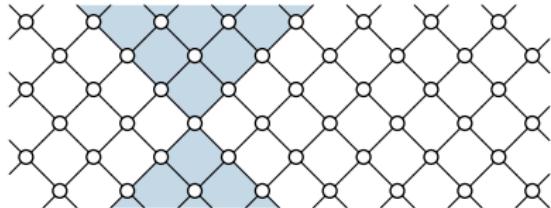
Zoo of biunitary circuits



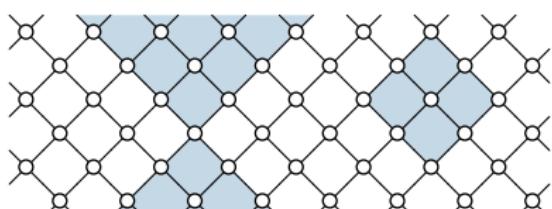
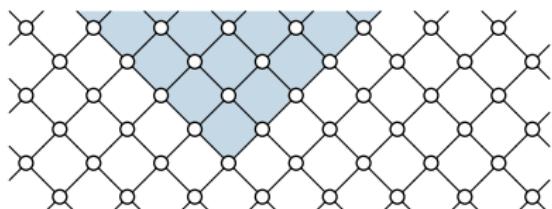
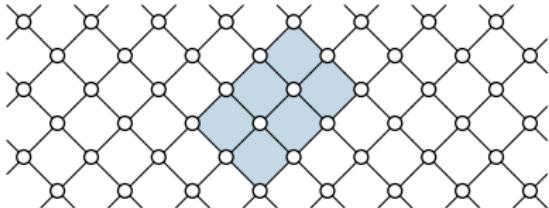
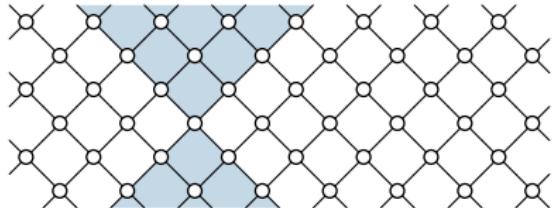
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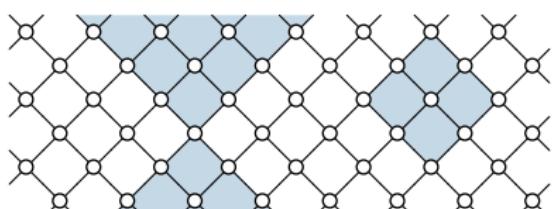
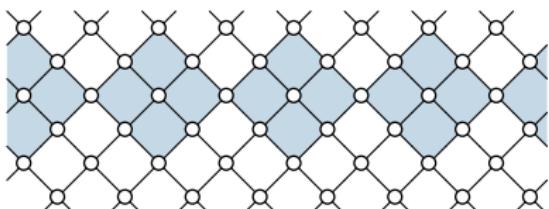
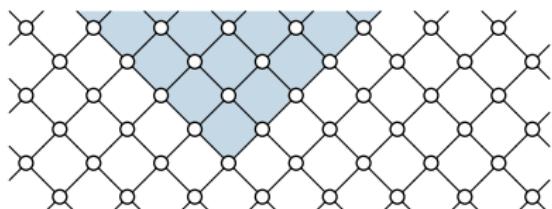
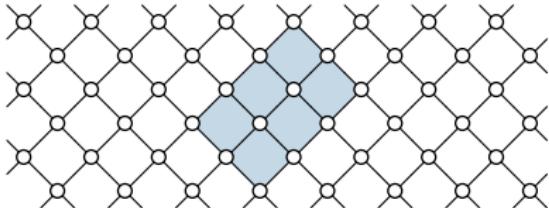
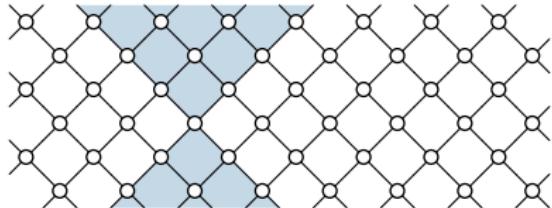
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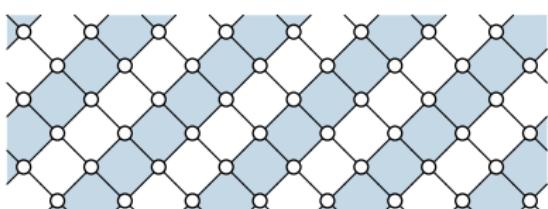
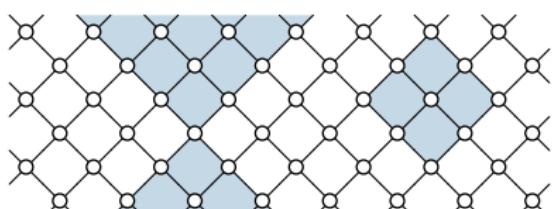
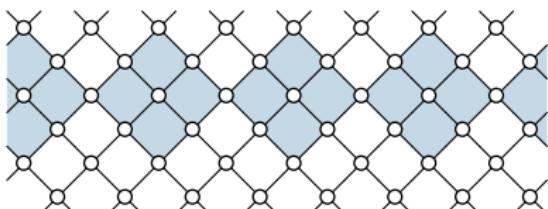
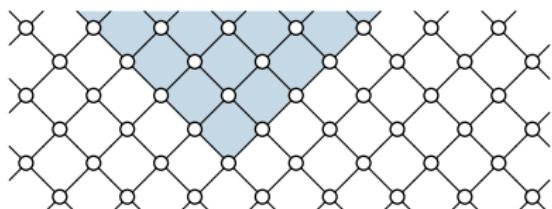
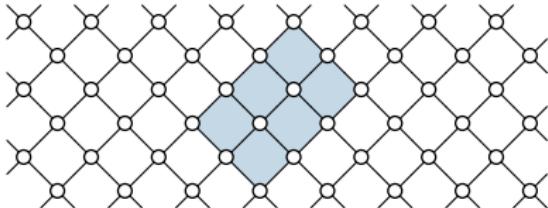
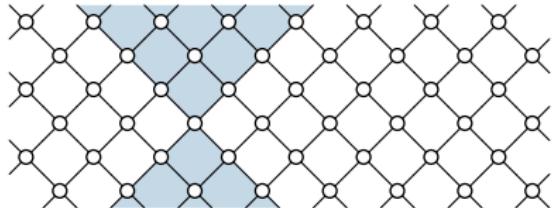
Zoo of biunitary circuits



