Question A3

A light bulb has a solid cylindrical filament of length L and radius a, and consumes power P. You are to design a new light bulb, using a cylindrical filament of the same material, operating at the same voltage, and emitting the same spectrum of light, which will consume power nP. What are the length and radius of the new filament? Assume that the temperature of the filament is approximately uniform across its cross-section; the filament doesn't emit light from the ends; and energy loss due to convection is minimal.

Question A4

In this problem we consider a simplified model of the electromagnetic radiation inside a cubical box of side length L. In this model, the electric field has spatial dependence

$$E(x, y, z) = E_0 \sin(k_x x) \sin(k_y y) \sin(k_z z)$$

where one corner of the box lies at the origin and the box is aligned with the x, y, and z axes. Let h be Planck's constant, k_B be Boltzmann's constant, and c be the speed of light.

a. The electric field must be zero everywhere at the sides of the box. What condition does this impose on k_x, k_y, and k_z? (Assume that any of these may be negative, and include cases where one or more of the k_i is zero, even though this causes E to be zero.)

b. In the model, each permitted value of the triple (k_x, k_y, k_z) corresponds to a quantum state. These states can be visualized in a *state space*, which is a notional three-dimensional space with axes corresponding to k_x, k_y, and k_z. How many states occupy a volume s of state space, if s is large enough that the discreteness of the states can be ignored?

c. Each quantum state, in turn, may be *occupied* by photons with frequency $\omega = \frac{f}{2\pi} = c|\mathbf{k}|$, where

$$|\mathbf{k}| = \sqrt{{k_x}^2 + {k_y}^2 + {k_z}^2}$$

In the model, if the temperature inside the box is T, no photon may have energy greater than $k_B T$. What is the shape of the region in state space corresponding to occupied states?

d. As a final approximation, assume that each occupied state contains exactly one photon. What is the total energy of the photons in the box, in terms of h, k_B, c, T, and the volume of the box V? Again, assume that the temperature is high enough that there are a very large number of occupied states. (*Hint*: divide state space into thin regions corresponding to photons of the same energy.)

Note that while many details of this model are extremely inaccurate, the final result is correct except for a numerical factor.

Part B

Question B1

An AC power line cable transmits electrical power using a sinusoidal waveform with frequency 60 Hz. The load receives an RMS voltage of 500 kV and requires 1000 MW of average power. For this problem, consider only the cable carrying current in one of the two directions, and ignore effects due to capacitance or inductance between the cable and with the ground.

a. Suppose that the load on the power line cable is a residential area that behaves like a pure resistor.

   i. What is the RMS current carried in the cable?

   ii. The cable has diameter 3 cm, is 500 km long, and is made of aluminum with resistivity $2.8 \times 10^{-8}\ \Omega \cdot \text{m}$. How much power is lost in the wire?

b. A local rancher thinks he might be able to extract electrical power from the cable using electromagnetic induction. The rancher constructs a rectangular loop of length a and width $b < a$, consisting of N turns of wire. One edge of the loop is to be placed on the ground; the wire is straight and runs parallel to the ground at a height h much less than the length of the wire. Write the current in the wire as $I = I_0 \sin \omega t$, and assume the return wire is far away.

   i. Determine an expression for the magnitude of the magnetic field at a distance r from the power line cable in terms of I, r, and fundamental constants.

   ii. Where should the loop be placed, and how should it be oriented, to maximize the induced emf in the loop?

   iii. Assuming the loop is placed in this way, determine an expression for the emf induced in the loop (as a function of time) in terms of any or all of I_0, h, a, b, N, ω, t, and fundamental constants.

   iv. Suppose that $a = 5$ m, $b = 2$ m, and $h = 100$ m. How many turns of wire N does the rancher need to generate an RMS emf of 120 V?

c. The load at the end of the power line cable changes to include a manufacturing plant with a large number of electric motors. While the average power consumed remains the same, it now behaves like a resistor in parallel with a 0.25 H inductor.

   i. Does the power lost in the power line cable increase, decrease, or stay the same? (You need not calculate the new value explicitly, but you should show some work to defend your answer.)

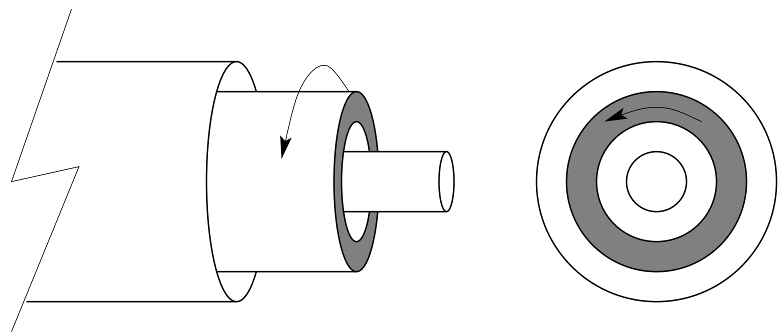
   ii. The power company wishes to make the load behave as it originally did by installing a capacitor in parallel with the load. What should be its capacitance?

