MAX-PLANCK-GESELLSCHAFT

mpipks

Exactly-solvable many-body quantum dynamics in biunitary circuits

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Quantum 6, 738 (2022)

Phys. Rev. Res. 4, 043212 (2022)

arXiv:2302.07280 (2023)

In collaboration with

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University of Cambridge

Jamie Vicary
University of Cambridge

Quantum 6, 738 (2022)
Phys. Rev. Res. 4, 043212 (2022)
arXiv:2302.07280 (2023)

1. Dual-unitary circuits
2. Shaded calculus
3. Biunitary circuits
4. Solvable states

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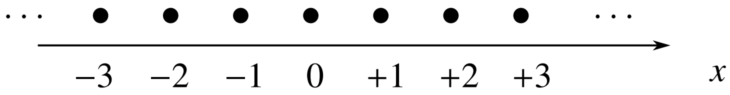
2. Shaded calculus

3. Biunitary circuits

4. Solvable states

Local unitary dynamics

- **Setting**: 1+1D lattice models



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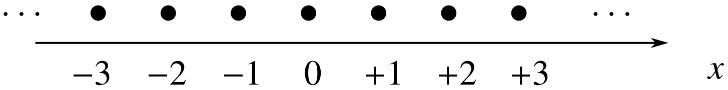

- Hamiltonian evolution with **local** interactions

$$U(t) = \exp\left[-iHt\right] \quad \text{with} \quad H = \sum_i h_{i,i+1}$$

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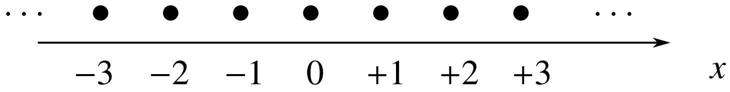


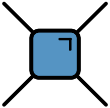
- Hamiltonian evolution with **local** interactions

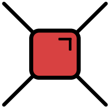
$$U(t) = \exp\left[-iHt\right] \quad \text{with} \quad H = \sum_i h_{i,i+1}$$

$\Longrightarrow$ Can we find **minimal models** for many-body dynamics?

Unitary circuits

- Consider time evolution generated by two-site **unitary circuits**

$U =$ 

$U^\dagger =$ 

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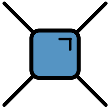
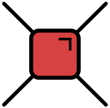
$U =$  $U^\dagger =$ 

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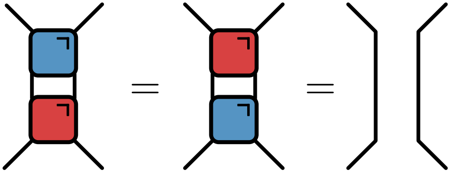

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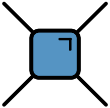
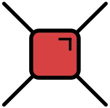
$U =$  $U^\dagger =$ 

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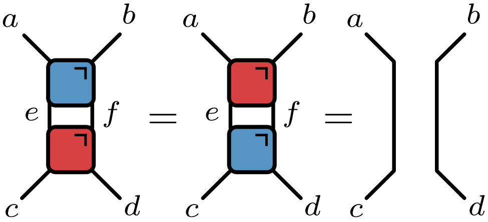

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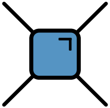
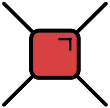
$U =$  $U^\dagger =$ 

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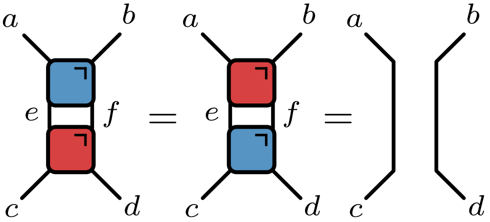

$$\sum_{ef} U_{ab,ef}U^\dagger_{ef,cd} = \sum_{ef} U^\dagger_{ab,ef}U_{ef,cd} = \delta_{ac}\delta_{bd}$$

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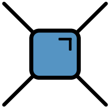
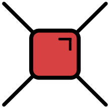
$U =$  $U^\dagger =$ 

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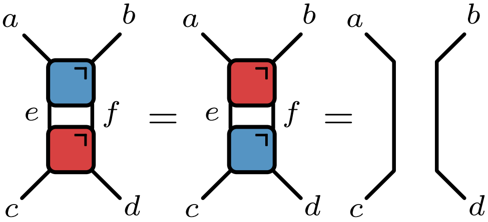

$$UU^\dagger = U^\dagger U = \mathbb{1}$$

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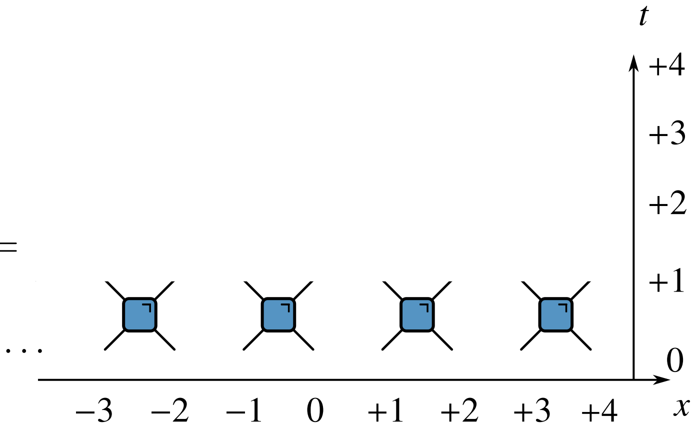

- Infinite lattice, discrete time

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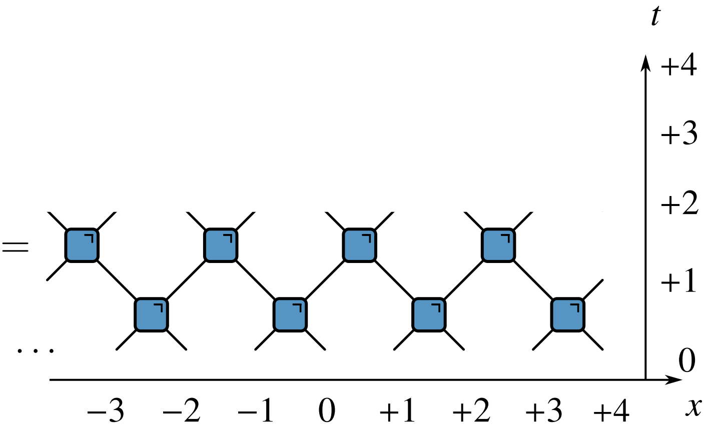

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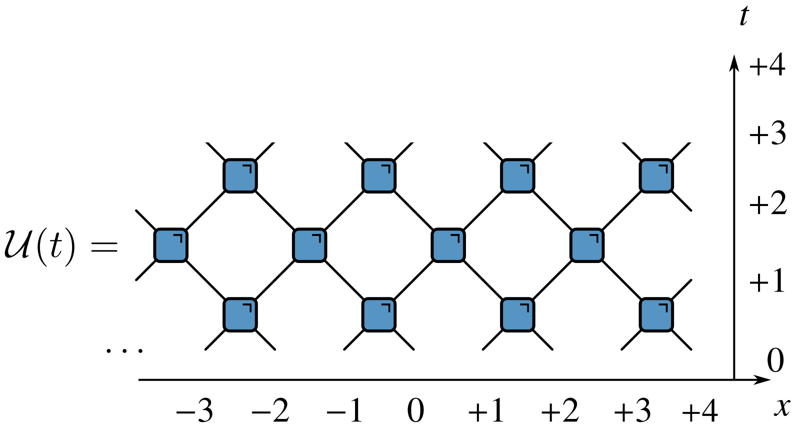

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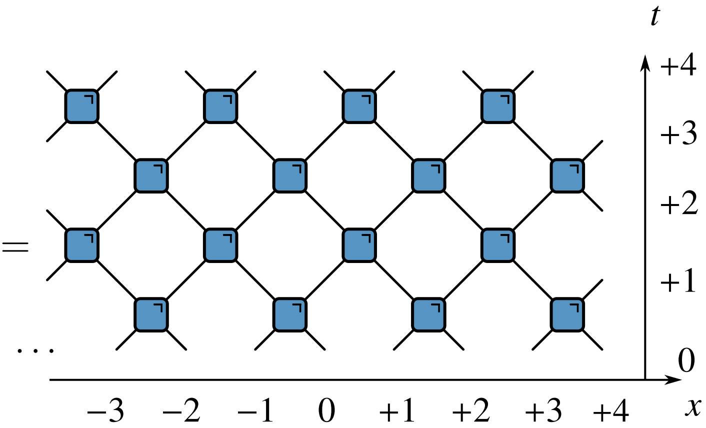

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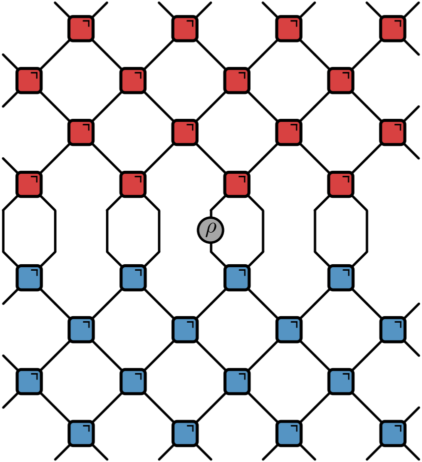
Causal light cone

- Light cone baked in!

$$\mathcal{U}^\dagger(t)\rho(0)\mathcal{U}(t) = \cdots$$



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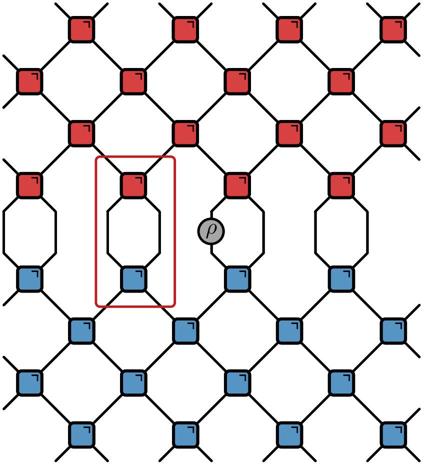
$$\mathcal{U}^\dagger(t)\rho(0)\mathcal{U}(t) = \cdots$$  $$\cdots$$

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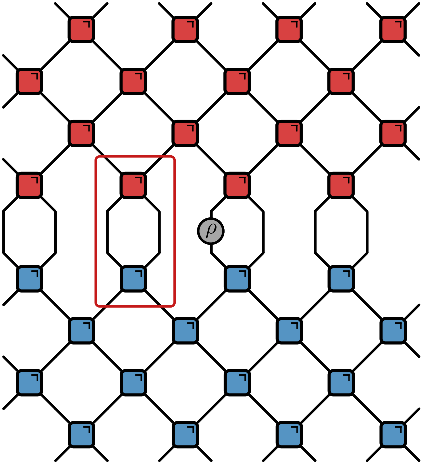

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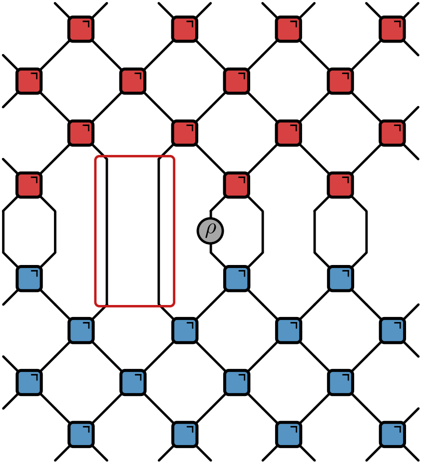