Zoo of biunitary circuits



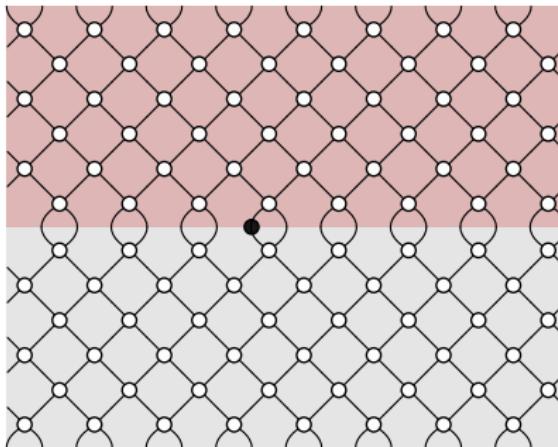
Zoo of biunitary circuits



Dynamics of correlation functions

- Proof from dual-unitarity **directly extends** to biunitary circuits

$$\mathcal{U}(t)^\dagger \rho \mathcal{U}(t) =$$

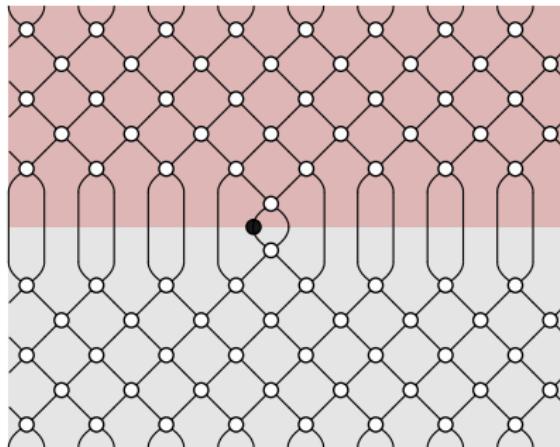


Vertical unitarity implies causal light cone

Dynamics of correlation functions

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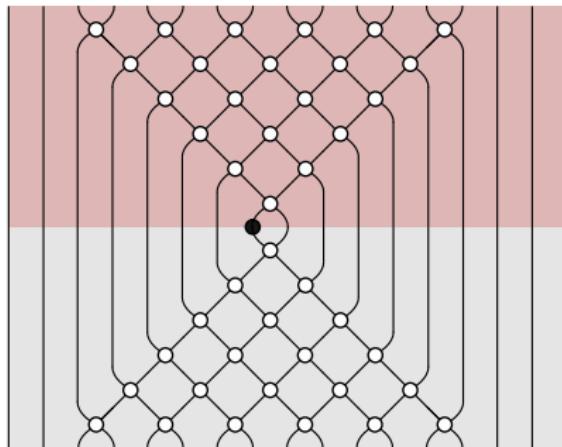


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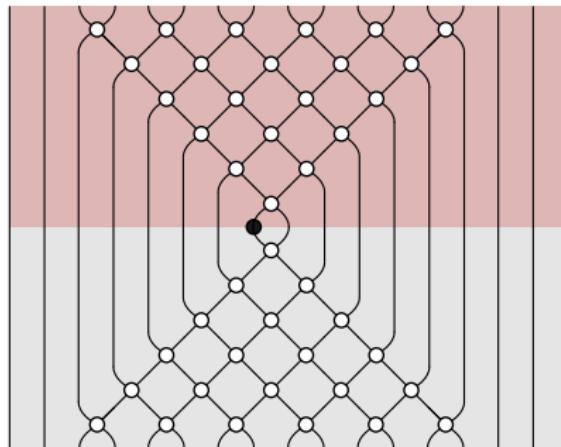


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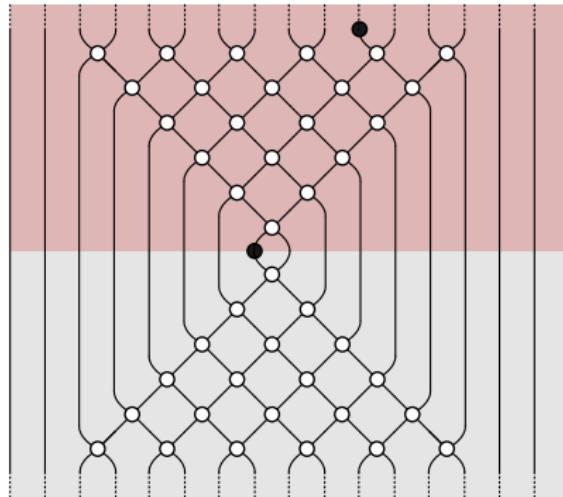
Vertical unitarity implies causal light cone

⇒ Correlations vanish outside the light cone

Dynamics of correlation functions

- Proof from dual-unitarity directly extends to biunitary circuits

$$\text{Tr} [\sigma \mathcal{U}(t)^\dagger \rho \mathcal{U}(t)] =$$

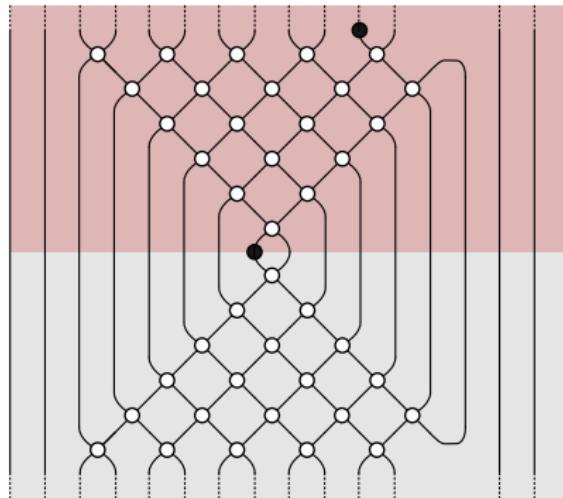


Now using **horizontal unitarity**

Dynamics of correlation functions

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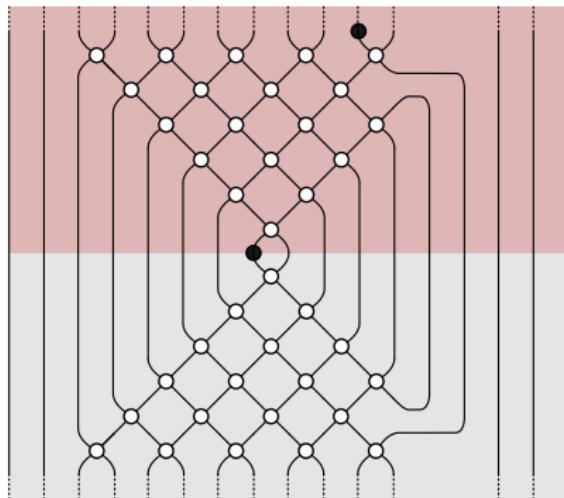


