1. (10 pts) Hermione needs your help with her wizardly homework. She's trying to come up with an example of a directed graph $G = (V, E)$, a source vertex $s \in V$ and a set of tree edges $E_\pi \subseteq E$ such that for each vertex $v \in V$, the unique path in the graph (V, E_π) from s to v is a shortest path in G, yet the set of edges E_π cannot be produced by running a breadth-first search on G, no matter how the vertices are ordered in each adjacency list. Include an explanation of why your example satisfies the requirements.

2. (15 pts) Prof. Dumbledore needs your help to compute the in- and out-degrees of all vertices in a directed multigraph G. However, he is not sure how to represent the graph so that the calculation is most efficient. For each of the three possible representations, express your answers in asymptotic notation (the only notation Dumbledore understands), in terms of V and E, and justify your claim.

   (a) An *edge list* representation. Assume vertices have arbitrary labels.

   (b) An *adjacency list* representation. Assume the vector's length is known.

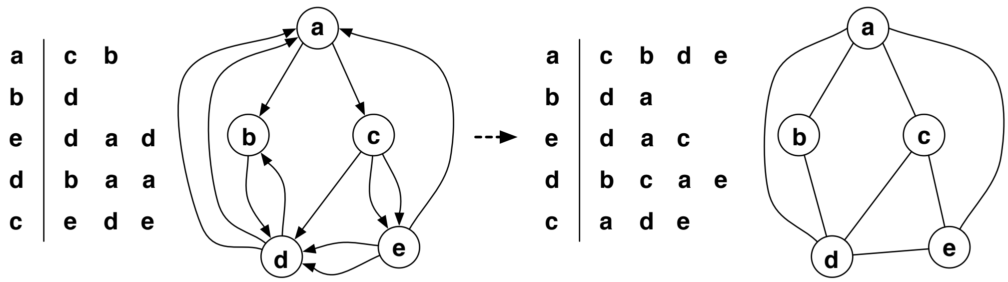
   (c) An *adjacency matrix* representation. Assume the size of the matrix is known.

3. (25 pts) Professor Snape gives you the following unweighted graph and asks you to construct a weight function w on the edges, using positive integer weights only, such that the following conditions are true regarding minimum spanning trees and single-source shortest path trees:

   - The MST is distinct from any of the seven SSSP trees.
   - The order in which Jarník/Prim's algorithm adds the safe edges is different from the order in which Kruskal's algorithm adds them.

   Justify your solution by (i) giving the edges weights, (ii) showing the corresponding MST and all the SSSP trees, and (iii) giving the order in which edges are added by each of the three algorithms. (For Borůvka's algorithm, be sure to denote which edges are added simultaneously in a single round.)

4. (25 pts extra credit) Deep in the heart of the Hogwarts School of Witchcraft and Wizardry, there lies a magical Sphinx that demands that any challenger efficiently convert directed multigraphs into undirected simple graphs. If the wizard can correctly solve a series of arbitrary instances of this problem, the Sphinx will unlock a secret passageway.



An example of transforming $G \to G'$

Let $G = (E, V)$ denote a directed multigraph. An undirected simple graph is a $G' = (V, E')$, such that E' is derived from the edges in E so that (i) every directed multi-edge, e.g., $\{(u, v), (u, v)\}$ or even simply $\{(u, v)\}$, has been replaced by a single pair of directed edges $\{(u, v), (v, u)\}$ and (ii) all self-loops (u, u) have been removed.

Describe and analyze an algorithm (explain how it works, give pseudocode if necessary, derive its running time and space usage, and prove its correctness) that takes $O(V + E)$ time and space to convert G into G', and thereby will solve any of the Sphinx's questions. Assume both G and G' are stored as adjacency lists.

Hermione's hints: Don't assume adjacencies `Adj[u]` are ordered in any particular way, and remember that you can add edges to the list and then remove ones you don't need.