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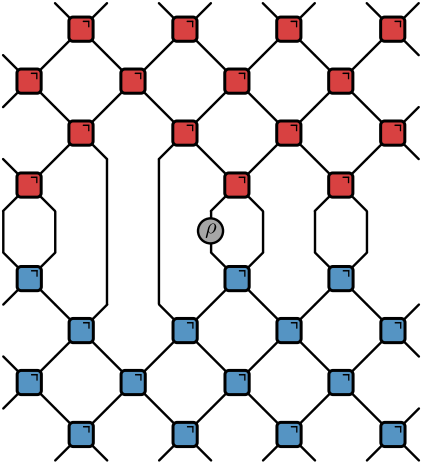

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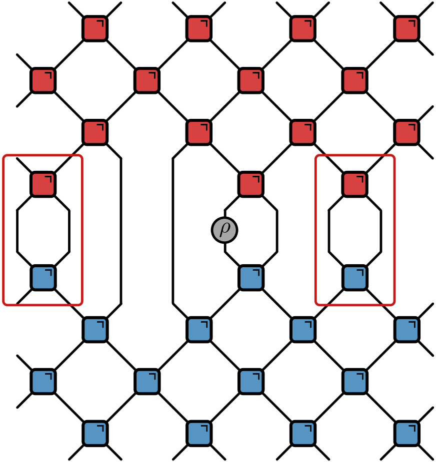

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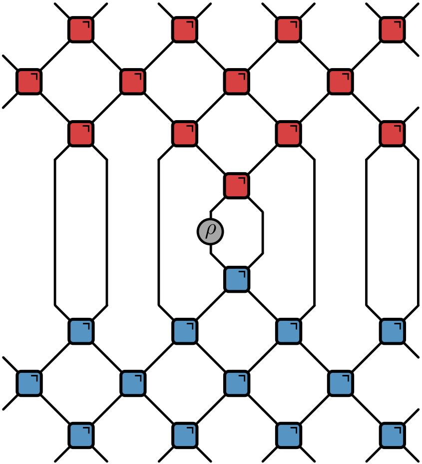

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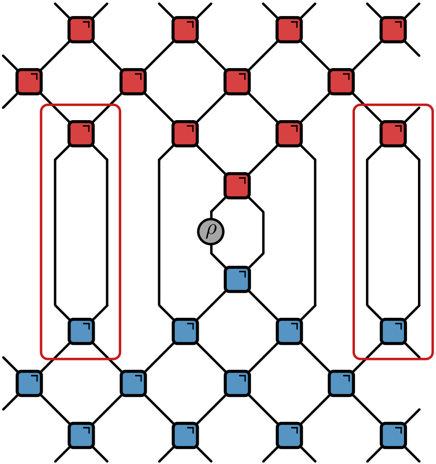

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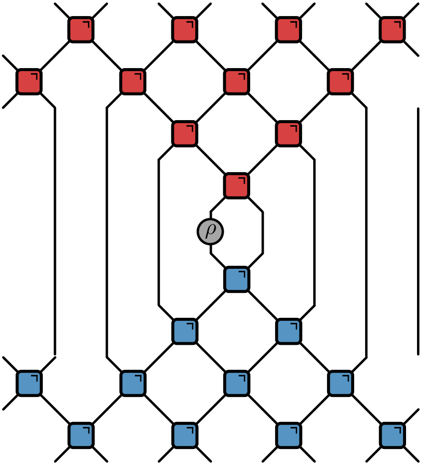

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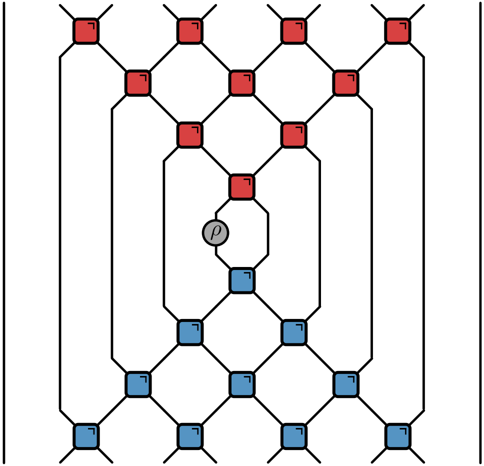
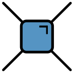
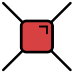


Dual-unitary gates

- Time evolution generated by two-site **unitary gates**

$U =$  $U^\dagger =$ 

B. Bertini, P. Kos, and T. Prosen, Phys. Rev. Lett. 123, 210601 (2019)

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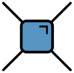
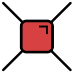
$U =$  $\qquad U^\dagger =$ 

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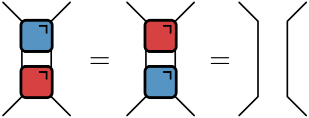

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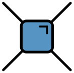
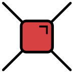
$U =$  $U^\dagger =$ 

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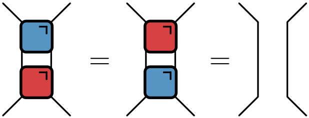
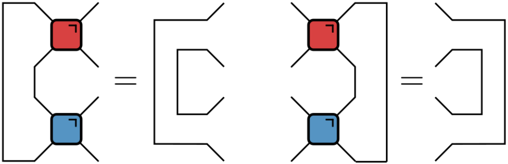


- Dual-unitary gates additionally satisfy



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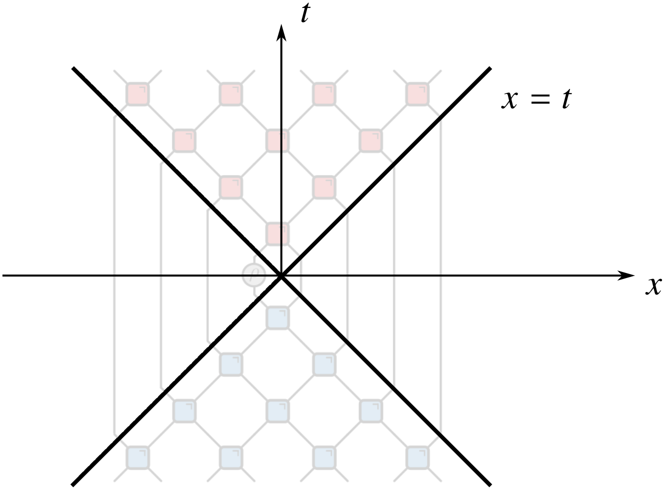
Effect of dual-unitarity

- Correlation functions $c_{\rho\sigma}(x,t) \sim \mathrm{Tr}\left[\rho(0,t)\sigma(x)\right]$



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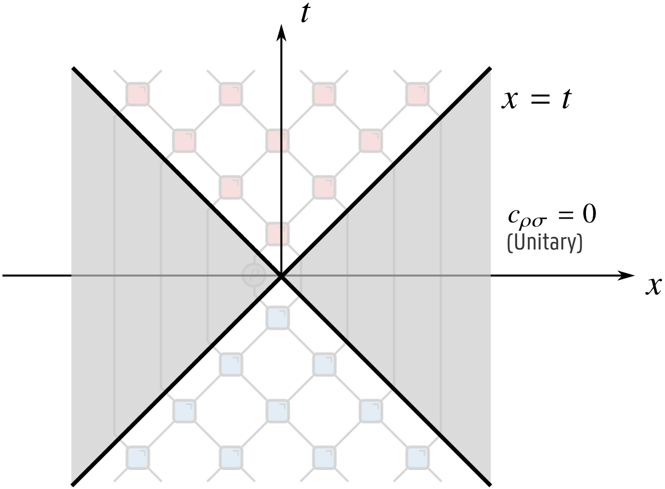

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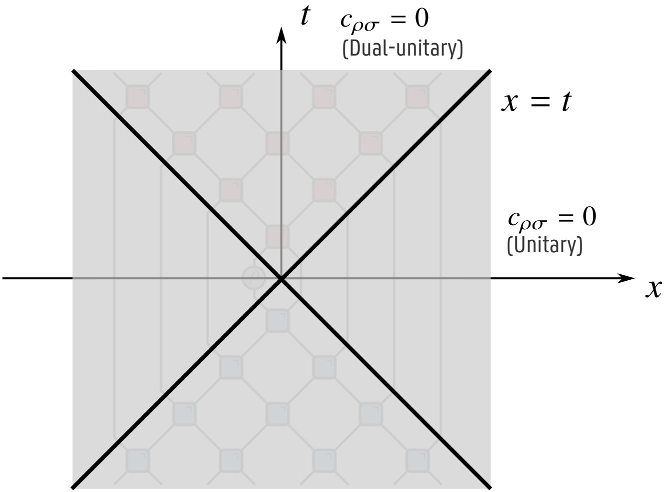

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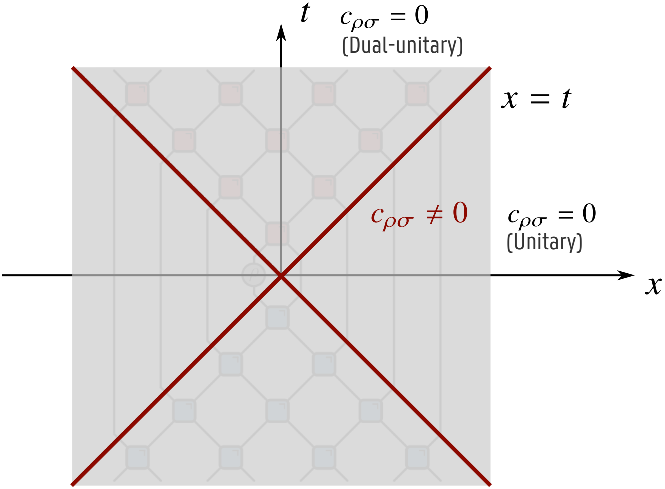

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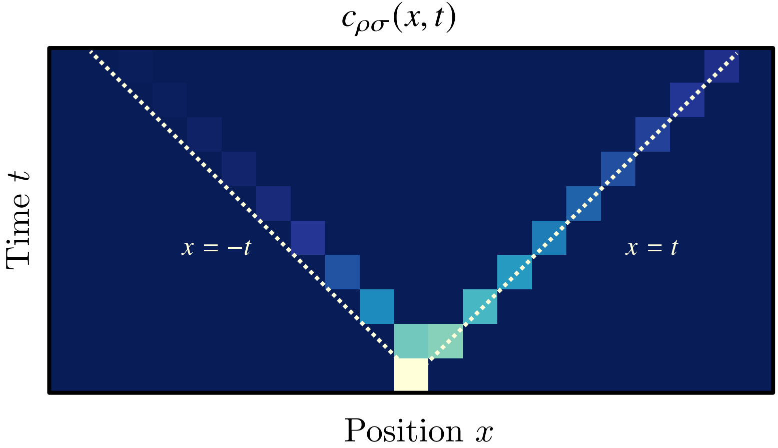
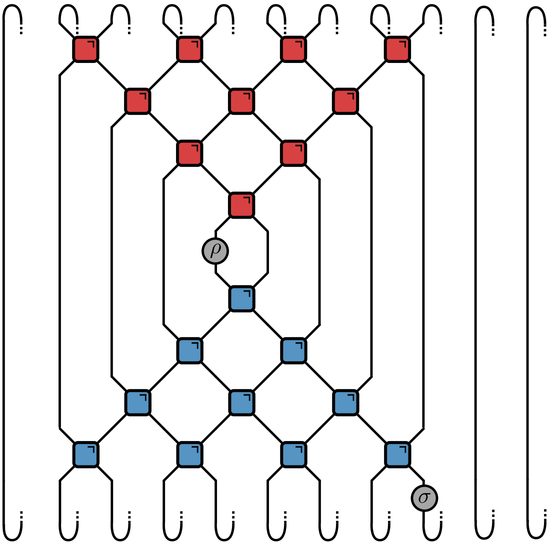


