Austin Bennett
Final #3: Linear Recurrences

1.

$x_0 = 1$
$x_1 = 2$
$x_2 = 3(2)+2(1) = 8$
$x_3 = 3(8)+2(2) = 28$
$x_4 = 3(28)+2(8) = 100$
$x_5 = 3(100)+2(28) = 356$
$x_6 = 3(356)+2(100) = 1268$
$x_7 = 3(1268)+2(356) = 4516$
$x_8 = 3(4516)+2(1268) = 16084$
$x_9 = 3(16084)+2(4516) = 57284$

2. $\forall n \geq 0: x_{n+1} > x_n$

Assume $x_n = 3x_{n-1} + 2x_{n-2}$
Then $x_{n+1} = 3x_n + 2x_{n-1}$
Thus $x_{n+1} = 3(3x_{n-1} + 2x_{n-2}) + 2\cancel{x_{n-1}}$
$= 9x_{n-1} + 6x_{n-2} + 2\cancel{x_{n-1}}$
$= 11x_{n-1} + 6x_{n-2}$

$11x_{n-1} + 6x_{n-2} > 3x_{n-1} + 2x_{n-2} \quad \forall n \geq 0$

3. $\forall n \geq 0: x_n \leq 4^n$

base cases: $n = 0, \quad x_n = 1 \leq 4^n$ ✓
$n = 1, \quad x_n = 2 \leq 4^n$ ✓

IS Assume $x_n \leq 4_n$ holds
Then, $x_{n+1} = 3x_n + 2x_{n-1} \leq 4^n + 4^{n-1}$
$\leq 4^n \quad \frac{1}{4}4^n$
$\leq 5/4 \cdot 4^n$
$x_{n+1} \leq 15/4 \cdot 4^n \leq 4^{n+1}$

4.

$X_n = 3X_{n-1} + 2X_{n-2} \le 3\beta^{n-1} + 2\beta^{n-2}$

$\beta^{n-2}(3\beta+2) \le \beta^n$

$3\beta + 2 \le \beta^2$

$\beta^2 - 3\beta - 2 = 0$

$\frac{3 \pm \sqrt{9-4(1)(-2)}}{2} = \beta = \frac{3 \pm \sqrt{17}}{2}$

$\Rightarrow O\left(\left(\frac{3 \pm \sqrt{17}}{2}\right)^n\right)$

5. $\alpha \cdot \beta^0 \le X_0$, $\alpha \cdot \beta^1 \le X_1$

$\alpha \cdot \beta^0 \le 1$, $\alpha \cdot \beta^1 \le 2$

To find the tightest lower bound of X_n they must satisfy the two inequalities produced by our base cases. Since we are trying to find the tightest bound β should be the greatest value possible to still remain valid in our inequalities. Thus, $\beta = 2$

$\beta^0 = 2^0 \le 1$ ✓

$\beta^1 = 2^1 \le 2$ ✓

$\beta^2 = 2^2 \le 8$ ✓

If β increased at all from $\beta = 2$ then the bound would fail to contain our base cases.

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6.

$Y_0 = 1.0000000000$
$Y_1 = 2.0000000000$
$Y_2 = 1.2500000000$
$Y_3 = 1.8125000000$
$Y_4 = 1.3906250000$
$Y_5 = 1.7070312500$
$Y_6 = 1.4697265250$
$Y_7 = 1.6477050781$
$Y_8 = 1.5142211914$
$Y_9 = 1.6143341064$

7. $\forall n \geq 0;\ 1 \leq Y_n \leq 2$

$Y_n = \frac{1}{4} Y_{n-1} + \frac{3}{4} Y_{n-2}$

$$Y_{n+1} = \frac{1}{4} Y_n + \frac{3}{4} Y_{n-1}$$
$$= \frac{1}{4}\left(\frac{1}{4} Y_{n-1} + \frac{3}{4} Y_{n-2}\right) + \frac{3}{4} Y_{n-1}$$
$$= \frac{1}{16} Y_{n-1} + \frac{3}{16} Y_{n-2} + \frac{3}{4} Y_{n-1}$$
$$= \frac{13}{16} Y_{n-1} + \frac{3}{16} Y_{n-2}$$

