Recitation 9/13

1. $T(n) = T(n-1) + cn \quad \forall\, n > 1$

$T(1) = c$

We start unrolling the general equation

$$T(n) = T(n-1) + cn. \quad \text{— } (1)$$

Substituting n as $(n-1)$,

$$T(n-1) = T[(n-1)-1] + c(n-1)$$
$$= T(n-2) + c(n-1)$$

Resubstituting $T(n-1)$ in (1),

$$T(n) = [T(n-2) + c(n-1)] + cn$$
$$= T(n-2) + c[n+(n-1)] \quad \text{— } (2)$$

Unrolling $T(n-2)$,

$$T(n) = [T(n-3) + c(n-2)] + c[n+(n-1)]$$
$$= T(n-3) + c[n+(n-1)+(n-2)] \quad \text{— } (3)$$

So now we need a general unrolling expression for some 'k' steps.

Let's write down our results from ①, ②, ③ for the first 3 steps.

Step 1 : $T(n) = T(n-1) + cn$

Step 2 : $T(n) = T(n-2) + c[n+(n-1)]$

Step 3 : $T(n) = T(n-3) + c[n+(n-1)+(n-2)]$

⋮

Step k : $T(n) = T(n-k)$
$+ c[n+(n-1)+(n-2)+(n-3) + \dots + (n-k+1)]$ —— ④

At each step, the recurrence expression $T(n-k)$ has a decreasing index $-(n-k)$, i.e.

$(n-1), (n-2), (n-3) \dots (n-k), (n-k-1),$
$\dots (n-(n-3)), (n-(n-2)), (n-(n-1))$

Thus, $(n-k)$ varies from $(n-1)$ at the start to $[n-(n-1)] = 1$ towards the end.

~~If we substitue~~

Let's rewrite our generic equation ④ for $(n-k) = 1$

$$\therefore \quad T(n) = T(1) + c[n + (n-2) + (n-3) + \dots + 2]$$

Since $T(1) = c$,

$$T(n) = c[1 + 2 + 3 + \dots + (n-2) + (n-1)]$$

$$= c \sum_{i=1}^{n} i$$

$$= \frac{cn(n+1)}{2}$$

$$= \frac{cn^2 + cn}{2}$$

$$= O(n^2)$$

$$\therefore \quad T(n) = O(n^2).$$

2.) $T(n) = 2T(n-1) + 1 \quad \forall n > 1$

$T(1) = 2$

Unrolling,

$$T(n) = 2T(n-1) + 1 \quad \text{— } (1)$$

$$T(n-1) = 2T(n-2) + 1$$

$\therefore$ Resubstituting,

$$T(n) = 2\left[2T(n-2) + 1\right] + 1$$

$$= 2^2 T(n-2) + 2 + 1$$

$$= 2^2 T(n-2) + [1+2] \quad \text{— } (2)$$

Notice that I'm not reducing the sum $[1+2]$

Unrolling further,

$$T(n) = 2^2 T(n-2) + [1+2]$$

$$= 2^2 \left[2T(n-3) + 1\right] + [1+2]$$

$$= 2^3 T(n-3) + [1 + 2 + 2^2]$$

Summarizing the first 3 steps,

Step 1 : $T(n) = 2T(n-1) + 1$

Step 2 : $T(n) = 2^2 T(n-2) + (1+2)$

Step 3 : $T(n) = 2^3 T(n-3) + (1+2+2^2)$

Step 4 would be

$$T(n) = 2^4 T(n-4) + (1+2+2^2+2^3)$$

$\therefore$ Step ~~5~~ k is :

$$T(n) = 2^k T(n-k) + (1+2+2^2+\ldots 2^{k-1})$$

We know what $T(1)$ is, so we can find k such that $T(n-k) = T(1)$. This happens when,

$$n-k = 1$$

i.e. $k = (n-1)$

$$\therefore \quad T(n) = 2^{(n-1)} \cdot T(n-(n-1))$$

$$+ \sum_{i=0} + (1+2+2^2+\ldots 2^{n-2})$$

$$= 2^{n-1} \cdot T(1) + \sum_{i=0}^{n-2} 2^i$$

$$= 2^{n-1} \cdot 2 + \sum_{i=0}^{n-2} 2^i$$

$$= 2^n + \frac{(2^{n-1} - 1)}{(2-1)}$$

The final substitution comes from Geometric Progression sum –

$$\sum_{i=0}^{n} (a^i) = \frac{a^{n+1} - 1}{a - 1}$$

$$\therefore T(n) = 2^n + 2^{n-1} - 1$$

$$= O(2^n)$$