Light cone correlations

- Correlations on light cone $\mathrm{Tr}\left[\mathcal{U}^\dagger(t)\rho(0)\mathcal{U}(t)\sigma(x=t)\right]$



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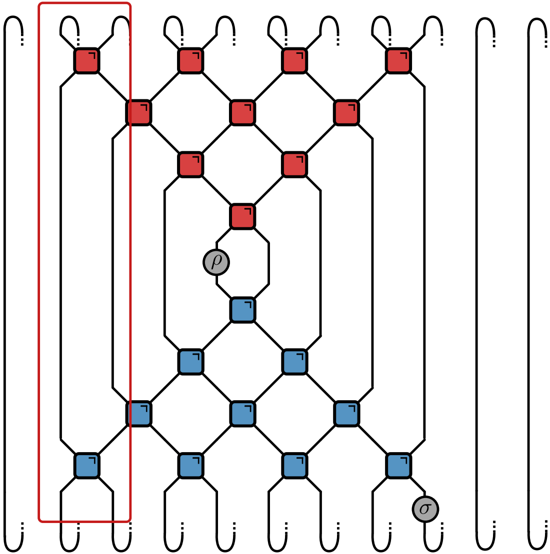

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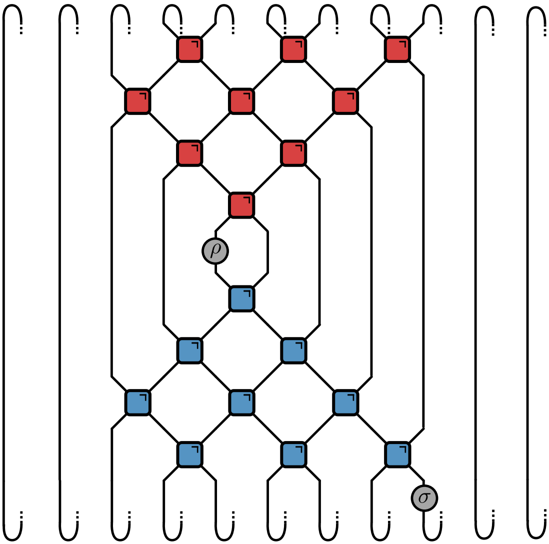

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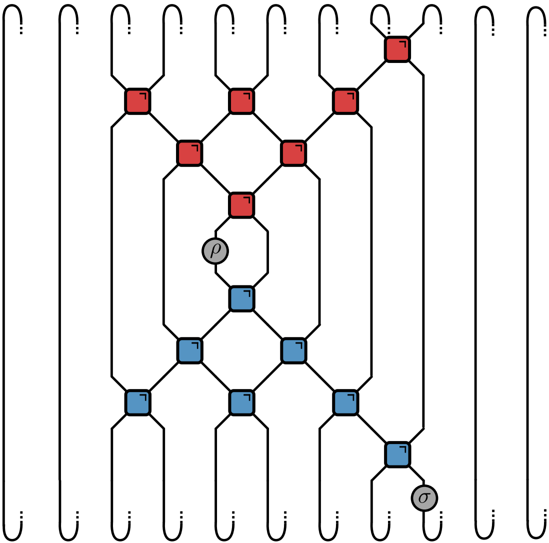

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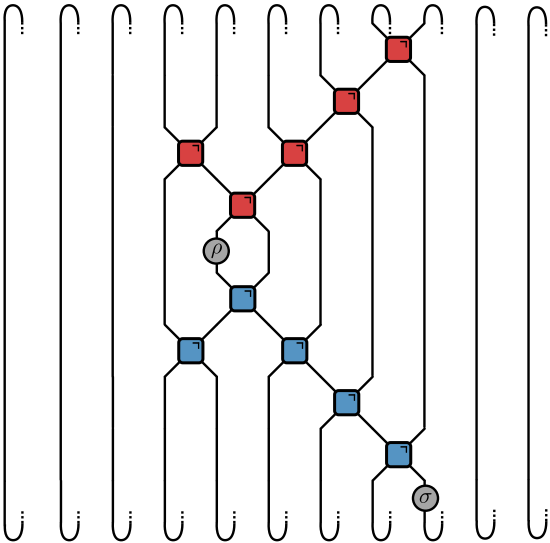

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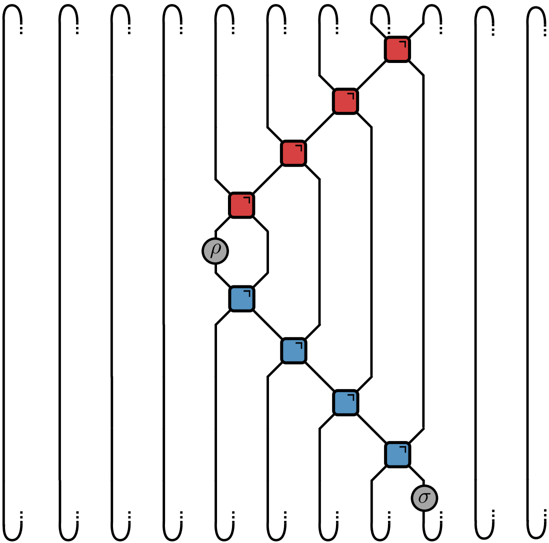
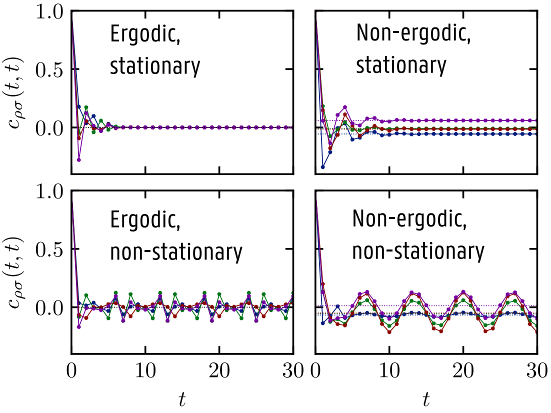


Light-cone correlations

- Can be analytically calculated for arbitrary long times
- Different (dual-unitary) circuits $\Longrightarrow$ different dynamics



Bertini, Kos, and Prosen, Phys. Rev. Lett. 123, 210601 (2019)

Claeys and Lamacraft, Phys. Rev. Lett. 126, 100603 (2021)

Motivation

Dual-unitary circuits are a minimal model of **local** and **unitary** many-body dynamics with many interesting properties

S. Gopalakrishnan and A. Lamacraft, Phys. Rev. B 100, 064309 (2019)
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P. W. Claeys and A. Lamacraft, Phys. Rev. Research 2, 033032 (2020)
L. Piroli, B. Bertini, J. I. Cirac, and T. Prosen, Phys. Rev. B 101, 094304 (2020)
... and many more

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- **Exact solvability**. Supporting exact calculations of thermalization, scrambling, entanglement growth
- **Maximally chaotic.** Allow for analytic connections with random matrix theory

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B. Bertini, P. Kos, and T. Prosen, Phys. Rev. Lett. 123, 210601 (2019)
B. Bertini, P. Kos, and T. Prosen, SciPost Physics 8, 067 (2020)
P. W. Claeys and A. Lamacraft, Phys. Rev. Research 2, 033032 (2020)
L. Piroli, B. Bertini, J. I. Cirac, and T. Prosen, Phys. Rev. B 101, 094304 (2020)
... and many more

Motivation

Dual-unitary circuits are a minimal model of **local** and **unitary** many-body dynamics with many interesting properties

- **Exact solvability**. Supporting exact calculations of thermalization, scrambling, entanglement growth
- **Maximally chaotic.** Allow for analytic connections with random matrix theory
- **Maximal entanglement velocity**. Entanglement spreads at fastest possible rate.

S. Gopalakrishnan and A. Lamacraft, Phys. Rev. B 100, 064309 (2019)
B. Bertini, P. Kos, and T. Prosen, Phys. Rev. Lett. 123, 210601 (2019)
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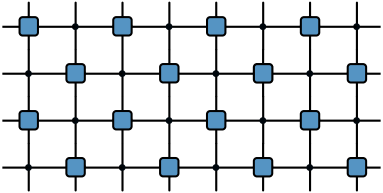
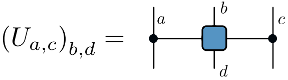
Dual-unitarity round-a-face

- Circuits introduced in Prosen, Chaos 31, 093101 (2021) with similar properties, built from 1-site 2-controlled gates





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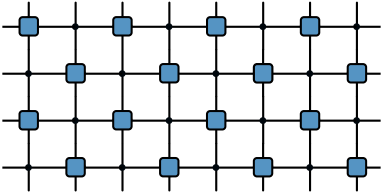
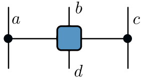


$(U_{a,c})_{b,d} =$ 

- **Dual-unitarity**: enforce unitarity when exchanging indices

  $\Longrightarrow$ $\tilde{U}_{a,c}$ unitary with $(\tilde{U}_{a,c})_{b,d} = (U_{b,d})_{a,c}$

Prosen, Chaos 31, 093101 (2021)

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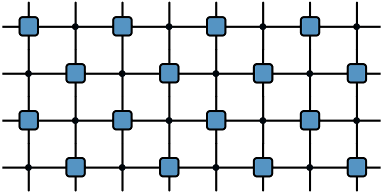

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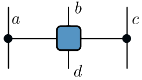

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- Similar **light-cone dynamics** as in dual-unitary brickwork circuits

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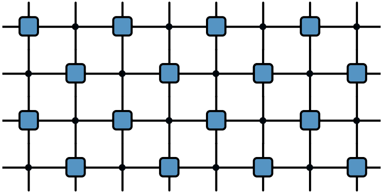

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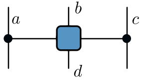

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  $\implies$ $\tilde{U}_{a,c}$ unitary with $(\tilde{U}_{a,c})_{b,d} = (U_{b,d})_{a,c}$

- Similar **light-cone dynamics** as in dual-unitary brickwork circuits
- Dual-unitary interactions round-a-face (DUIRF or 'clockwork')

Prosen, Chaos 31, 093101 (2021)

1. Dual-unitary circuits
2. **Shaded calculus**
3. Biunitary circuits
4. Solvable states

Shaded tensor networks

- Planar string diagram where some of the regions are shaded

D. J. Reutter and J. Vicary, Higher Structures 3, 109 (2019)

Shaded tensor networks

- Planar string diagram where some of the regions are shaded
- String diagram for a subcategory of the **2Hilb**, the 2-category of 2–Hilbert spaces

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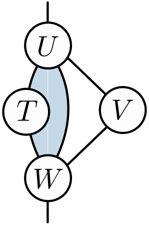

Shaded region = index

Closed shaded region = implicit summation

D. J. Reutter and J. Vicary, Higher Structures 3, 109 (2019)

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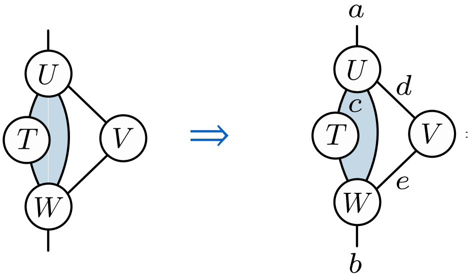