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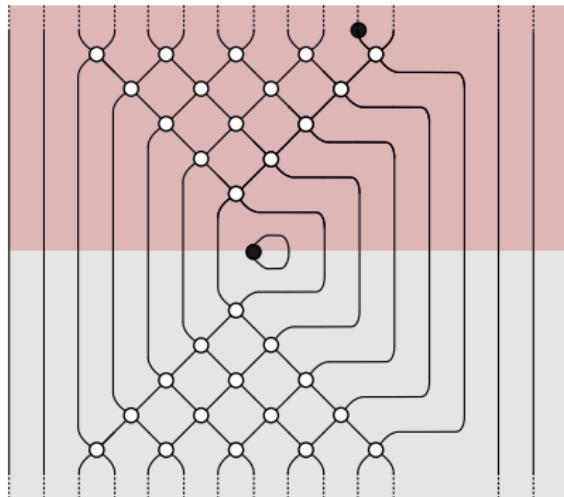


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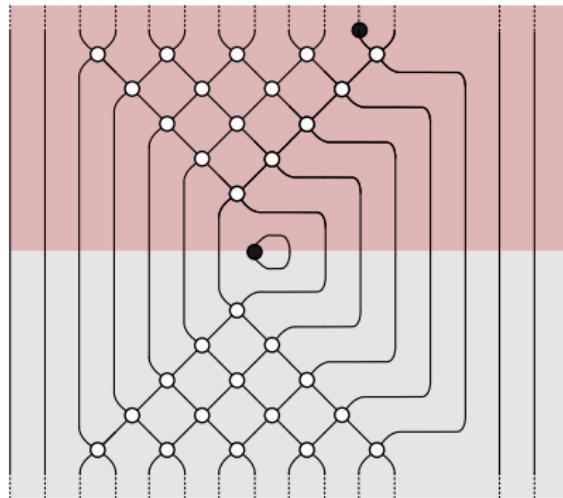


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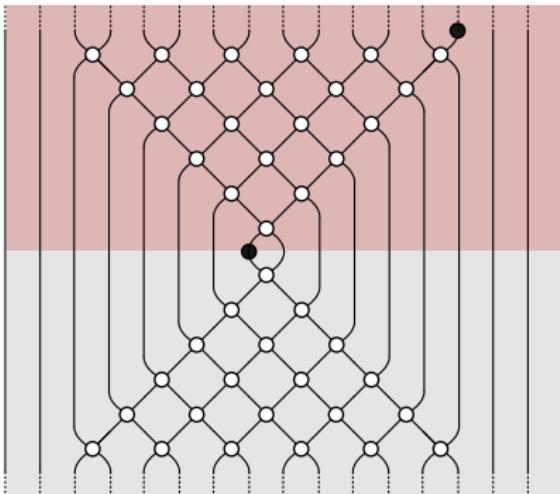


Now using **horizontal unitarity**

⇒ Correlations vanish inside the light cone

Light-cone dynamics

- Light-cone correlation functions can be efficiently calculated



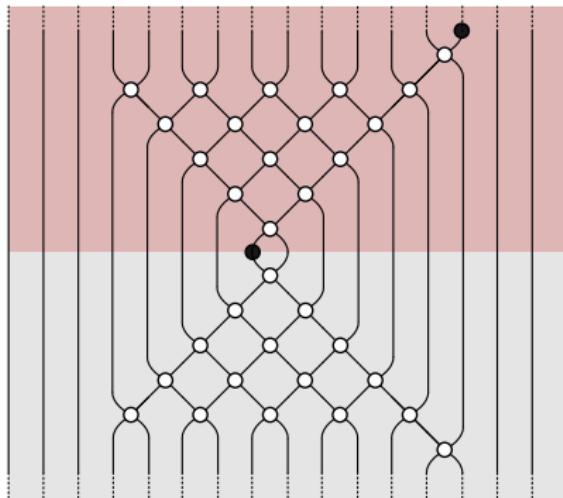
$$\text{tr} [\mathcal{U}^\dagger(t) \rho(0) \mathcal{U}(t) \sigma(x = t)] =$$

... in the exact same way as for dual-unitarity

Light-cone dynamics

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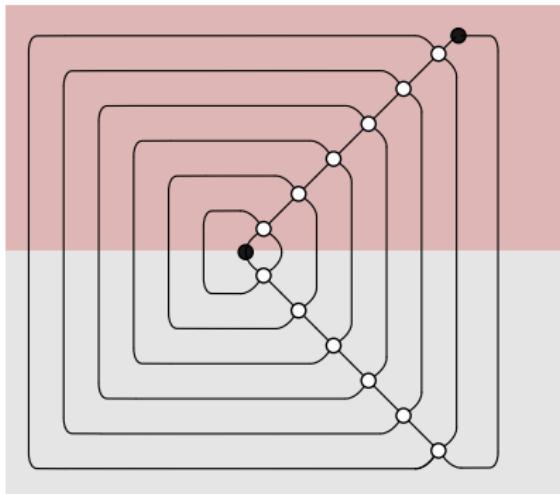


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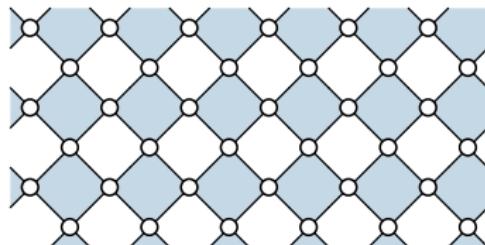
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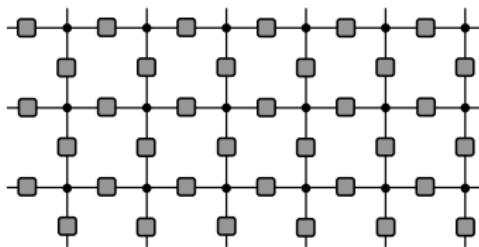
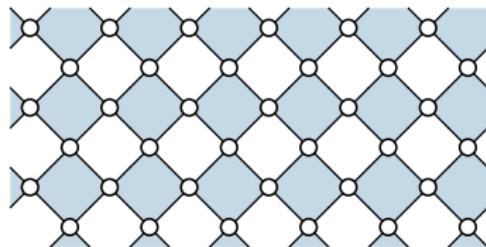
Complex Hadamard circuits

- Construct circuit exclusively out of complex Hadamard matrices



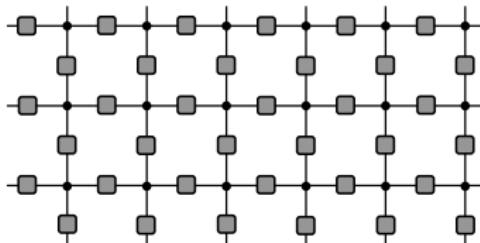
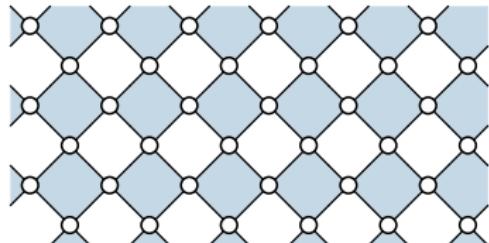
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Complex Hadamard circuits

