Drag Forces

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1 Introduction

Drag forces occur when an object is moving throuh a fluid and the fluid opposes the motion of the object. The simplest form for a general drag force can be represented as

$$\vec{F}_{drag} = -b\vec{v} - c\vec{v}^2.$$

The minus signs serve as reminders that drag forces *oppose* the direction of motion. The $-b\vec{v}$ is called the **viscous** drag, and the $-c\vec{v}^2$ is called the **pressure** or **form** drag. Of course, there are many other types of drag, which can result in other equations, but these two are pretty common yet simple.

Working with both the viscous and pressure term can become challenging, so we often simplify it depending on the speed of the object. When the magnitude of the velocity is large, the magnitude of the v^2 term dominates over v, so we represent the force as $\vec{F} = -c\vec{v}^2$. On the other hand, when the magnitude of v is small, the v term dominates over v^2, so we can represent the force as $\vec{F} = -b\vec{v}$.

2 Viscous Drag

Let's start our discussion with viscous drag. As an object slowly moves through a fluid, the fluid is able to move in mostly parallel layers (called lamina), known as laminar or streamline flow. To get a quantitative idea of the difference in flow velocity between the layers, called the velocity gradient $\frac{\partial u}{\partial y}$, we need a property of the fluid known as its **viscosity**, μ (the symbol η is also commonly used). Here, u represents the flow velocity and y represents distance between the layers. The relationship between these quantities and the force acting on the layers is

$$\frac{F}{A} = \mu \frac{\partial u}{\partial y},$$

where A is the area of which the force acts on. The viscosity of some common fluids are listed below (found in viscosity tables).

Fluid	Viscosity(Pa·sec)
Air	1.83×10^{-5}
Corn Syrup	1.38
Olive Oil	8.1×10^{-2}
Oxygen	2.02×10^{-5}
Water (0° C)	1.797×10^{-3}
Water (100° C)	0.28×10^{-3}

Since the velocity gradient is proportional to the force, we see that the drag force can be written in the form

$$\vec{F}_d = -b\vec{v}.$$

Let's take a simple example of an object of mass m starting from rest and falling downward through a fluid. Call the downward direction positive. The forces acting on the object are gravity (downward), and the drag force (upward). Then we can write the net force in one dimension as

$$F_{net} = mg - bv.$$

By Newton's Second Law this is

$$ma = mg - bv.$$

Since $a = \frac{dv}{dt}$, we have the differential equation

$$m\frac{dv}{dt} = mg - bv.$$

Separating variables and taking integrals gives

$$\int_0^{v(t)} \frac{m}{mg - bv} dv = \int_0^t dt.$$

Since the object started from rest, we simply have

$$-\frac{m}{b}(\ln(mg - bv(t)) - \ln(mg)) = t.$$

After basic manipulations, we obtain

$$v(t) = \frac{mg}{b}(1 - e^{-\frac{b}{m}t}).$$

Note that $\lim_{t\to\infty} v(t) = \dfrac{mg}{b}$, which is called the **terminal velocity**. This makes sense because if we substitute $v = \dfrac{mg}{b}$ back into the original differential equation, we obtain $F_{net} = 0$ and $a = 0$, as expected when an object reaches its terminal velocity.

3 Pressure Drag

When a faster object is moving through a fluid, the fluid hits the front of the object, and is unable to continue moving in parallel layers, as it needs to go around the object. As a result, the flow velocity does not change smoothly as before, but abruptly, called turbulent flow. While we simplify the force in one dimension to

$$F_d = -cv^2,$$

the coefficient c actually has a lot more terms within it. In particular, the drag equation gives the magnitude of the drag force

$$F_d = \frac{1}{2}C_d\rho u^2 A,$$

where ρ is the density of the fluid, u is the flow velocity relative to the object, A is the cross sectional area of the object, and C_d is the **drag coefficient** for the object. The drag coefficient is mainly determined by an object's shape, although there are also other factors, the most general of which is Reynold's number (look this up if you're interested).

Now let's consider an object startin from rest and falling downward through a fluid while experiencing quadratic drag. Again, take the positive direction to be downward, so that gravity is positive and the drag force is negative. Then we can write the net force in one dimension as

$$F_{net} = mg - cv^2.$$

By Newton's Second Law, this is

$$m\frac{dv}{dt} = mg - cv^2,$$

and separating variables and taking integrals gives

$$\int_0^{v(t)} \frac{m}{mg - cv^2}\ dv = \int_0^t dt.$$

We will need to take the partial fraction decomposition of the left side, so let's write this more conveniently as

$$\int_0^{v(t)} \frac{1}{1 - \frac{c}{mg}v^2}\ dv = \int_0^t g\ dt.$$

We are now ready to take the partial fraction decomposition of the left side to obtain

$$\int_0^{v(t)} \frac{1}{2}\left(\frac{1}{1 + \sqrt{\frac{c}{mg}}v} + \frac{1}{1 - \sqrt{\frac{c}{mg}}v}\right) dv = \int_0^t g\ dt.$$

Applying initial conditions, we get

$$\sqrt{\frac{mg}{c}}\left(\ln\left(1 + \sqrt{\frac{c}{mg}}v(t)\right) - \ln\left(1 - \sqrt{\frac{c}{mg}}v(t)\right)\right) = 2gt.$$

After some manipulations, we obtain

$$v(t) = \sqrt{\frac{mg}{c}}\left(\frac{e^{2\sqrt{\frac{cg}{m}}t} - 1}{e^{2\sqrt{\frac{cg}{m}}t} + 1}\right).$$

Note that $\lim_{t\to\infty} v(t) = \sqrt{\frac{mg}{c}}$, which is our terminal velocity. This makes sense because if we substitute $v = \sqrt{\frac{mg}{c}}$ back into the original differential equation, we obtain $F_{net} = 0$ and $a = 0$, which is expected when an object reaches its terminal velocity.