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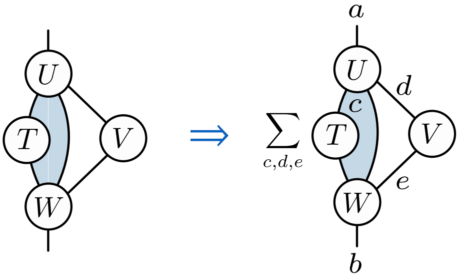

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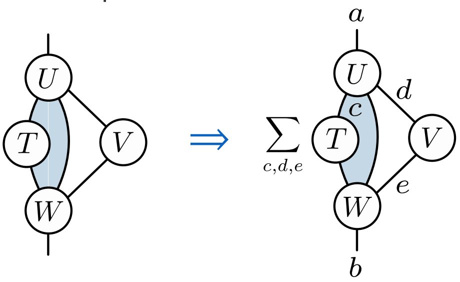

$$= \sum_{c,d,e} T_c U_{acd} V_{de} W_{ceb}$$

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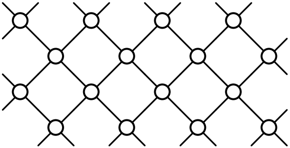
D. J. Reutter and J. Vicary, Higher Structures 3, 109 (2019)

Shaded calculus representation of circuits

No shading corresponds to **brickwork circuits**



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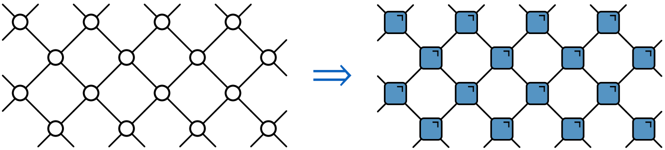

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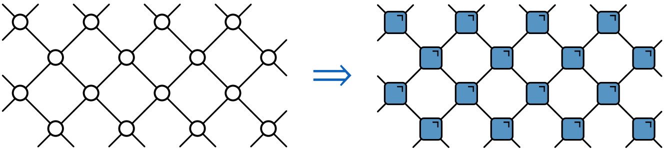
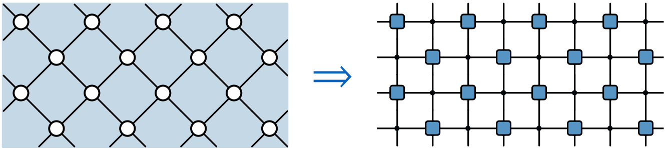


Fully shaded corresponds to **clockwork circuits**



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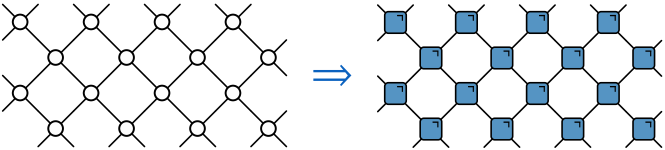

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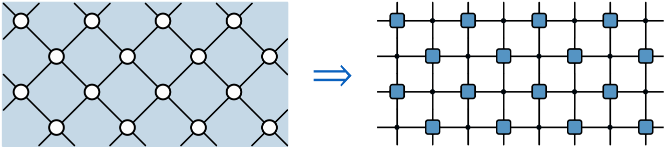
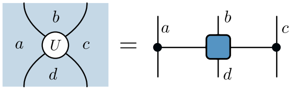


...writing



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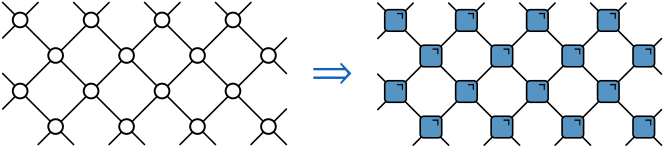

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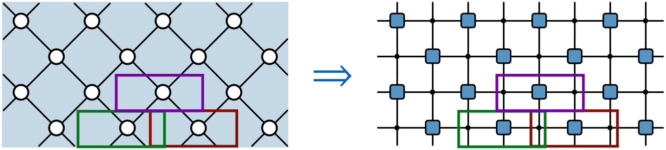

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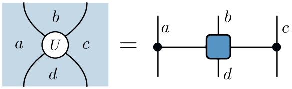
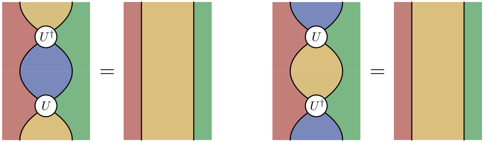


1. Dual-unitary circuits
2. Shaded calculus
3. **Biunitary circuits**
4. Solvable states

Biunitarity

- In shaded tensor networks, a 4-valent vertex U can be **vertically unitary:**



D. J. Reutter and J. Vicary, Higher Structures 3, 109 (2019)

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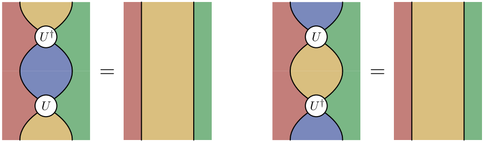
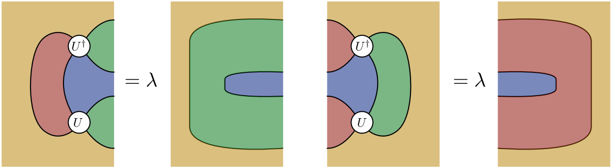


- It can also be **horizontally unitary**:



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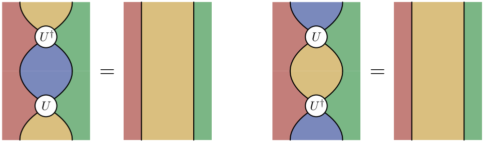

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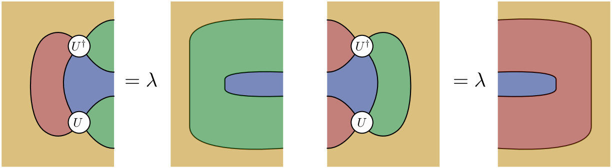


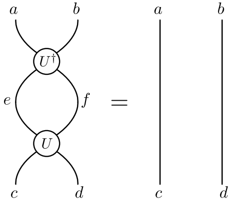
A 4-valent map is **biunitary** when it is **vertically and horizontally unitary.**

D. J. Reutter and J. Vicary, Higher Structures 3, 109 (2019)

Dual-unitarity from biunitarity

- A biunitary without any shading corresponds to a **dual-unitary gate**

Vertical unitarity:



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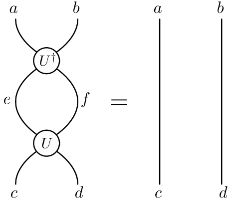

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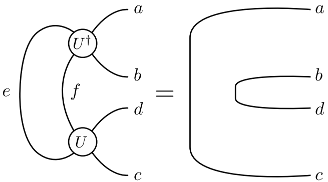

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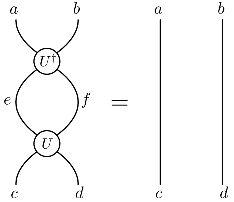
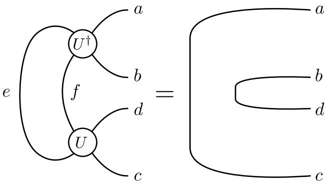


$$\sum_{e,f=1}^{q} U^{\dagger}_{ab,ef} U_{ef,cd} = \delta_{ac}\delta_{bd}$$

Horizontal unitarity:



Dual-unitarity from biunitarity

- A fully shaded biunitary corresponds to **dual-unitary clockwork**

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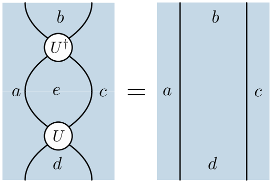

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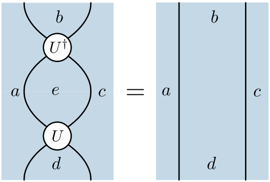

$$\sum_e (U_{a,c})^\dagger_{b,e} (U_{a,c})_{e,d} = \delta_{bd}, \qquad \forall a, c$$

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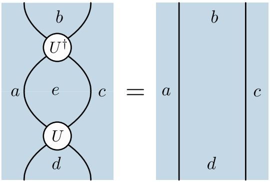
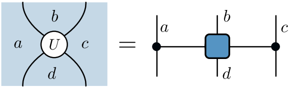


$$\sum_e \left(U_{a,c}\right)^\dagger_{b,e} \left(U_{a,c}\right)_{e,d} = \delta_{bd}, \qquad \forall a, c$$

- Can express fully shaded biunitary as **2-controlled 1-site unitary**

$$\left(U_{a,c}\right)_{b,d} =$$



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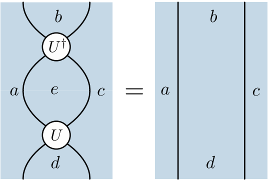

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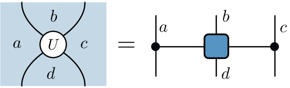

- **Horizontal** unitarity enforces unitarity of $\tilde{U}_{a,c}$ with

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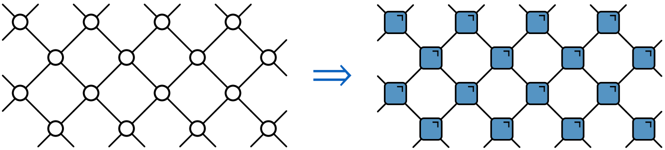
Shaded calculus representation of circuits

No shading corresponds to **brickwork circuits**



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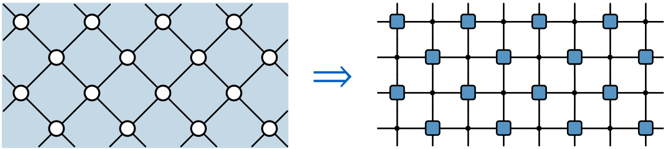
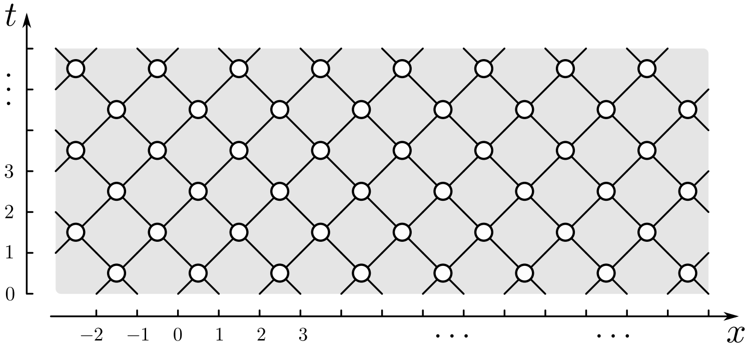


Biunitary circuits

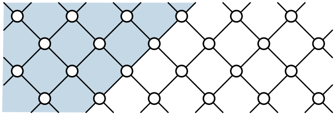
- We consider lattices where **every vertex is biunitary**



Gray background = region can be **shaded or not**

Heterogeneous biunitary circuits

- We can vary the shading pattern...