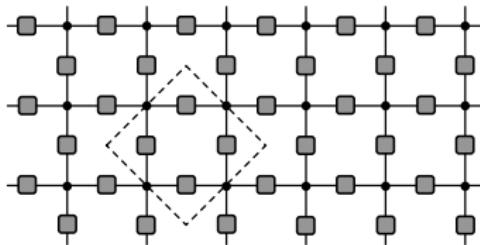
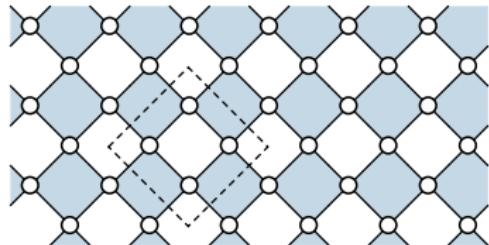
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- Can be reinterpreted as **brickwork** or **clockwork** depending on unit cell

Complex Hadamard circuits

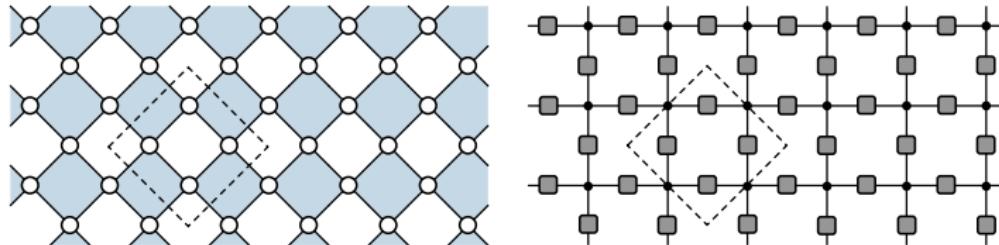
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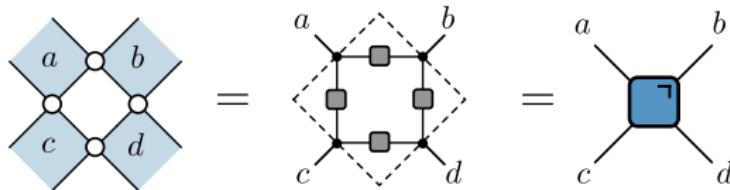
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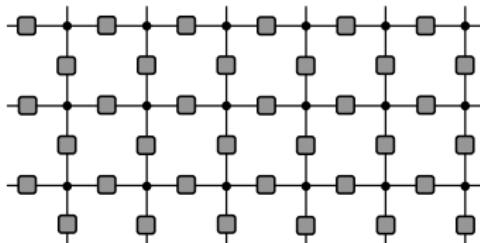
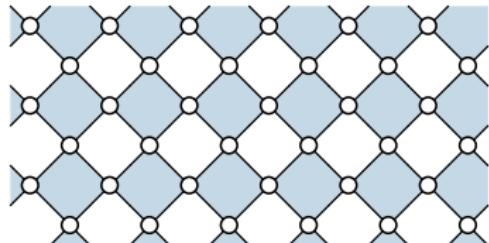


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Complex Hadamard circuits

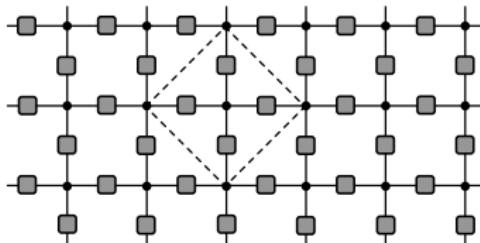
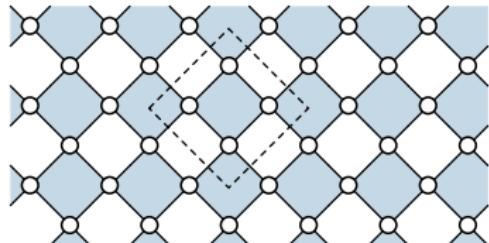
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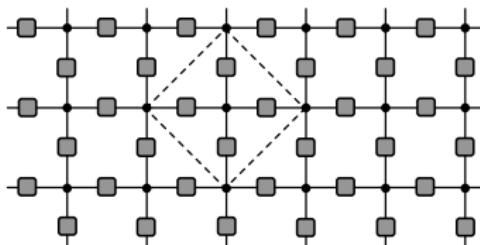
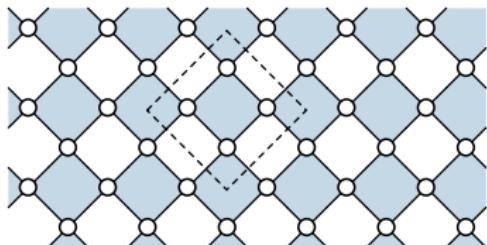
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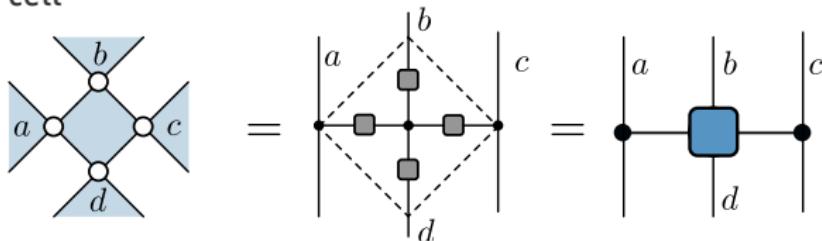
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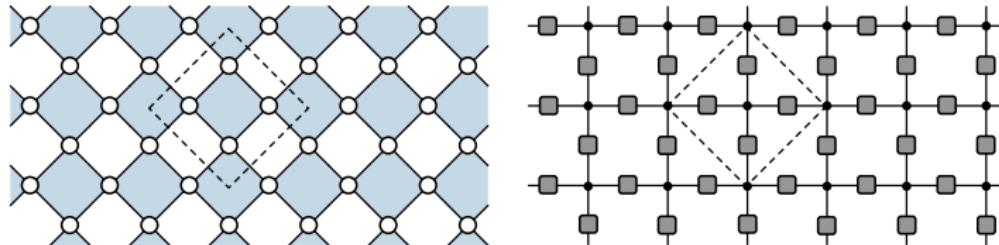


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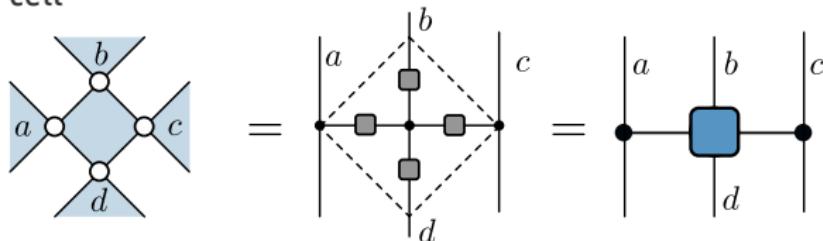


Complex Hadamard circuits

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- Can be reinterpreted as **brickwork** or **clockwork** depending on unit cell



Biunitaries combine into biunitaries!

