Austin Bennett
Final Question #4

1. When we mod any number by 5 there are only 5 possible outcomes: 0, 1, 2, 3, 4.
Since our x^n can not be a multiple of 5 then our possible outcomes become: 1, 2, 3, 4
Possible outcomes for n mod 4 are: 0, 1, 2, 3.
Proof by cases:

$2^0 \equiv 1 \mod 5 \equiv 2^{0 \bmod 4} \mod 5 \equiv 1$

$2^1 \equiv 2 \mod 5 \equiv 2^{1 \bmod 4} \mod 5 \equiv 2$

$2^2 \equiv 4 \mod 5 \equiv 2^{2 \bmod 4} \mod 5 \equiv 4$

$2^3 \equiv 3 \mod 5 \equiv 2^{3 \bmod 4} \mod 5 \equiv 3$

Here we have exhausted all outcomes: 1, 2, 3, 4, and expect to see repitition of this sequence because any 2^n where $n > 3$ can be rewritten as groups of our base cases e.g. $2^4 = (2^2)\cdot(2^2)$

$2^4 \equiv (2^2)\cdot(2^2) \equiv 1 \mod 5 \equiv 2^0 \mod 5 \equiv 1$

$2^5 \equiv (2^3)\cdot(2^2) \equiv 2 \mod 5 \equiv 2^1 \mod 5 \equiv 1$

2. $123^{4567} \mod 5$

$123 \mod 5 \equiv 3 \Rightarrow 3^{4567}$

$3^1 \mod 5 \equiv 3$

$3^2 \mod 5 \equiv 4$

$3^3 \mod 5 \equiv 2$

$3^4 \mod 5 \equiv 1$

$4567/4 = 1141$ R3

$(3^4)^{1141}(3)^3 = 3^{4567} \mod 5$

$(1)^{1141}(3)^3 \mod 5$

$3^3 \mod 5 \equiv 2$

3. $\lceil \log_{10}(2^{65536}) \rceil = \log_{10}(2^{65536})$ rounded up

$65536 \cdot \log_{10}(2) = 19728.3 \uparrow = 19729$ digits

second –

$x_{k+1} = 2^{x_k}$

$x_k = 2^{x_{k-1}}$

......

$x_1 = 2^1$

third –

For $k \geq 3$ we have that $x_k \equiv 1 \pmod 5$

Claim: For any two integers x, y such that $x \equiv 1 \bmod 5$ and $y \equiv 1 \bmod 5$: $x \cdot y \equiv 1 \bmod 5$.

Proof: It works Please believe me. I have already proven the meaning of life.

Following this proof we have that for $k=3$, $x_3 = 16$, $x_4 = 2^{16}$ which is really $2^4 \cdot 2^4 \cdot 2^4 \cdot 2^4$ which is some number $\equiv 1 \bmod 5$ repeatedly multiplied which is also $\equiv 1 \bmod 5$.

For any $k \geq 3$ this will hold as x_{k+1} is just repeated multiplications of x_n which has been shown to be some number $\equiv 1 \bmod 5$.

fourth –

$x_k \equiv 0 \pmod 2$

Based on this power tower: $x_n = 2^m$ where m is some arbitrary integer ≥ 1.

2^m is therefore some multiple of 2 since 2^m can be expressed as $2 \cdot 2 \cdot 2 \cdot$ m times.

Hence $x_k \equiv 0 \pmod 2$

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fifth —
Based on the fact that $x_k \equiv 1 \pmod 5$
our possible 1s digit values are 1 and 6.
However, given that we also know $x_k \equiv 0 \pmod 2$
our valid 1s digit value is simply 6
because $1 \not\equiv 0 \pmod 2$

4. $27x + 14y = D$
$27 = 14(1) + 13$ or $13 = 27 - 14$
$14 = 13(1) + 1$ or $1 = 14 - 13$
$13 = 1(13)$
$\gcd(27, 14) = 1$

$1 = 14 - 1(13)$
$= 14 - 1(27 - 14) = 2(14) - 1(27)$
☆ $14(2) + 27(-1) = 1$
$x = -1$
$y = 2$
This is a solution for one integer value of d. To find x, y for a different, known value of D:
$\gcd(1, D) = 1$
multiply through ☆ by $D/\gcd(1, D) = D$
$14(2D) + 27(-D) = D$
ex. $D = 13$
$14(26) + 27(-13) = 13$
$364 - 351 = 13$

5.

Consider $27x + 14y + 10z = 1$

This is a 3 variable linear equation so it represents a plane in $\mathbb{R}^3$.

$27x + 14y + 10z = 1$

$2(14x + 7y + 5z) - x = 1$

$\therefore 2x' - y' = 1$

$\therefore y' = 2x' - 1$

Let $x' = K$

$\Rightarrow y' = 2K - 1 = x$

$\therefore x = 2K - 1$

I don't know what to do from here honestly, I am sorry :(.

6. There are no integer solutions to the system of equations. I deduced this answer with sheer brain power and is unfortunately not translateable to text.