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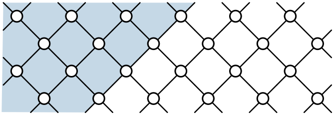

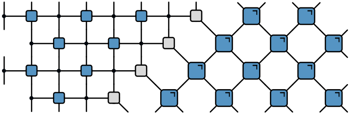

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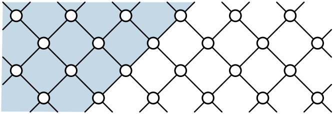
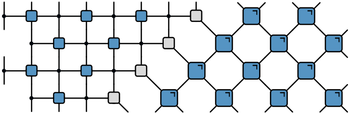
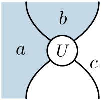
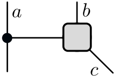




- .... by introducing new elements

$(U_{a,b})_c =$ 

$=$ 

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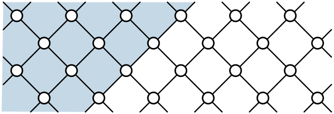

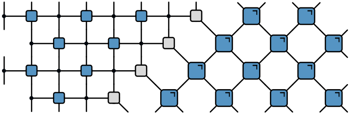

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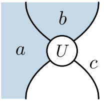
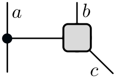
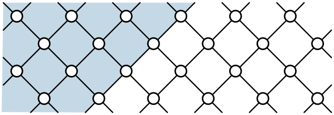
$(U_{a,b})_c =$  $=$ 

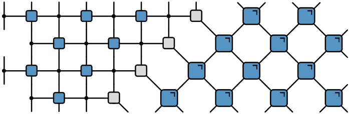
$\Longrightarrow$ Biunitarity implies that these vertices correspond to **quantum Latin squares**

B. Musto and J. Vicary, Quantum Info. Comput. 16, 1318 (2016)

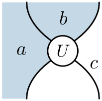
Heterogeneous biunitary circuits

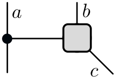
- We can vary the shading pattern...





- .... by introducing new elements

$(U_{a,b})_c =$ 

$=$ 

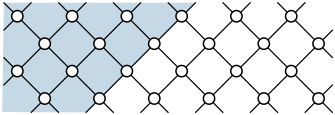
$\Longrightarrow$ Biunitarity implies that these vertices correspond to **quantum Latin squares**

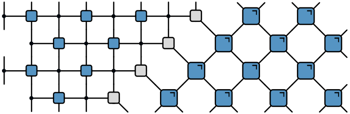
$\Longrightarrow$ Matrix of vectors $U_{a,b}$ for which every row and column forms an orthonormal basis

B. Musto and J. Vicary, Quantum Info. Comput. 16, 1318 (2016)

Heterogeneous biunitary circuits

- We can vary the shading pattern...





Latin square

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$$

Quantum Latin square

$$U = \begin{pmatrix} |1\rangle & |2\rangle & |3\rangle \\ |3\rangle & |1\rangle & |2\rangle \\ |2\rangle & |3\rangle & |1\rangle \end{pmatrix}$$

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Heterogeneous biunitary circuits

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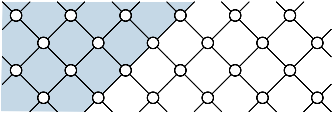
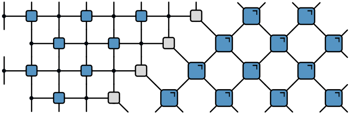


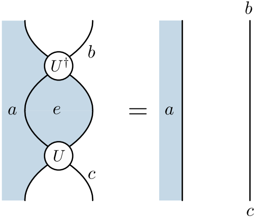


$\|0\rangle$	$\|1\rangle$	$\|2\rangle$	$\|3\rangle$
$\frac{1}{\sqrt{2}}(\|1\rangle - \|2\rangle)$	$\frac{1}{\sqrt{5}}(i\|0\rangle + 2\|3\rangle)$	$\frac{1}{\sqrt{5}}(2\|0\rangle + i\|3\rangle)$	$\frac{1}{\sqrt{2}}(\|1\rangle + \|2\rangle)$
$\frac{1}{\sqrt{2}}(\|1\rangle + \|2\rangle)$	$\frac{1}{\sqrt{5}}(2\|0\rangle + i\|3\rangle)$	$\frac{1}{\sqrt{5}}(i\|0\rangle + 2\|3\rangle)$	$\frac{1}{\sqrt{2}}(\|1\rangle - \|2\rangle)$
$\|3\rangle$	$\|2\rangle$	$\|1\rangle$	$\|0\rangle$

B. Musto and J. Vicary, Quantum Info. Comput. 16, 1318 (2016)

Quantum Latin squares

- **Vertical unitarity** determines the properties of the rows



D. J. Reutter and J. Vicary, Higher Structures 3, 109 (2019)

Quantum Latin squares

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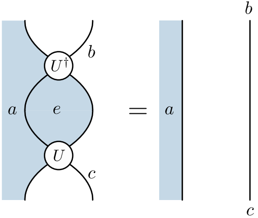

$$\sum_e \left(U_{a,e}\right)^\dagger_b \left(U_{a,e}\right)_c = \delta_{bc}, \quad \forall a$$

D. J. Reutter and J. Vicary, Higher Structures 3, 109 (2019)

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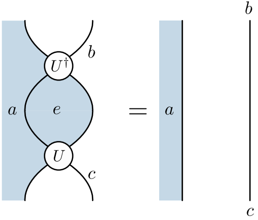

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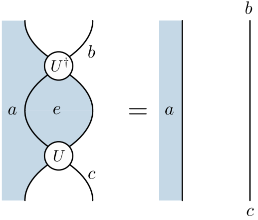

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Completeness

D. J. Reutter and J. Vicary, Higher Structures 3, 109 (2019)

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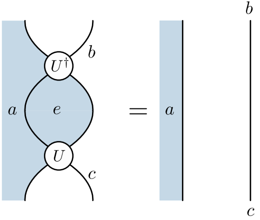

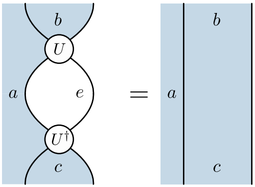

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D. J. Reutter and J. Vicary, Higher Structures 3, 109 (2019)

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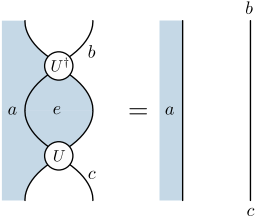

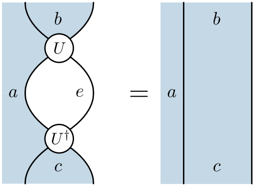

$$\sum_e (U_{a,e})^\dagger_b \, (U_{a,e})_c = \delta_{bc}, \quad \forall a$$

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Completeness

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Quantum Latin squares

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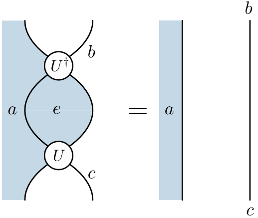
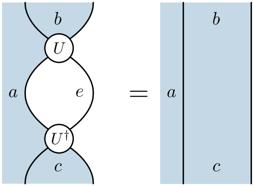




$$\sum_e \left(U_{a,e}\right)^\dagger_b \left(U_{a,e}\right)_c = \delta_{bc}, \quad \forall a$$

$$\sum_e \left|U_{a,e}\right\rangle \left\langle U_{a,e}\right| = \mathbb{1}$$

Completeness

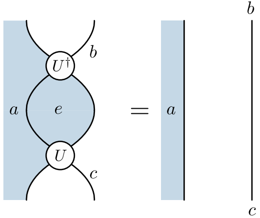
$$\sum_e \left(U_{a,b}\right)^\dagger_e \left(U_{a,c}\right)_e = \delta_{bc}, \quad \forall a$$

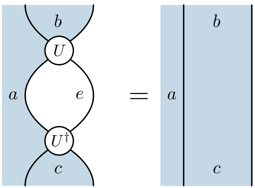
$$\langle U_{a,b} | U_{a,c} \rangle = \delta_{bc}$$

D. J. Reutter and J. Vicary, Higher Structures 3, 109 (2019)

Quantum Latin squares

- **Vertical unitarity** determines the properties of the rows





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Orthonormality

D. J. Reutter and J. Vicary, Higher Structures 3, 109 (2019)

Quantum Latin squares

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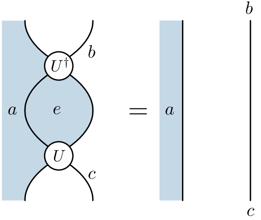

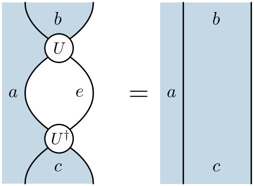

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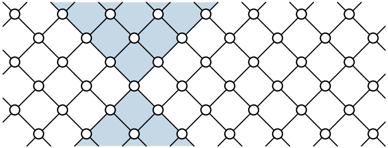
Orthonormality

- **Horitontal unitarity** fixes same property for the columns

D. J. Reutter and J. Vicary, Higher Structures 3, 109 (2019)

Heterogeneous biunitary circuits

- Introducing additional shading patterns



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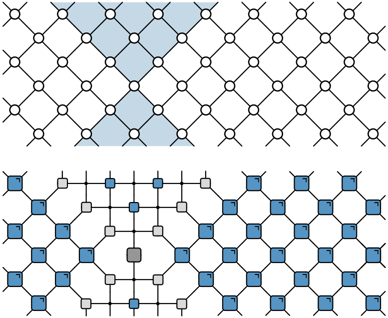

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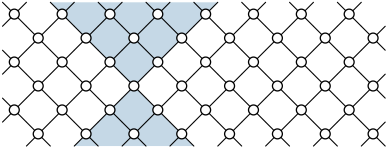
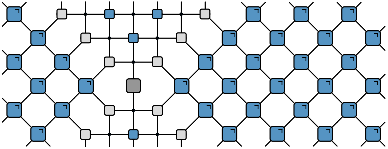
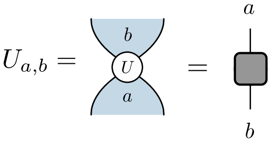


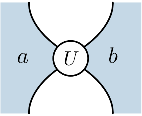


- ... by introducing new elements



Complex Hadamard matrix

- A biunitary with two opposite shaded regions

$U_{ab} =$ 

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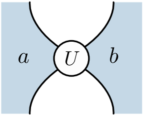
$U_{ab} =$ 

- Corresponds to a complex Hadamard matrix

$$U^\dagger U = U U^\dagger = q\,\mathbb{1} \qquad \text{and} \qquad |U_{ab}| = 1, \forall a, b$$

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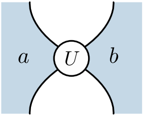
$U_{ab} =$ 

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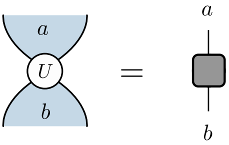
- Represents either **one-site unitary** or **two-site controlled phase**



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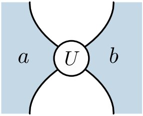
$U_{ab} =$ 

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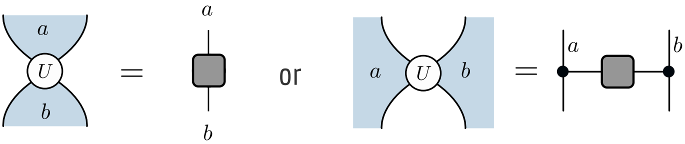

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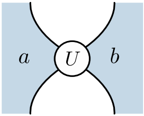
$U_{ab} =$ 

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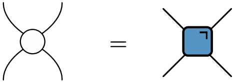
1-site unitary	2-controlled phase
$\sqrt{\frac{1}{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$	$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$