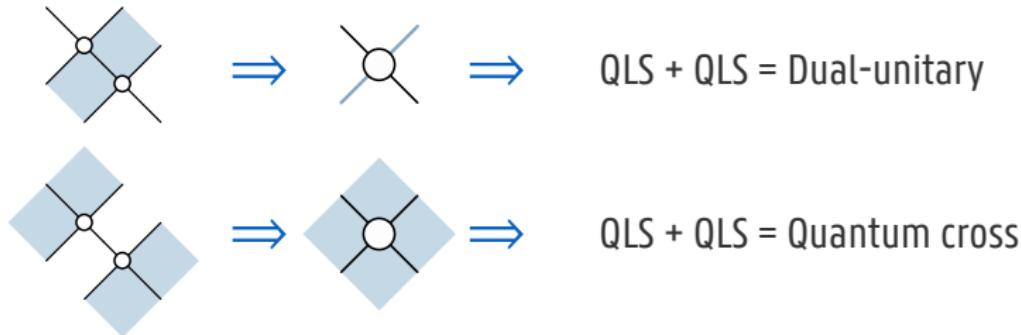
Compositions

- Diagonal compositions of biunitaries return biunitaries



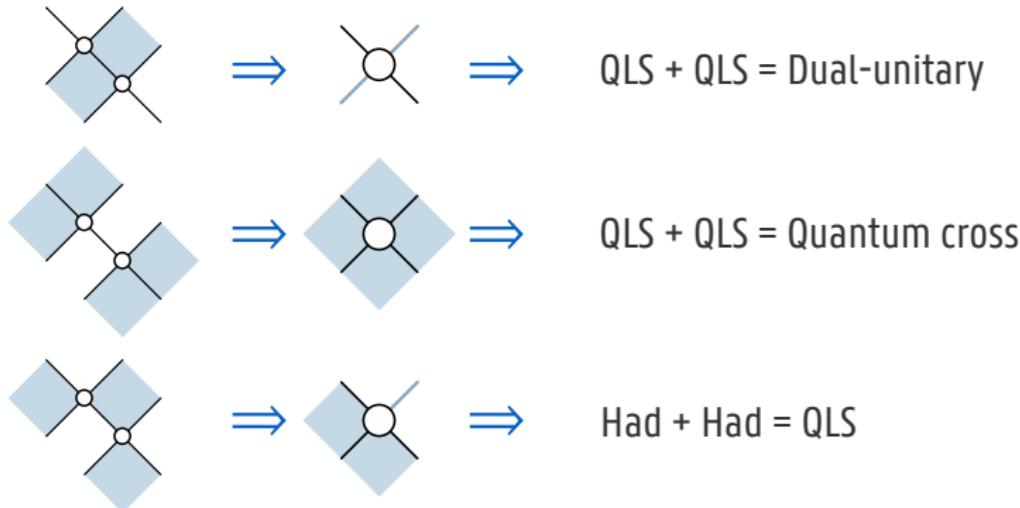
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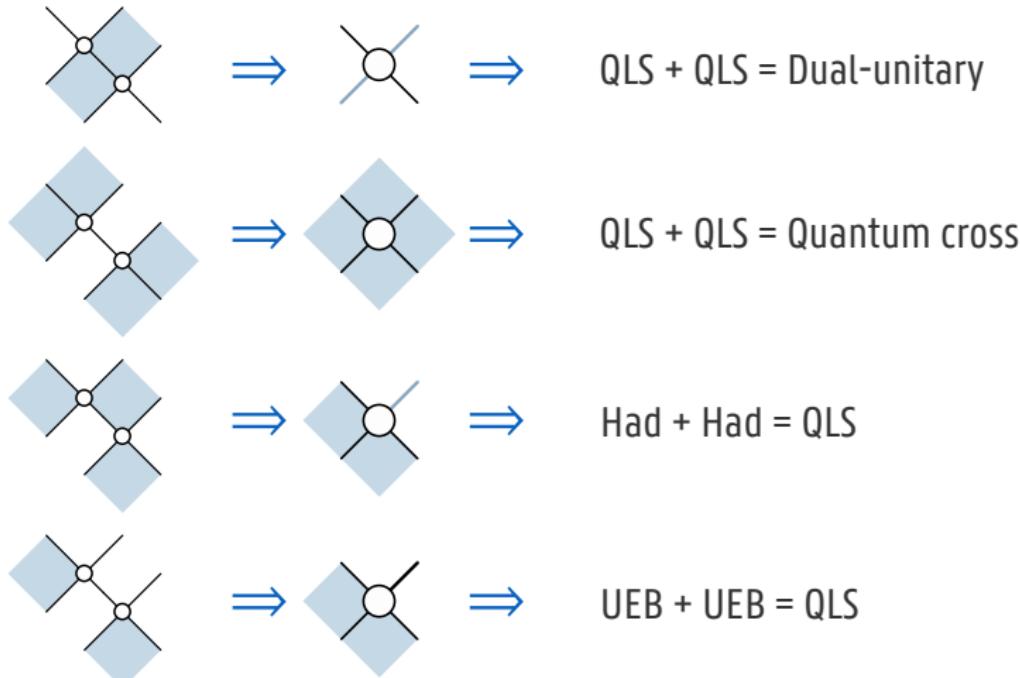
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Compositions

- Diagonal compositions of biunitaries return biunitaries



1. Dual-unitary circuits
2. Shaded calculus
3. Biunitary circuits
- 4. Solvable states**

Solvable initial states

- Dynamics so far restricted to initial (ultralocal) operators

Solvable initial states

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- What about initial states?

Solvable initial states

- Dynamics so far restricted to initial (ultralocal) operators
- What about initial states?