Question A3

This problem inspired by the 2008 Guangdong Province Physics Olympiad

Two infinitely long concentric hollow cylinders have radii a and $4a$. Both cylinders are insulators; the inner cylinder has a uniformly distributed charge per length of $+\lambda$; the outer cylinder has a uniformly distributed charge per length of $-\lambda$.

An infinitely long dielectric cylinder with permittivity $\epsilon = \kappa\epsilon_0$, where κ is the dielectric constant, has a inner radius $2a$ and outer radius $3a$ is also concentric with the insulating cylinders. The dielectric cylinder is rotating about its axis with an angular velocity $\omega \ll c/a$, where c is the speed of light. Assume that the permeability of the dielectric cylinder and the space between the cylinders is that of free space, μ_0.

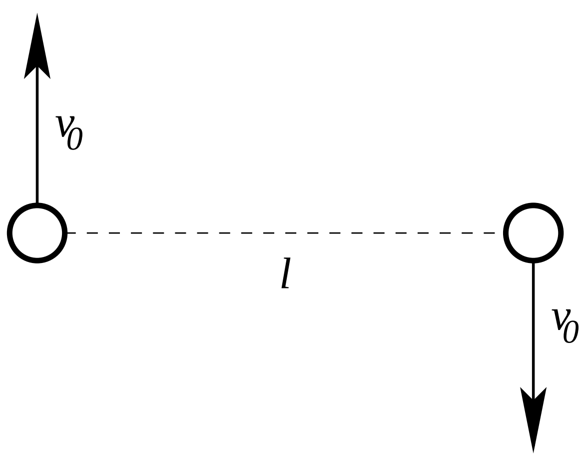


a. Determine the electric field for all regions.

b. Determine the magnetic field for all regions.

Question A4

Two masses m separated by a distance l are given initial velocities v_0 as shown in the diagram. The masses interact only through universal gravitation.



a. Under what conditions will the masses eventually collide?

b. Under what conditions will the masses follow circular orbits of diameter l?

c. Under what conditions will the masses follow closed orbits?

d. What is the minimum distance achieved between the masses along their path?

Question B2

For this problem, assume the existence of a hypothetical particle known as a *magnetic monopole.* Such a particle would have a "magnetic charge" q_m, and in analogy to an electrically charged particle would produce a radially directed magnetic field of magnitude

$$B = \frac{\mu_0}{4\pi}\frac{q_m}{r^2}$$

and be subject to a force (in the absence of electric fields)

$$F = q_m B$$

A magnetic monopole of mass m and magnetic charge q_m is constrained to move on a vertical, nonmagnetic, insulating, frictionless U-shaped track. At the bottom of the track is a wire loop whose radius b is much smaller than the width of the "U" of the track. The section of track near the loop can thus be approximated as a long straight line. The wire that makes up the loop has radius $a \ll b$ and resistivity ρ. The monopole is released from rest a height H above the bottom of the track.

Ignore the self-inductance of the loop, and assume that the monopole passes through the loop many times before coming to a rest.

a. Suppose the monopole is a distance x from the center of the loop. What is the magnetic flux ϕ_B through the loop?

b. Suppose in addition that the monopole is traveling at a velocity v. What is the emf $\mathcal{E}$ in the loop?

c. Find the change in speed Δv of the monopole on one trip through the loop.

d. How many times does the monopole pass through the loop before coming to a rest?

e. **Alternate Approach:** You may, instead, opt to find the above answers to within a dimensionless multiplicative constant (like $\frac{2}{3}$ or π^2). If you only do this approach, you will be able to earn up to 60% of the possible score for each part of this question.

You might want to make use of the integral

$$\int_{-\infty}^{\infty} \frac{1}{(1+u^2)^3} du = \frac{3\pi}{8}$$

or the integral

$$\int_0^{\pi} \sin^4\theta \, d\theta = \frac{3\pi}{8}$$

Question B2

This problem concerns three situations involving the transfer of energy into a region of space by electromagnetic fields. In the first case, that energy is stored in the kinetic energy of a charged object; in the second and third cases, the energy is stored in an electric or magnetic field.

In general, whenever an electric and a magnetic field are at an angle to each other, energy is transferred; for example, this principle is the reason electromagnetic radiation transfers energy. The power transferred per unit area is given by the *Poynting vector*:

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$$

In each part of this problem, the last subpart asks you to verify that the rate of energy transfer agrees with the formula for the Poynting vector. Therefore, you should not use the formula for the Poynting vector before the last subpart!

a. A long, insulating cylindrical rod has radius R and carries a uniform volume charge density ρ. A uniform external electric field E exists in the direction of its axis. The rod moves in the direction of its axis at speed v.

   i. What is the power per unit length $\mathcal{P}$ delivered to the rod?
   ii. What is the magnetic field B at the surface of the rod? Draw the direction on a diagram.
   iii. Compute the Poynting vector, draw its direction on a diagram, and verify that it agrees with the rate of energy transfer.

b. A parallel plate capacitor consists of two discs of radius R separated by a distance $d \ll R$. The capacitor carries charge Q, and is being charged by a small, constant current I.