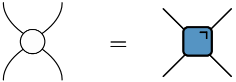
Classification of biunitaries

Dual-unitary gates 

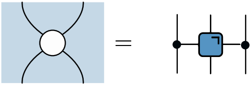
arXiv:2302.07280

Classification of biunitaries

Dual-unitary gates



Quantum crosses



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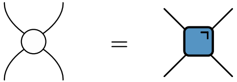

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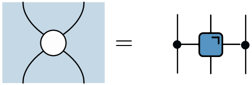
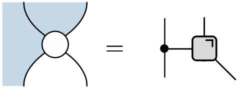


Quantum Latin squares



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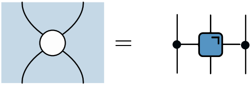

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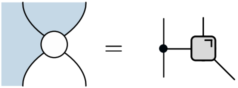
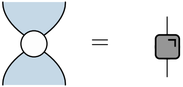


Complex Hadamard matrices



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Classification of biunitaries

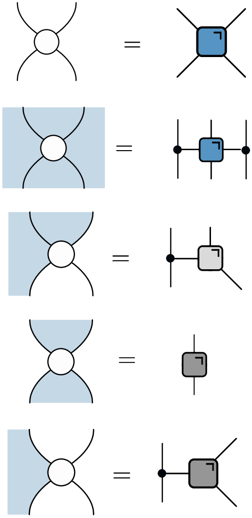
Dual-unitary gates

Quantum crosses

Quantum Latin squares

Complex Hadamard matrices

Unitary error bases



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Classification of biunitaries

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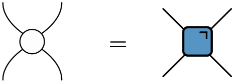

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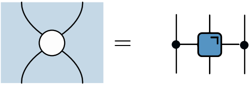
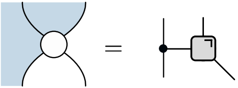
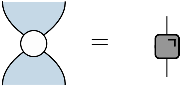


Quantum Latin squares

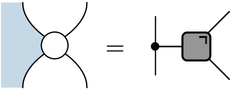


Complex Hadamard matrices



Unitary error bases

Orthogonal and complete set of unitaries, e.g. Pauli matrices



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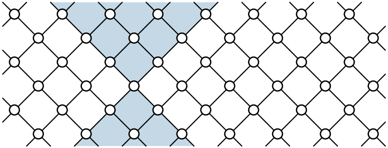
Zoo of biunitary circuits



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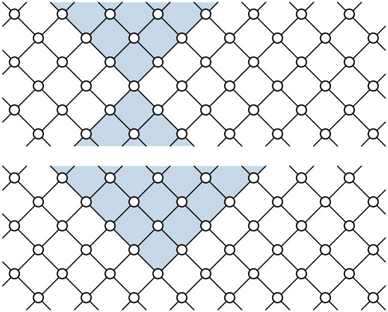

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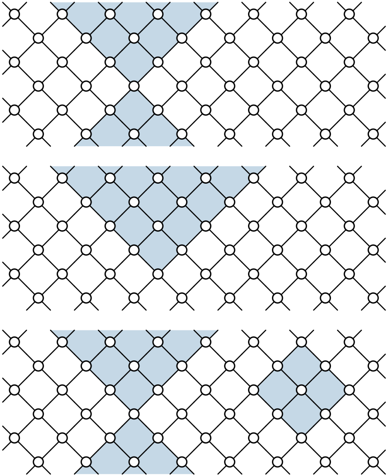

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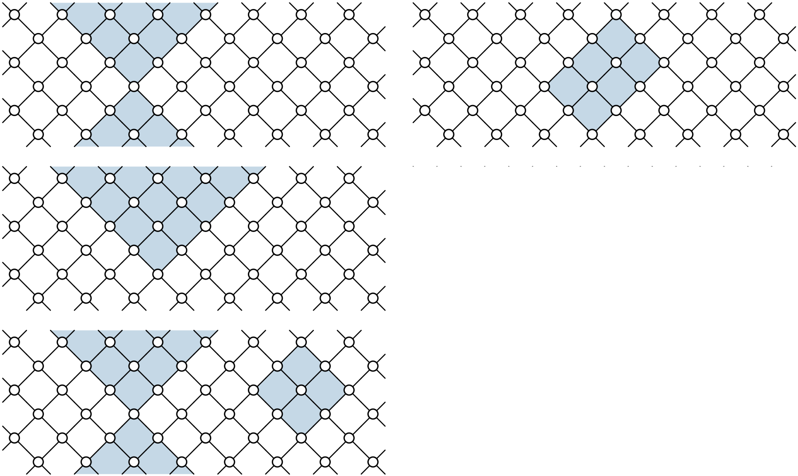
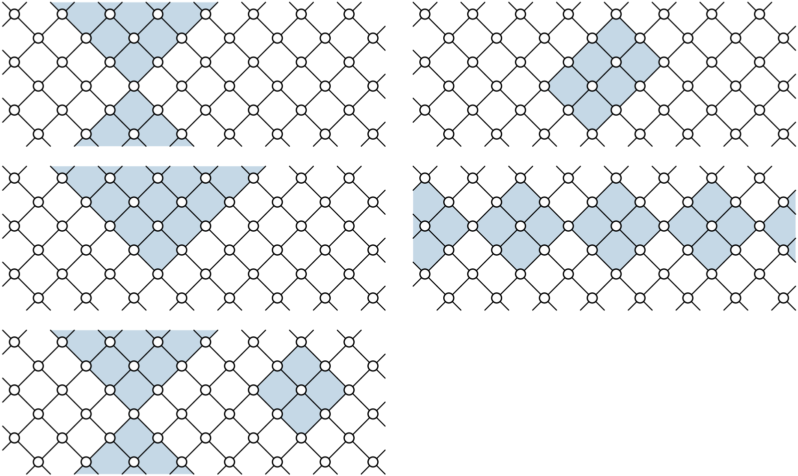


Zoo of biunitary circuits



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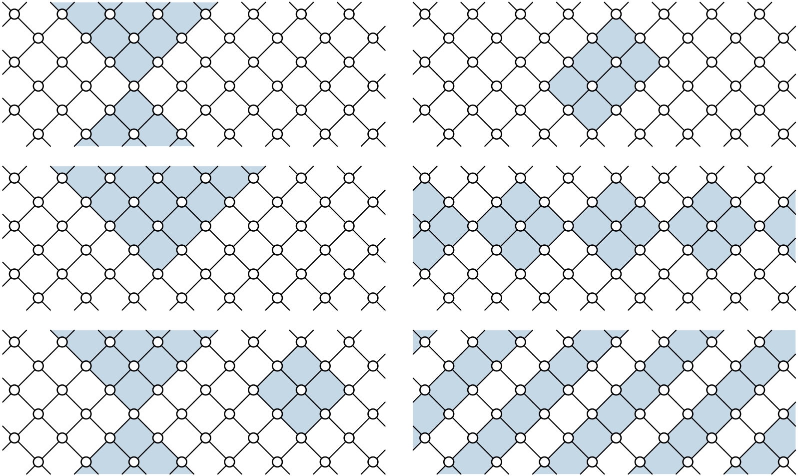
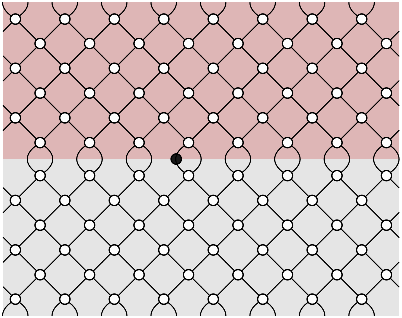


Dynamics of correlation functions

- Proof from dual-unitarity **directly extends** to biunitary circuits

$$\mathcal{U}(t)^\dagger \rho\, \mathcal{U}(t) =$$



Vertical unitarity implies **causal light cone**

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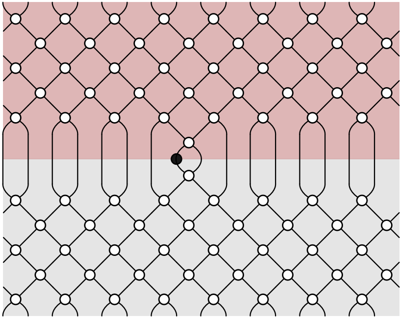

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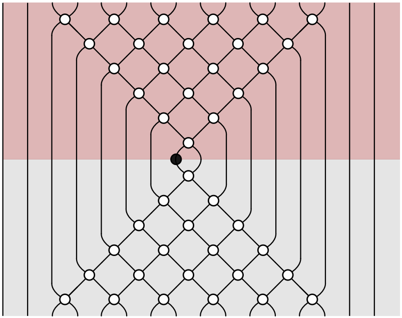
$$\mathcal{U}(t)^\dagger \rho \mathcal{U}(t) =$$ 

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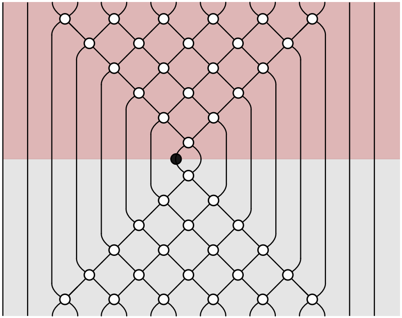

Vertical unitarity implies **causal light cone**

$\Longrightarrow$ Correlations vanish outside the light cone

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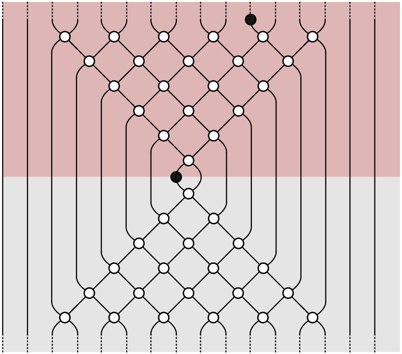
$$\mathrm{Tr}\left[\sigma\,\mathcal{U}(t)^\dagger \rho\, \mathcal{U}(t)\right] =$$ 

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Identical to Bertini et al., Phys. Rev. Lett 123, 210601 (2019)

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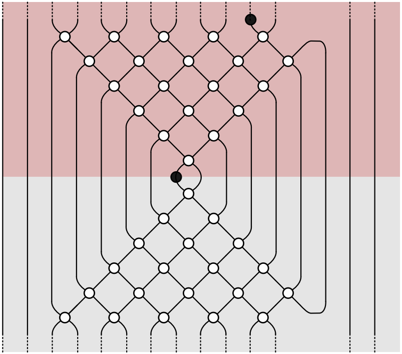

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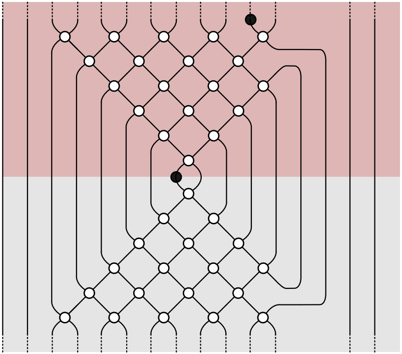

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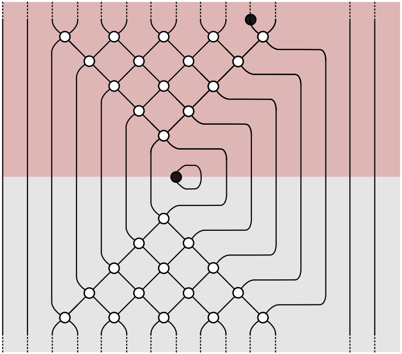