Class of solvable **matrix product states** introduced by Piroli et al.

$$\begin{aligned} |\Psi(\mathcal{N})\rangle &= \sum_{\{i_j\}}^q \dots \text{---} \overset{i_1}{|} \text{---} \overset{i_2}{|} \text{---} \mathcal{N} \text{---} \text{---} \overset{i_3}{|} \text{---} \overset{i_4}{|} \text{---} \mathcal{N} \text{---} \text{---} \overset{i_5}{|} \text{---} \overset{i_6}{|} \text{---} \mathcal{N} \text{---} \dots | \dots i_1 i_2 i_3 i_4 i_5 i_6 \dots \rangle \\ &= \sum_{\{i_j\}}^q \text{Tr} \left[\dots \mathcal{N}^{(i_1, i_2)} \mathcal{N}^{(i_3, i_4)} \mathcal{N}^{(i_5, i_6)} \dots \right] | \dots i_1 i_2 i_3 i_4 i_5 i_6 \dots \rangle \end{aligned}$$

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- Consistent with dual-unitarity if \mathcal{N} satisfies some (graphical) identities

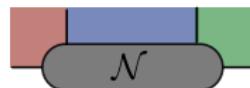
Solvable tensors

- Start from tensor \mathcal{N} with arbitrary shading pattern



Solvable tensors

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- Satisfies notion of **horizontal unitarity**

$$\begin{array}{c} \text{red} \\ \text{blue} \\ \text{green} \end{array} \begin{matrix} \text{---} \\ \text{---} \\ \text{---} \end{matrix} \begin{matrix} \mathcal{N}^\dagger \\ \mathcal{N} \end{matrix} = \begin{array}{c} \text{red} \end{array} \quad \text{and} \quad \begin{array}{c} \text{red} \\ \text{blue} \\ \text{green} \end{array} \begin{matrix} \text{---} \\ \text{---} \\ \text{---} \end{matrix} \begin{matrix} \mathcal{N}^\dagger \\ \mathcal{N} \end{matrix} = \begin{array}{c} \text{green} \end{array}$$

Solvable tensors

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$$\begin{array}{c} \mathcal{N}^\dagger \\ \diagdown \quad \diagup \\ \text{red} \quad \text{blue} \quad \text{green} \\ \diagup \quad \diagdown \\ \mathcal{N} \end{array} = \begin{array}{c} \text{red} \\ \text{dome} \end{array} \quad \text{and} \quad \begin{array}{c} \mathcal{N}^\dagger \\ \diagdown \quad \diagup \\ \text{red} \quad \text{blue} \quad \text{green} \\ \diagup \quad \diagdown \\ \mathcal{N} \end{array} = \begin{array}{c} \text{green} \\ \text{dome} \end{array}$$

- Transfer matrix has a **nondegenerate leading eigenvalue**

$$\begin{array}{c} \mathcal{N}^\dagger \\ \diagdown \quad \diagup \\ \text{red} \quad \text{blue} \quad \text{green} \\ \diagup \quad \diagdown \\ \mathcal{N} \end{array}$$

with

$$\begin{array}{c} \mathcal{N}^\dagger \\ \diagdown \quad \diagup \\ \text{red} \quad \text{blue} \quad \text{green} \\ \diagup \quad \diagdown \\ \mathcal{N} \end{array} = \begin{array}{c} \text{red} \\ \text{dome} \end{array}$$
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Identical to Piroli et al., Phys Rev. B 101, 094304 (2020)

Entanglement dynamics

- Initial state constructed from solvable tensors

$$|\Psi(\{\mathcal{N}\})\rangle = \text{[Diagram of a chain of 10 tensors]} \quad |$$

Entanglement dynamics

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$$|\Psi(\{\mathcal{N}\})\rangle = \text{[Diagram showing a chain of 10 rectangular blocks, where the first one is shaded dark grey and the others are light grey]} \quad \text{[Diagram showing a chain of 10 rectangular blocks, where the first one is shaded dark grey and the others are light grey]}$$

- Reduced density matrix $\rho_A(t) = \text{Tr}_{\bar{A}} (|\Psi(t, \{\mathcal{N}\})\rangle \langle \Psi(t, \{\mathcal{N}\})|)$

$$|\Psi(t, \{\mathcal{N}\})\rangle = \text{[Diagram showing a 5x5 grid of nodes connected by lines, with a horizontal row of 5 shaded rectangular blocks at the bottom]} \quad \text{[Diagram showing a 5x5 grid of nodes connected by lines, with a horizontal row of 5 shaded rectangular blocks at the bottom]}$$

Entanglement dynamics

- Initial state constructed from solvable tensors

$$|\Psi(\{\mathcal{N}\})\rangle = \text{[Diagram of a chain of 10 tensors, each represented by a grey rectangle with a small black dot at its center.]}$$

- Reduced density matrix $\rho_A(t) = \text{Tr}_{\bar{A}} (|\Psi(t, \{\mathcal{N}\})\rangle \langle \Psi(t, \{\mathcal{N}\})|)$

$$\rho_A(t) = \text{[Diagram showing the reduced density matrix. It consists of three horizontal layers. The top layer is a grid of nodes connected by lines, with a bracket above it labeled 'A'. The middle layer is a horizontal band of nodes, each connected to the grid above and below it. The bottom layer is a horizontal band of nodes, each connected to the middle layer above and below it. The bottom layer is shaded pink.]}$$

Entanglement dynamics

- Initial state constructed from solvable tensors

$$|\Psi(\{\mathcal{N}\})\rangle = \text{[Diagram showing a chain of 10 gray circles connected by vertical lines, representing a product state.]}$$

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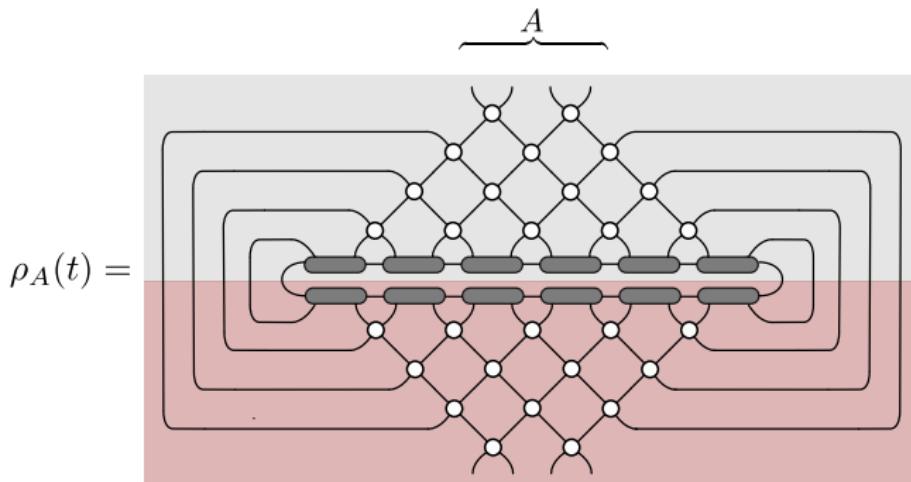
$$\rho_A(t) = \text{[Diagram showing two layers of a 2D lattice. The top layer is shaded gray and labeled A, with a bracket above it. The bottom layer is shaded pink. Both layers have nodes connected by a grid of lines. The top layer's connections are mostly within its own layer, while the bottom layer's connections are mostly within its own layer, illustrating entanglement between them.]}$$

Entanglement dynamics

- Initial state constructed from solvable tensors

$$|\Psi(\{\mathcal{N}\})\rangle = \text{[Diagram of a chain of 10 tensors, each represented by a grey oval with two horizontal lines extending to the left and right, connected sequentially.]}$$

