CS 205 Final Question 4: Diophantine Equations and Power Towers

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1) Prove that for an integer X that is not a multiple of 5, $X^n \equiv X^{n \bmod 4} \pmod 5$. **(5 points)**

2) Use this to compute, as efficiently as you can, $123^{4567} \pmod 5$. Show your steps. **(2 points)**

3) A fast way to generate large numbers is with power towers. The following are the first few power towers of 2:

$$\begin{aligned} X_1 &= 2 \\ X_2 &= 2^2 = 4 \\ X_3 &= 2^{2^2} = 2^4 = 16 \\ X_4 &= 2^{2^{2^2}} = 2^{16} = 65536 \\ X_5 &= 2^{2^{2^{2^2}}} = 2^{65536}. \end{aligned} \tag{1}$$

As you can see, these numbers get large fast. But even for these incredibly large numbers, we can still compute and determine things about them. Let X_k be the k-th power tower of 2.

- How many digits (base-10) does X_5 have? Give an exact value. **(1 points)**
- Give a recursive formulation for X_{k+1} in terms of X_k. **(2 points)**
- Prove that for all sufficiently large k, $X_k \equiv 1 \pmod 5$. **(5 points)**
- Prove that for all sufficiently large k, $X_k \equiv 0 \pmod 2$. **(2 points)**
- Based on the previous results, compute the 1s digit of the k-th power tower for all sufficiently large k. **(8 points)**

4) Prove that for any integer value of D, the equation $27x + 14y = D$ has integer solutions for x and y. **(10 points)**

5) Consider the equation $27x + 14y + 10z = 1$. Give parameterized solutions for all integer solutions x, y, z. How many parameters do you need? *Hint: What does this equation represent, geometrically?* **(8 points)**

6) Consider the following system of equations:

$$\begin{aligned} 27x + 14y + 10z &= 1 \\ 3x + 5y + 7z &= 1. \end{aligned} \tag{2}$$

Are there any integer solutions to this system of equations? If so, what are they? *Hint: What does the solution to this system of equations represent, geometrically?* **(7 points)**

- *Bonus: For $a = 1, 2, 3, 4, 5, 6, 7, 8, 9$, determine what the 1s digit of the k-th power tower base-a is, for all sufficiently large k.*