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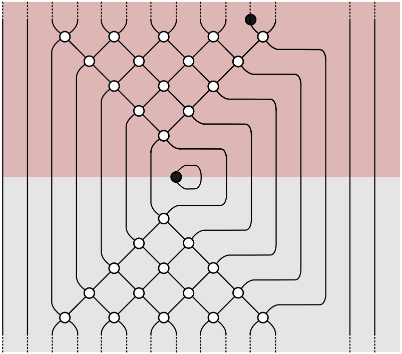


Now using **horizontal unitarity**

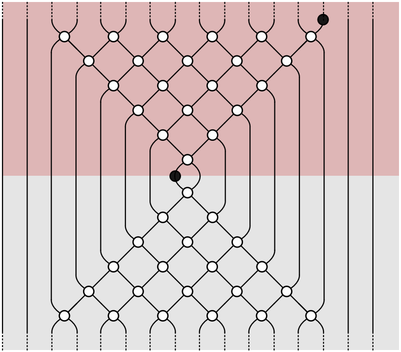
$\implies$ Correlations vanish inside the light cone

Identical to Bertini et al., Phys. Rev. Lett 123, 210601 (2019)

Light-cone dynamics

- Light-cone correlation functions can be efficiently calculated

$$\mathrm{tr}\left[\mathcal{U}^{\dagger}(t)\rho(0)\mathcal{U}(t)\sigma(x=t)\right] =$$



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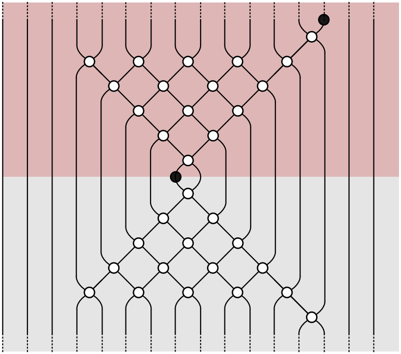

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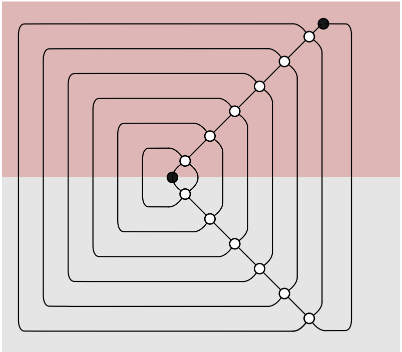

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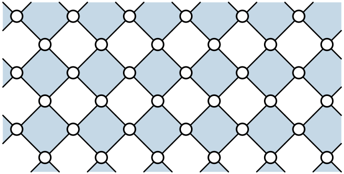
Complex Hadamard circuits

- Construct circuit exclusively out of complex Hadamard matrices



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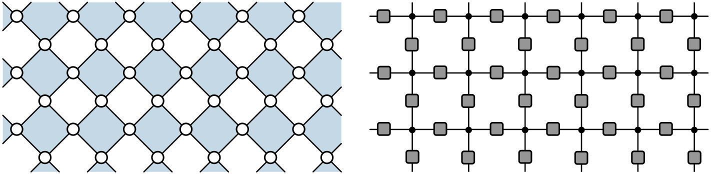

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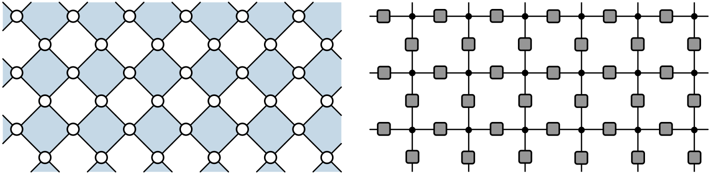

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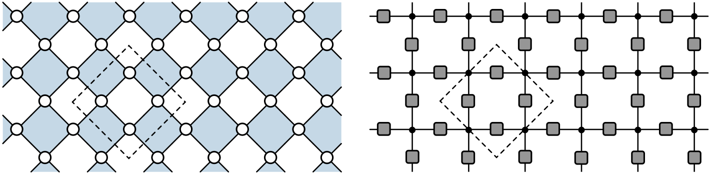

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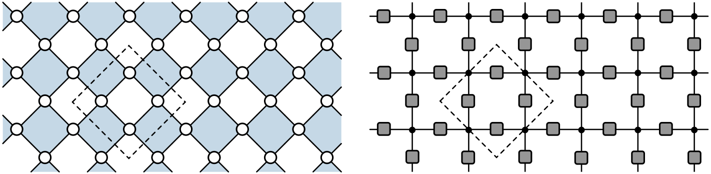

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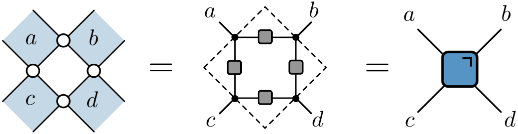

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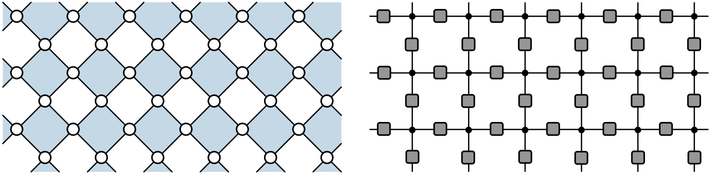

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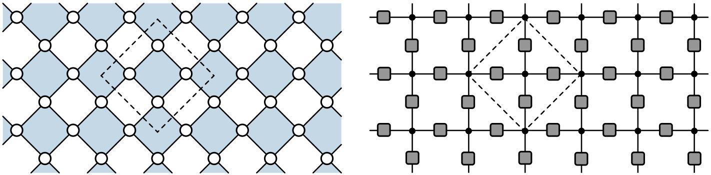

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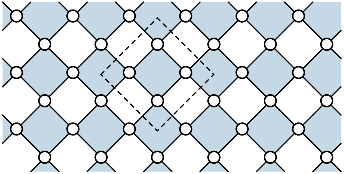

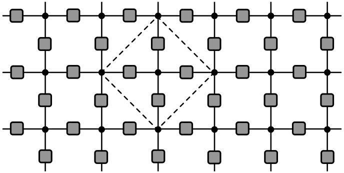

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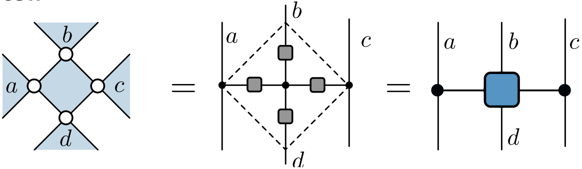

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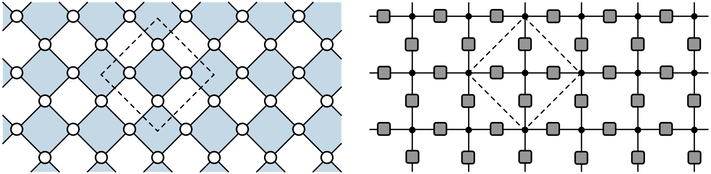

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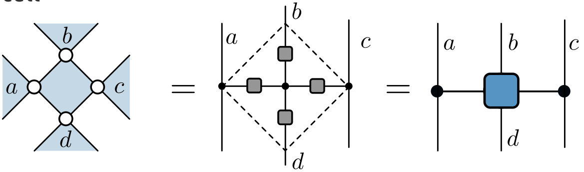
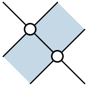


Biunitaries combine into biunitaries!

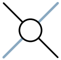
D. J. Reutter and J. Vicary, Higher Structures 3, 109 (2019)

Compositions

- **Diagonal compositions** of biunitaries return biunitaries



$\Longrightarrow$



$\Longrightarrow$ QLS + QLS = Dual-unitary

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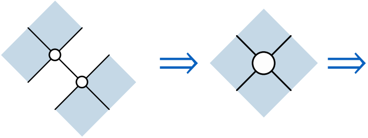

QLS + QLS = Quantum cross

D. J. Reutter and J. Vicary, Higher Structures 3, 109 (2019)

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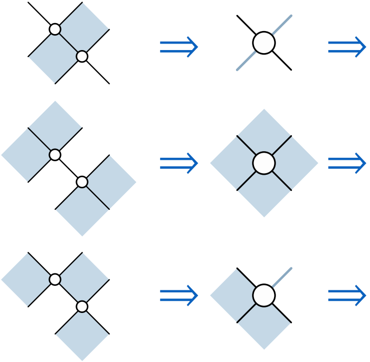

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D. J. Reutter and J. Vicary, Higher Structures 3, 109 (2019)

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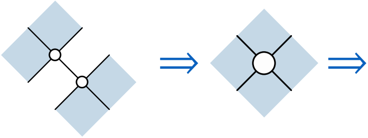

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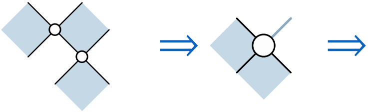

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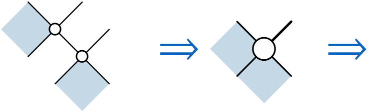

UEB + UEB = QLS

D. J. Reutter and J. Vicary, Higher Structures 3, 109 (2019)

1. Dual-unitary circuits
2. Shaded calculus
3. Biunitary circuits
4. **Solvable states**

Solvable initial states

- Dynamics so far restricted to initial (ultralocal) operators

Piroli et al. Phys Rev. B 101, 094304 (2020)

Solvable initial states

- Dynamics so far restricted to initial (ultralocal) operators
- What about initial states?

Piroli et al. Phys Rev. B 101, 094304 (2020)

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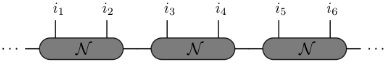
  Class of solvable **matrix product states** introduced by Piroli et al.

$$|\Psi(\mathcal{N})\rangle = \sum_{\{i_j\}}^{q}$$



$$|\ldots i_1\, i_2\, i_3\, i_4\, i_5\, i_6 \ldots\rangle$$

$$= \sum_{\{i_j\}}^{q} \mathrm{Tr}\left[\ldots \mathcal{N}^{(i_1,i_2)}\mathcal{N}^{(i_3,i_4)}\mathcal{N}^{(i_5,i_6)}\ldots\right] |\ldots i_1\, i_2\, i_3\, i_4\, i_5\, i_6 \ldots\rangle$$

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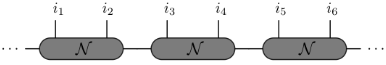

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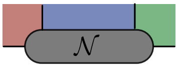
- Consistent with dual-unitarity if $\mathcal{N}$ satisfies some (graphical) identities

Piroli et al. Phys Rev. B 101, 094304 (2020)

Solvable tensors

- Start from tensor $\mathcal{N}$ with arbitrary shading pattern



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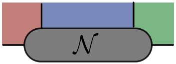

- Satisfies notion of **horizontal unitarity**

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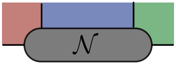

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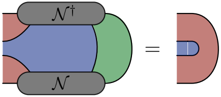
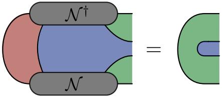
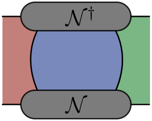


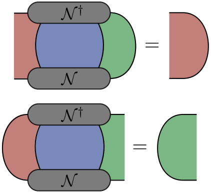
and



- Transfer matrix has a **nondegenerate leading eigenvalue**



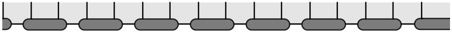
with



Identical to Piroli et al., Phys Rev. B 101, 094304 (2020)