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Entanglement dynamics

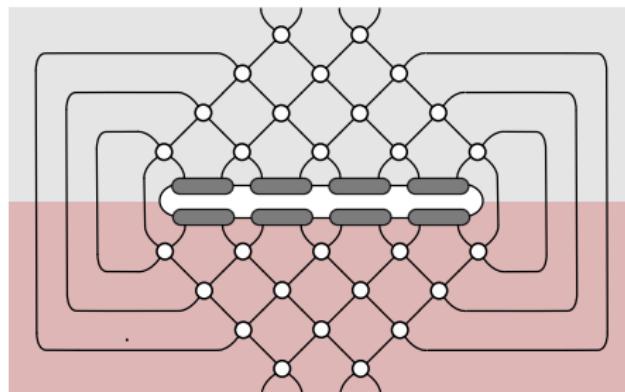
- Initial state constructed from solvable tensors

$$|\Psi(\{\mathcal{N}\})\rangle = \text{[Diagram of a chain of 10 tensors, each represented by a grey oval with a small vertical bar on its left side, connected by horizontal lines.]}$$

- Reduced density matrix $\rho_A(t) = \text{Tr}_{\bar{A}} (|\Psi(t, \{\mathcal{N}\})\rangle \langle \Psi(t, \{\mathcal{N}\})|)$

$\overbrace{\hspace{10em}}$
 A

$$\rho_A(t) =$$



Entanglement dynamics

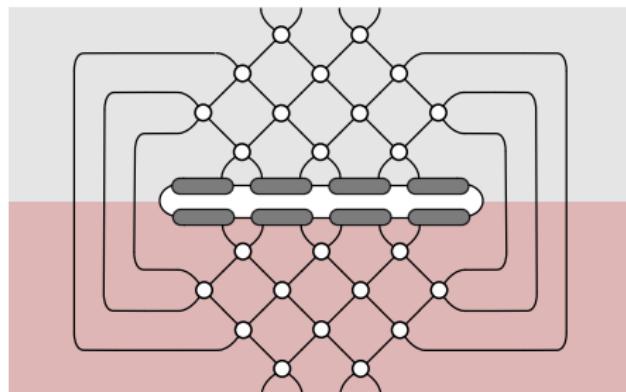
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$\overbrace{\hspace{10em}}$
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Entanglement dynamics

- Initial state constructed from solvable tensors

$$|\Psi(\{\mathcal{N}\})\rangle = \text{[Diagram of a tensor network with 11 nodes, where the first node is a small grey circle and the remaining 10 nodes are larger grey rectangles connected by vertical lines.]}$$

- Reduced density matrix $\rho_A(t) = \text{Tr}_{\bar{A}} (|\Psi(t, \{\mathcal{N}\})\rangle \langle \Psi(t, \{\mathcal{N}\})|)$

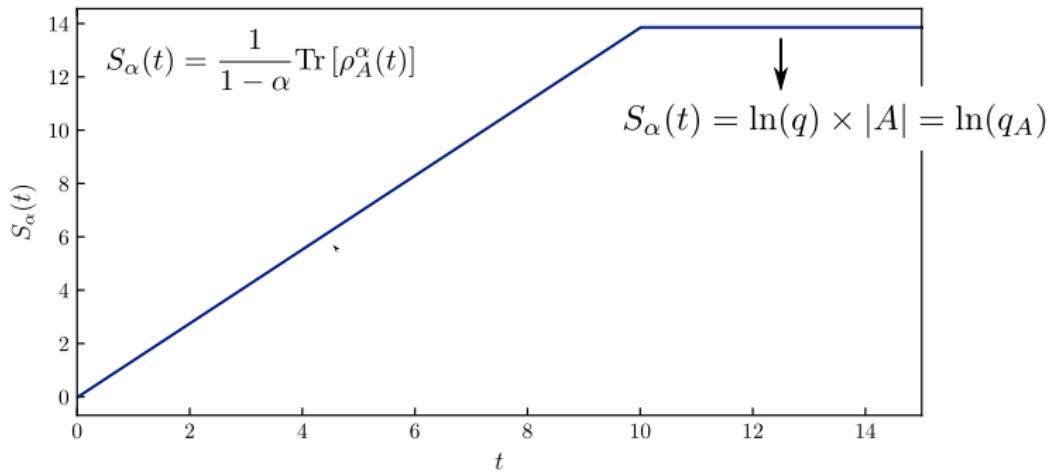
$$\rho_A(t) = \underbrace{\text{[Diagram of a tensor network with 5 nodes, where the top row consists of 5 vertical light blue rectangles labeled 'A' above them, and the bottom row consists of 5 vertical red rectangles.]}}$$

Entanglement dynamics

- Biunitary circuits **exactly thermalize** $\rho_A(t \geq |A|/2) \propto \mathbb{1}$

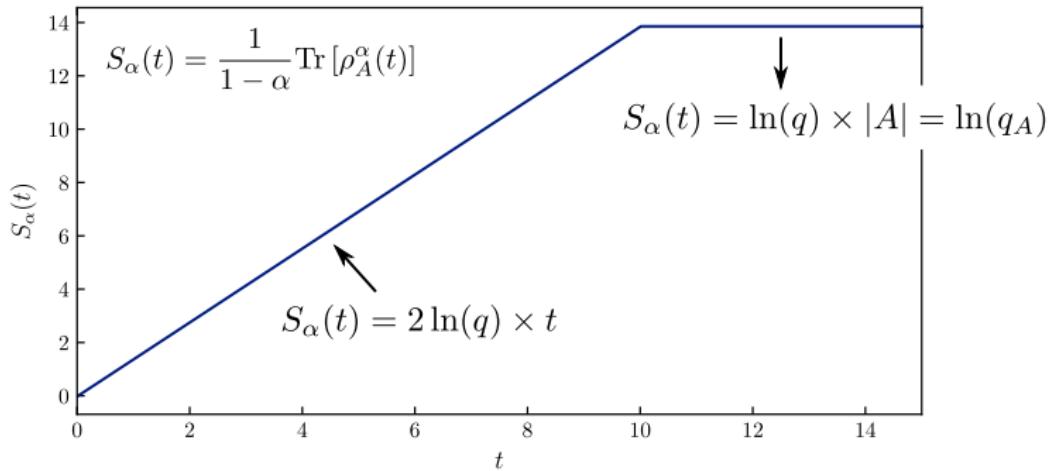
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- ... with **maximal entanglement entropy growth**