$1 \leq Y_0 \leq 2 \Rightarrow 1 \leq 1 \leq 2$ ✓
$1 \leq Y_1 \leq 2 \Rightarrow 1 \leq 2 \leq 2$ ✓
$1 \leq Y_n \leq 2 \Rightarrow 1 \leq \frac{1}{4} Y_{n-1} + \frac{3}{4} Y_{n-2} \leq 2$
Let $n = 2 \Rightarrow 1 \leq 1.25 \leq 2$
Assuming $Y_n = \frac{1}{4} Y_{n-1} + \frac{3}{4} Y_{n-2}$ holds
then $1 \leq Y_n \leq 2$
and $1 \leq Y_{n+1} \leq 2$

8. $Y_{n+1} - Y_n = \left(-\frac{3}{4}\right)^n$

$\frac{13}{16} Y_{n-1} + \frac{3}{16} Y_{n-2} - \frac{1}{4} Y_{n-1} - \frac{3}{4} Y_{n-2} = \left(-\frac{3}{4}\right)^n$

$\frac{9}{16} Y_{n-1} - \frac{9}{16} Y_{n-2} = \left(-\frac{3}{4}\right)^n$

base cases: let $n = 2 \Rightarrow \frac{9}{16}(2) - \frac{9}{16}(1) = \left(-\frac{3}{4}\right)^n$

$\frac{18}{16} - \frac{9}{16} = \left(-\frac{3}{4}\right)^2$

$\frac{9}{16} = \frac{9}{16}$ ✓

$n = 1 \Rightarrow \frac{5}{4} - 2 = \left(-\frac{3}{4}\right)^n$

$-\frac{3}{4} = -\frac{3}{4}$ ✓

$n = 0 \Rightarrow 2 - 1 = \left(-\frac{3}{4}\right)^0$

$1 = 1$ ✓

Inductive step: suppose $Y_{n+1} - Y_n = \left(-\frac{3}{4}\right)^n$ is true

$\frac{9}{16} Y_n - \frac{9}{16} Y_{n-1} = \left(-\frac{3}{4}\right)^{n+1}$

$\frac{9}{16}(Y_n - Y_{n-1}) = -\frac{3}{4} \cdot \left(-\frac{3}{4}\right)^n$

$\frac{-36}{48}(Y_n - Y_{n-1}) = \left(-\frac{3}{4}\right)^n$

$-\frac{3}{4}(Y_n - Y_{n-1}) = \left(-\frac{3}{4}\right)^n$

$Y_n - Y_{n-1} = \left(-\frac{3}{4}\right)^{n-1}$

$n = 1$ $\quad 2 - 1 = \left(-\frac{3}{4}\right)^0$

$1 = 1$ ✓

thus $n+1 \Rightarrow Y_{n+1} - Y_n = \left(-\frac{3}{4}\right)^n \quad \forall n \geq 0$

9. $Y_n = 1 + \sum_{k=0}^{n-1} \left(\frac{-3}{4}\right)^k$

From the previous problem we have

$Y_1 = -\frac{3}{4} = \left(-\frac{3}{4}\right)^1$

$Y_2 = \frac{9}{16} = \left(-\frac{3}{4}\right)^2$

$Y_3 = \frac{-27}{64} = \left(-\frac{3}{4}\right)^3$

From the previous problem we also have $Y_{n+1} - Y_n = \left(-\frac{3}{4}\right)^n$ which entails that the difference between the next element and the current element $= \left(-\frac{3}{4}\right)^n$

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This is simply $\sum_{k=0}^{n+1} \left(\frac{-3}{4}\right)^k$

the additional +1 is derived from the fact that $Y_{n+1} - Y_n$ has a minimum of $n=0$, such that we have $Y_1 - Y_0$ which does not account for our original base case.

Thus $Y_n = 1 + \sum_{k=0}^{n-1} \left(\frac{-3}{4}\right)^k$

10.

$\lim_{n\to\infty} Y_n = 1 + \sum_{k=0}^{n-1} \left(\frac{-3}{4}\right)^k$ = undefined

Because of our alternating positive/negative term: $\left(\frac{-3}{4}\right)^k$ as $n \to \infty$ our value of Y_n will alternate after every term approaching 1 or 2.