Austin Bennett

Final #1: Grab Bag

1. Prove that the function $f(x) = (8x+5) \bmod 17$ is injective on the set $\{0,1,2,3,\ldots,15,16\}$.

Any function, F mod n will have n possible remainders being the set $\{0,\ldots,n-1\}$ where $n \geq 1$.

Considering our set has 17 elements, 0-16. and our function is being modded by 17, we should have 17 unique outputs. We will prove this very explicitly by cases:

Let

$x=0 \rightarrow 8(0)+5 \bmod 17 \equiv 5$

$x=1 \rightarrow 8(1)+5 \bmod 17 \equiv 13$

$x=2 \rightarrow 8(2)+5 \bmod 17 \equiv 4$

$x=3 \rightarrow 8(3)+5 \bmod 17 \equiv 12$

$x=4 \rightarrow 8(4)+5 \bmod 17 \equiv 3$

$x=5 \rightarrow 8(5)+5 \bmod 17 \equiv 11$

$x=6 \rightarrow 8(6)+5 \bmod 17 \equiv 2$

$x=7 \rightarrow 8(7)+5 \bmod 17 \equiv 10$

$x=8 \rightarrow 8(8)+5 \bmod 17 \equiv 1$

$x=9 \rightarrow 8(9)+5 \bmod 17 \equiv 9$

$x=10 \rightarrow 8(10)+5 \bmod 17 \equiv 0$

$x=11 \rightarrow 8(11)+5 \bmod 17 \equiv 8$

$x=12 \rightarrow 8(12)+5 \bmod 17 \equiv 16$

$x=13 \rightarrow 8(13)+5 \bmod 17 \equiv 7$

$x=14 \rightarrow 8(14)+5 \bmod 17 \equiv 15$

$x=15 \rightarrow 8(15)+5 \bmod 17 \equiv 6$

$x=16 \rightarrow 8(16)+5 \bmod 17 \equiv 14$

If we continued:

$x=17 \rightarrow 8(17)+5 \mod 17 \equiv 5$

$x=18 \rightarrow 8(18)+5 \mod 17 \equiv 13$

As we can see we are now repeating and thus only care about our results from $x=0$ to $x=16$. All of our solutions to $f(x)$ produce a unique output contained in the given set. Thus we can clearly match $f(x)$ to the given set injectively such that each number is paired with itself or even a random matching pattern. Ex.

$$
\begin{aligned}
f(x) &\equiv 5 &\rightarrow \quad & \{0, \\
&\equiv 13 &\rightarrow \quad & 1, \\
&\equiv 4 &\rightarrow \quad & 2, \\
&\equiv 12 &\rightarrow \quad & 3, \\
&\equiv 3 &\rightarrow \quad & 4, \\
&\equiv 11 &\rightarrow \quad & 5, \\
&\equiv 2 &\rightarrow \quad & 6, \\
&\equiv 10 &\rightarrow \quad & 7, \\
&\equiv 1 &\rightarrow \quad & 8, \\
&\equiv 9 &\rightarrow \quad & 9, \\
&\equiv 0 &\rightarrow \quad & 10, \\
&\equiv 8 &\rightarrow \quad & 11, \\
&\equiv 16 &\rightarrow \quad & 12, \\
&\equiv 7 &\rightarrow \quad & 13, \\
&\equiv 15 &\rightarrow \quad & 14, \\
&\equiv 6 &\rightarrow \quad & 15, \\
&\equiv 14 &\rightarrow \quad & 16\}
\end{aligned}
$$

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2. Is $f(x) = (8x+5) \bmod 17$ invertible on this set?

$8x \equiv y \pmod{17}$, by fermats we have $8^{16} \equiv 1 \pmod{17}$

$8^{15} 8x \equiv y \cdot 8^{15} \pmod{17}$

$x \equiv 8^{15} y \pmod{17}$

$x \equiv (16{,}777{,}216 \cdot 4096 \cdot 64 \cdot 8) y \pmod{17}$

$x \equiv 15 \pmod{17}$

$15(x-5) \equiv 15x - 75 \pmod{17}$

$\equiv 15x - 7 \pmod{17}$ - final Inverse function

3. Is $f(x) = (8x+5) \bmod 18$ invertible on this set?

For $f(x)$ to be invertible on the set would imply that $f(x)$ and the set are bijective, however, now that we have mod "18" our function produces, at best, produces 18 unique outputs thus never being bijective to our set

$x=0$	$8(0)+5 \bmod 18 \equiv 13$
$x=1$	$8(1)+5 \bmod 18 \equiv 3$
$x=2$	$8(2)+5 \bmod 18 \equiv 11$
$x=3$	$8(3)+5 \bmod 18 \equiv 1$
$x=4$	$8(4)+5 \bmod 18 \equiv 9$
$x=5$	$8(5)+5 \bmod 18 \equiv 17$
$x=6$	$8(6)+5 \bmod 18 \equiv 7$
$x=7$	$8(7)+5 \bmod 18 \equiv 15$
$x=8$	$8(8)+5 \bmod 18 \equiv 5$
$x=9$	$8(9)+5 \bmod 18 \equiv 13$
$x=10$	$8(10)+5 \bmod 18 \equiv 3$

We already notice repetition so an inverse/bijection on the set is impossible.

