CS 205 Final Question 3: Linear Recurrences

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Let a sequence $X_0, X_1, X_2, \ldots$ be defined in the following way:

$$\begin{aligned} X_0 &= 1 \\ X_1 &= 2 \\ X_n &= 3X_{n-1} + 2X_{n-2}. \end{aligned} \tag{1}$$

1) Compute the first 10 terms of this sequence. **(2 points)**
2) Prove that this sequence is strictly increasing, i.e., $\forall n \geq 0 : X_{n+1} > X_n$. **(2 points)**
3) Prove that $\forall n \geq 0 : X_n \leq 4^n$. What are the base cases? What is the inductive step? **(5 points)**
4) The above result suggests that this sequence grows in the worst case exponentially, i.e., $X_n = O(4^n)$. Consider trying to tighten this bound in the following way: what are the smallest values α and β such that the above proof still works for $\forall n \geq 0 : X_n \leq \alpha * \beta^n$? Give the tightest bounds on X_n you can. **(10 points)**
5) What is the tightest *lower* bound on X_n you can achieve in this way? Characterize the long term / asymptotic behavior of X_n as best you can. **(5 points)**

Consider the following sequence $Y_0, Y_1, \ldots$ defined in the following way:

$$\begin{aligned} Y_0 &= 1 \\ Y_1 &= 2 \\ Y_n &= \frac{1}{4}Y_{n-1} + \frac{3}{4}Y_{n-2}. \end{aligned} \tag{2}$$

6) Compute the first 10 terms of this sequence (out to ten decimal places). **(3 points)**
7) Prove that this sequence satisfies $\forall n \geq 0 : 1 \leq Y_n \leq 2$. **(3 points)**
8) Prove that the following holds for all $n \geq 0$:

$$Y_{n+1} - Y_n = \left(-\frac{3}{4}\right)^n. \tag{3}$$

What is your base case? What is your inductive step? **(10 points)**

9) Argue that

$$Y_n = 1 + \sum_{k=0}^{n-1} \left(-\frac{3}{4}\right)^k. \tag{4}$$

Hint: What does the previous problem say about Y_1? What does it say about Y_2, Y_3? **(7 points)**

10) What is the limit of Y_n as $n \to \infty$? **(3 points)**

Bonus: Show that regardless of the initial value of Y_0 and Y_1, that Y_n converges to a constant in the limit. What is the limit? What part of the above results generalize, and which do not?

Bonus 2: Generalize the above results to arbitrary Y_0, Y_1, and $Y_n = pY_{n-1} + (1-p)Y_{n-2}$ for $0 < p < 1$.