Entanglement dynamics

- Initial state constructed from solvable tensors

$|\Psi(\{\mathcal{N}\})\rangle =$ 

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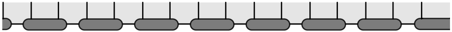
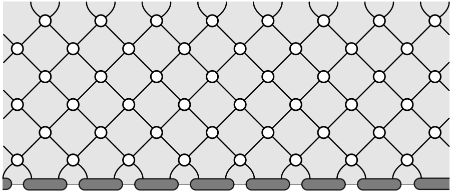
$|\Psi(\{\mathcal{N}\})\rangle =$ 

- Reduced density matrix $\rho_A(t) = \mathrm{Tr}_{\bar{A}} \left(|\Psi(t,\{\mathcal{N}\})\rangle \langle \Psi(t,\{\mathcal{N}\})| \right)$

$|\Psi(t,\{\mathcal{N}\})\rangle =$ 

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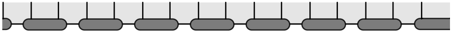
$|\Psi(\{\mathcal{N}\})\rangle =$ 

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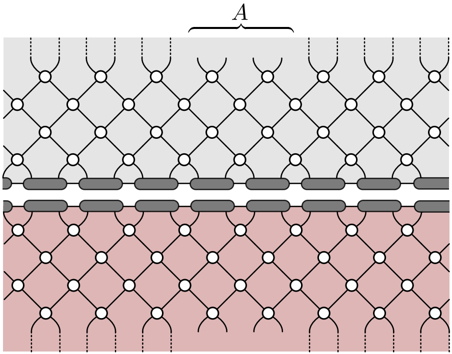
$\rho_A(t) =$ 

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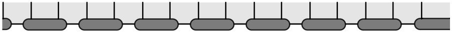

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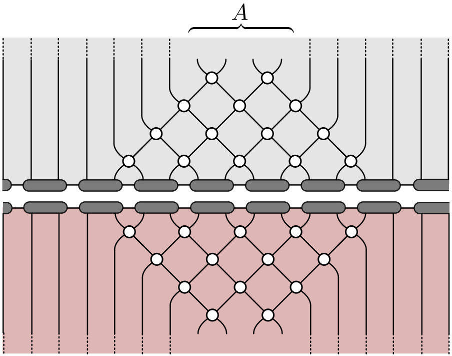

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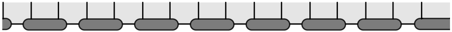
$|\Psi(\{\mathcal{N}\})\rangle =$ 

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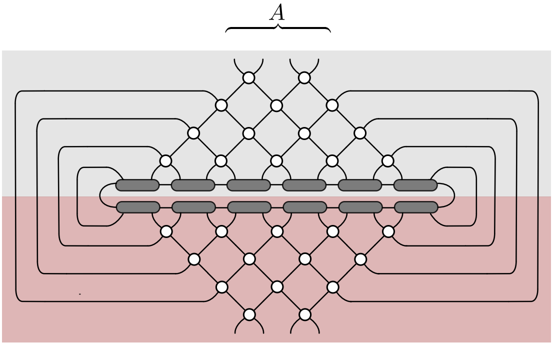
$\rho_A(t) =$ 

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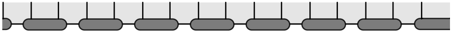
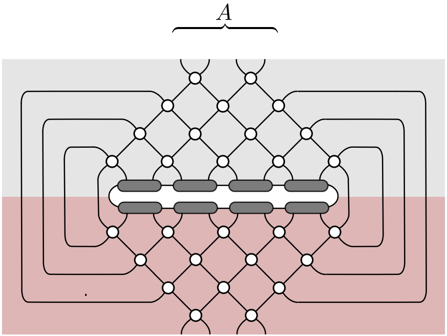
$|\Psi(\{\mathcal{N}\})\rangle =$ 

- Reduced density matrix $\rho_A(t) = \mathrm{Tr}_{\bar{A}}\left(|\Psi(t,\{\mathcal{N}\})\rangle\langle\Psi(t,\{\mathcal{N}\})|\right)$

$\rho_A(t) =$ 

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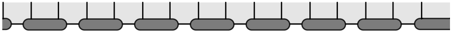
$|\Psi(\{\mathcal{N}\})\rangle =$ 

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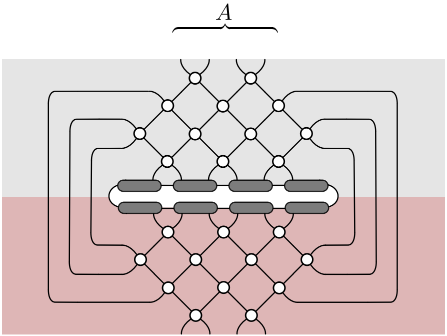
$\rho_A(t) =$ 

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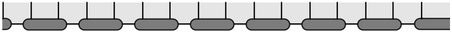
$|\Psi(\{\mathcal{N}\})\rangle =$ 

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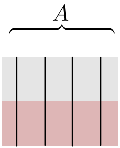
$\rho_A(t) =$ 

Entanglement dynamics

- Biunitary circuits **exactly thermalize** $\rho_A(t \geq |A|/2) \propto \mathbb{1}$

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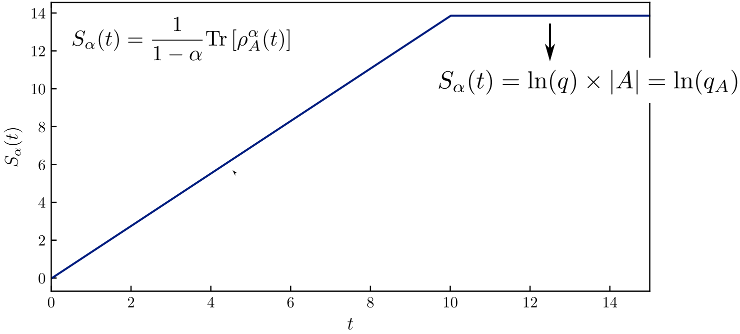

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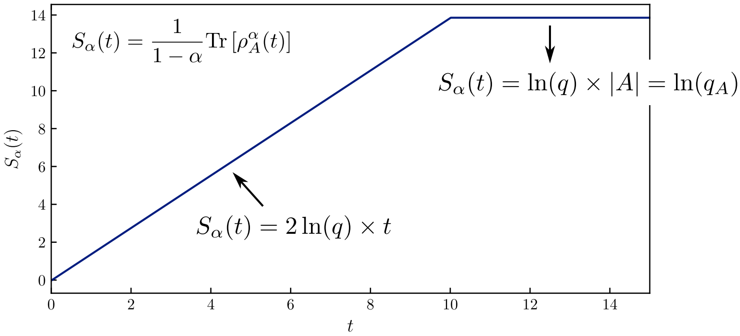

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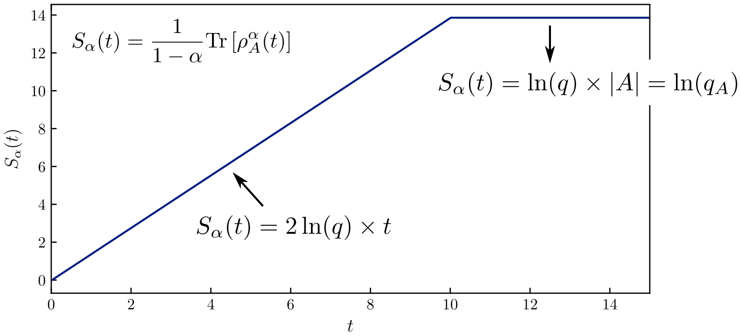
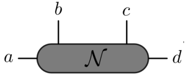


- In scaling limit $x, t \to \infty$ with x/t fixed

Solvable initial states

- No shading: returns solvable matrix product state

$\mathcal{N}_{ad}^{(b,c)} =$ 

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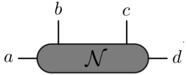
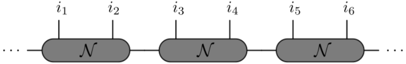


- Corresponding state

$$|\Psi(\mathcal{N})\rangle = \sum_{\{i_j\}}^{q}$$



$$|\dots i_1\, i_2\, i_3\, i_4\, i_5\, i_6 \dots\rangle$$

$$= \sum_{\{i_j\}}^{q} \mathrm{Tr}\left[\dots \mathcal{N}^{(i_1,i_2)} \mathcal{N}^{(i_3,i_4)} \mathcal{N}^{(i_5,i_6)} \dots\right] |\dots i_1\, i_2\, i_3\, i_4\, i_5\, i_6 \dots\rangle$$

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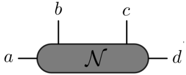

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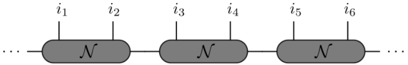

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- Horizontal unitarity corresponds to unitarity of $\mathcal{W}$

$$\mathcal{W}_{ab,cd} = \mathcal{N}_{ad}^{(b,c)}$$

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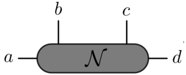

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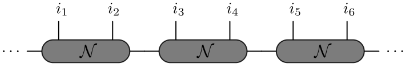

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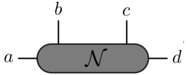

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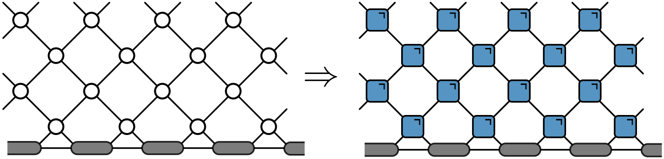
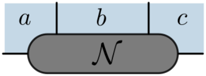


Solvable initial states

- Fully shaded = solvable initial state for clockwork circuits

$\mathcal{N}^{(b)}_{ac} =$ 

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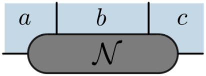

- Corresponding state follows as

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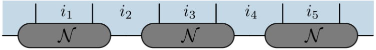

$$\dots |\dots i_1\, i_2\, i_3\, i_4\, i_5 \dots\rangle$$

$$= \sum_{\dots,i_1,i_2\dots}^{q} \left[\dots \mathcal{N}^{(i_1)}_{\dots i_2} \mathcal{N}^{(i_3)}_{i_2,i_4} \mathcal{N}^{(i_5)}_{i_4\dots} \dots\right] |\dots i_1\, i_2\, i_3\, i_4\, i_5\, i_6 \dots\rangle\ ,$$

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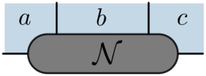

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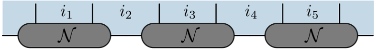

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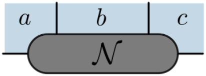
$\mathcal{N}_{ac}^{(b)} =$ 

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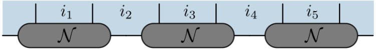
$|\Psi(\mathcal{N})\rangle = \sum_{i_1, i_2, \ldots}^{q} \ldots$ 

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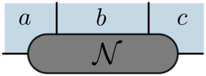
$\mathcal{N}_{ac}^{(b)} =$ 

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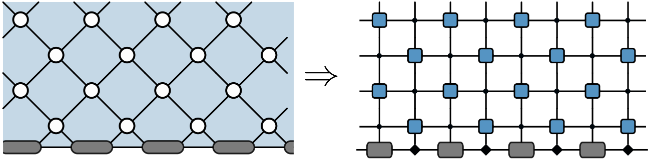

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Quantum 6, 738 (2022)
Phys. Rev. Res. 4, 043212 (2022)
arXiv:2302.07280 (2023)

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THANK YOU FOR YOUR ATTENTION!

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