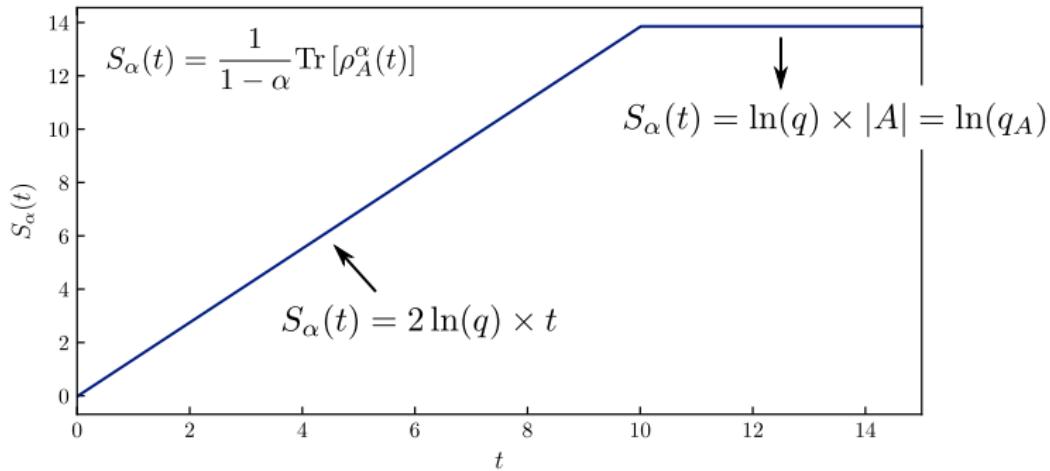
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- In scaling limit $x, t \rightarrow \infty$ with x/t fixed

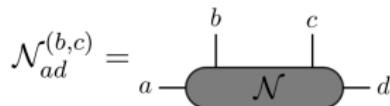
Solvable initial states

- No shading: returns solvable matrix product state

$$\mathcal{N}_{ad}^{(b,c)} = \begin{array}{c} b \\ | \\ \text{\textcolor{gray}{N}} \\ | \\ c \\ a - - d \end{array}$$

Solvable initial states

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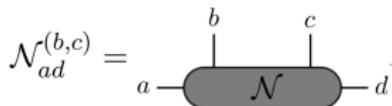
$$\mathcal{N}_{ad}^{(b,c)} = \begin{array}{c} b \\ | \\ a - \text{---} \text{---} \text{---} \\ | \\ c \\ | \\ d \end{array}$$
A diagram showing a horizontal grey oval labeled 'N' with two vertical lines extending from its left end to a point 'a' and two vertical lines extending from its right end to a point 'd'. Above the oval, there is a vertical line with a dot at its top labeled 'b' and another vertical line with a dot at its top labeled 'c'.

- Corresponding state

$$\begin{aligned} |\Psi(\mathcal{N})\rangle &= \sum_{\{i_j\}}^q \dots - \text{---} \text{---} \text{---} \\ &\quad | i_1 \rangle | N \rangle | i_2 \rangle | N \rangle | i_3 \rangle | N \rangle | i_4 \rangle | N \rangle | i_5 \rangle | N \rangle | i_6 \rangle \dots | \dots i_1 i_2 i_3 i_4 i_5 i_6 \dots \rangle \\ &= \sum_{\{i_j\}}^q \text{Tr} \left[\dots \mathcal{N}^{(i_1, i_2)} \mathcal{N}^{(i_3, i_4)} \mathcal{N}^{(i_5, i_6)} \dots \right] | \dots i_1 i_2 i_3 i_4 i_5 i_6 \dots \rangle \end{aligned}$$

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- Horizontal unitarity corresponds to unitarity of \mathcal{W}

$$\mathcal{W}_{ab,cd} = \mathcal{N}_{ad}^{(b,c)}$$

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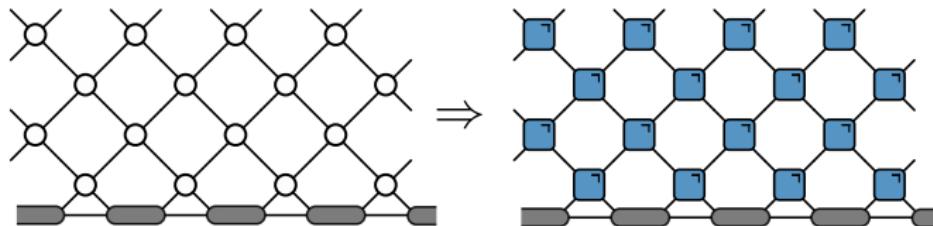
⇒ Parametrized by **unitary matrix**

Solvable initial states

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- Corresponding state



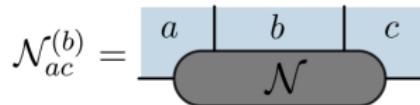
Solvable initial states

- Fully shaded = solvable initial state for clockwork circuits

$$\mathcal{N}_{ac}^{(b)} = \begin{array}{c} a \quad b \quad c \\ | \quad | \quad | \\ \text{\scriptsize \mathcal{N}} \end{array}$$

Solvable initial states

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- Corresponding state follows as

$$\begin{aligned} |\Psi(\mathcal{N})\rangle &= \sum_{i_1, i_2, \dots}^q \dots \begin{array}{c} | \quad i_1 \quad | \quad i_2 \quad | \quad i_3 \quad | \quad i_4 \quad | \quad i_5 \quad | \quad \dots \\ \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \\ \mathcal{N} \quad \mathcal{N} \quad \mathcal{N} \quad \mathcal{N} \quad \mathcal{N} \quad \dots \end{array} | \dots i_1 i_2 i_3 i_4 i_5 \dots \rangle \\ &= \sum_{..., i_1, i_2, \dots}^q \left[\dots \mathcal{N}_{..., i_2}^{(i_1)} \mathcal{N}_{i_2, i_4}^{(i_3)} \mathcal{N}_{i_4, \dots}^{(i_5)} \dots \right] | \dots i_1 i_2 i_3 i_4 i_5 i_6 \dots \rangle , \end{aligned}$$

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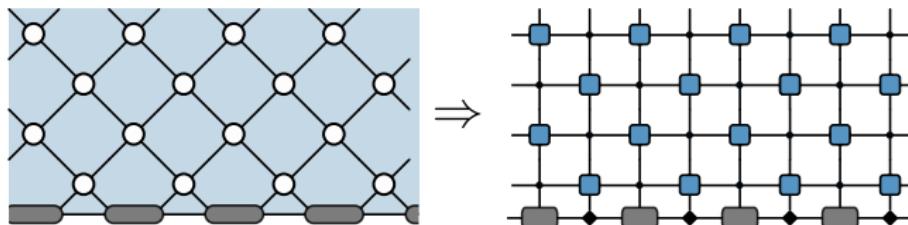
➡ Parametrized by **set of unitary matrices**

Solvable initial states

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Conclusions

- **Dual-unitary models** present a class of exactly solvable models for many-body dynamics...

Quantum 6, 738 (2022)

Phys. Rev. Res. 4, 043212 (2022)

arXiv:2302.07280 (2023)

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Dual-unitary gates, quantum crosses, unitary error bases, complex Hadamard matrices, quantum Latin squares

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- Any graphical proof for dual-unitarity **directly extends to biunitarity**

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THANK YOU FOR YOUR ATTENTION!

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