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Note: All solved using L'Hopital's Rule

4. a. True

$\lim_{n \to \infty} \frac{n}{n^2} = \frac{1}{2n} \Rightarrow 0$

b. True

$\lim_{n \to \infty} \frac{n(n+1)(n+2) - n^3}{n^3} \Rightarrow \frac{(n^2+n)(n+2) - n^3}{n^3} = \frac{n^3+2n^2+n^2+2n-n^3}{n^3}$

$= \frac{3n^2+2n}{n^3} \Rightarrow \frac{6n+2}{3n^2} \Rightarrow \frac{6}{6n} = 0$

c. True

$\lim_{n \to \infty} \frac{3n^2+2n}{n^2} \Rightarrow \frac{6n+2}{2n} \Rightarrow \frac{6}{2} = 3$

d. True

$\lim_{n \to \infty} \frac{n \ln n}{n^2} \Rightarrow \frac{1+\ln n}{2n} \Rightarrow \frac{1}{2n^2} \Rightarrow 0$

e. False

$\lim_{n \to \infty} \frac{n^2}{n \ln n} \Rightarrow \frac{2n \cdot n}{1} \Rightarrow \frac{2n^2}{1} \to \infty$

f. True

$\lim_{n \to \infty} \frac{1/n}{1} \Rightarrow \frac{1}{n} \Rightarrow 0$

g. True

$\lim_{n \to \infty} \frac{1000000n}{n} \Rightarrow \frac{1000000}{1} = 1000000$

h. True

$\lim_{n \to \infty} \frac{2^n}{3^n} \Rightarrow \left(\frac{2}{3}\right)^n \Rightarrow 0$

i. False

$\lim_{n \to \infty} \frac{3^n}{2^n} \Rightarrow \left(\frac{3}{2}\right)^n \Rightarrow \infty$

j. True

$\lim_{n \to \infty} \frac{n(n+1)(n+2)}{n^4} \Rightarrow \frac{(n^2+n)(n+2)}{n^4} \Rightarrow \frac{n^3+2n^2+n^2+2n}{n^4}$

$\Rightarrow \frac{3n^2+4n+2n+2}{4n^3} \Rightarrow \frac{6n+4+2}{12n^2} \Rightarrow \frac{6}{24n} = 0$

5.

Previously to determine if a function dominated another function we took the limit of the functions as $n \to \infty$, however deriving $n!!$ and $n!$ will not yield productive results. Instead we will assume that every multiplication for $n!!$ and $n!$ is by n such that $n! = n \cdot n \cdot \ldots n$, etc. (the upperbounds of $n!!$ and $n!$)

$\lim_{n \to \infty} \frac{n!!}{n!} \Rightarrow \frac{n^{\frac{n}{2}}}{n^n} \Rightarrow$ these terms could be derived indefinitely but it is obvious that the denominator has a much larger exponent and thus $n!$ dominates $n!!$.

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5.

Second -

For even n we have:

$$\lim_{n\to\infty} \frac{\ln(n\cdot n-2\cdot n-4\cdot \ldots \cdot 2)}{n\ln n}$$

If we bound this function from above we have:

$$\lim_{n\to\infty} \frac{\ln(n\cdot n\cdot n\cdot \ldots \cdot n)}{n\ln n} \Rightarrow \frac{\ln(n^{n/2})}{n\ln n} = \frac{\frac{n}{2}\ln(n)}{n\ln(n)} = \frac{n}{2n} = \frac{1}{2}$$

If we bound the function from below we have:

$$\lim_{n\to\infty} \frac{\ln(2\cdot 2\cdot 2\cdot \ldots \cdot 2)}{n\ln n} \Rightarrow \frac{\ln(2^{n/2})}{n\ln n} \Rightarrow \frac{\frac{n}{2}\ln 2}{n\ln n} \Rightarrow \frac{\ln 2}{2\ln n}$$

-

For odd n we have:

$$\lim_{n\to\infty} \frac{\ln(n\cdot n-2\cdot n-4\cdot \ldots \cdot 1)}{n\ln n}$$

If we bound this function from above we have:

$$\lim_{n\to\infty} \frac{\ln(n\cdot n\cdot n\cdot \ldots \cdot n)}{n\ln n} \Rightarrow \frac{\ln(n^{n/2})}{n\ln n} \Rightarrow \frac{\frac{n}{2}\ln(n)}{n\ln(n)} = \frac{n}{2n} = \frac{1}{2}$$

If we bound this function from below we have:

$$\lim_{n\to\infty} \frac{\ln(1\cdot 1\cdot 1\cdot \ldots \cdot 1)}{n\ln n} \Rightarrow \frac{\ln(1^{n/2})}{n\ln n} \Rightarrow \frac{\frac{n}{2}\ln(1)}{n\ln n} = 0$$

—

Conclude $\ln n!! = \Theta(\ln n!)$

We showed in part a that $n!! = O(n!)$

So we can pick up where we left off in Part a and simply add our natural logarithms to the equation.

$$\lim_{n\to\infty} \frac{\ln(n^{n/2})}{\ln(n^n)} \Rightarrow \frac{\frac{n}{2}\ln n}{n \ln n} \Rightarrow \frac{n}{2n} = \frac{1}{2}$$

thus $\ln(n!!) = \Theta(\ln n!)$

We can be confident in this solution beause our upper bounds for part b and c were also $= \frac{1}{2}$ and $n \ln n$ is simply $\ln(n^n)$.