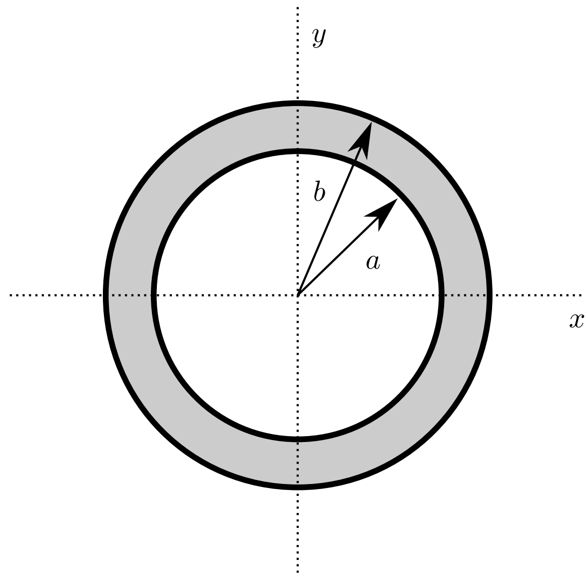
   i. What is the power P delivered to the capacitor?
   ii. What is the magnetic field B just inside the edge of the capacitor? Draw the direction on a diagram. (Ignore fringing effects in the electric field for this calculation.)
   iii. Compute the Poynting vector, draw its direction on a diagram, and verify that it agrees with the rate of energy transfer.

c. A long solenoid of radius R has $\mathcal{N}$ turns of wire per unit length. The solenoid carries current I, and this current is increased at a small, constant rate $\frac{dI}{dt}$.

   i. What is the power per unit length $\mathcal{P}$ delivered to the solenoid?
   ii. What is the electric field E just inside the surface of the solenoid? Draw its direction on a diagram.
   iii. Compute the Poynting vector, draw its direction on a diagram, and verify that it agrees with the rate of energy transfer.

Question A4

A positive point charge q is located inside a neutral hollow spherical conducting shell. The shell has inner radius a and outer radius b; $b - a$ is not negligible. The shell is centered on the origin.



a. Assume that the point charge q is located at the origin in the very center of the shell.

   i. Determine the magnitude of the electric field outside the conducting shell at $x = b$.

   ii. Sketch a graph for the magnitude of the electric field along the x axis on the answer sheet provided.

   iii. Determine the electric potential at $x = a$.

   iv. Sketch a graph for the electric potential along the x axis on the answer sheet provided.

b. Assume that the point charge q is now located on the x axis at a point $x = 2a/3$.

   i. Determine the magnitude of the electric field outside the conducting shell at $x = b$.

   ii. Sketch a graph for the magnitude of the electric field along the x axis on the answer sheet provided.

   iii. Determine the electric potential at $x = a$.

   iv. Sketch a graph for the electric potential along the x axis on the answer sheet provided.

   v. Sketch a figure showing the electric field lines (if any) inside, within, and outside the conducting shell on the answer sheet provided. You should show at least eight field lines in any distinct region that has a non-zero field.

Question B2

In parts a and b of this problem assume that velocities v are much less than the speed of light c, and therefore ignore relativistic contraction of lengths or time dilation.

a. An infinite uniform sheet has a surface charge density σ and has an infinitesimal thickness. The sheet lies in the xy plane.

   i. Assuming the sheet is at rest, determine the electric field $\vec{\mathbf{E}}$ (magnitude and direction) above and below the sheet.

   ii. Assuming the sheet is moving with velocity $\vec{\mathbf{v}} = v\hat{\mathbf{x}}$ (parallel to the sheet), determine the electric field $\vec{\mathbf{E}}$ (magnitude and direction) above and below the sheet.

   iii. Assuming the sheet is moving with velocity $\vec{\mathbf{v}} = v\hat{\mathbf{x}}$, determine the magnetic field $\vec{\mathbf{B}}$ (magnitude and direction) above and below the sheet.

   iv. Assuming the sheet is moving with velocity $\vec{\mathbf{v}} = v\hat{\mathbf{z}}$ (perpendicular to the sheet), determine the electric field $\vec{\mathbf{E}}$ (magnitude and direction) above and below the sheet.

   v. Assuming the sheet is moving with velocity $\vec{\mathbf{v}} = v\hat{\mathbf{z}}$, determine the magnetic field $\vec{\mathbf{B}}$ (magnitude and direction) above and below the sheet.

b. In a certain region there exists only an electric field $\vec{\mathbf{E}} = E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}}$ (and no magnetic field) as measured by an observer at rest. The electric and magnetic fields $\vec{\mathbf{E}}'$ and $\vec{\mathbf{B}}'$ as measured by observers in motion can be determined entirely from the local value of $\vec{\mathbf{E}}$, regardless of the charge configuration that may have produced it.

   i. What would be the observed electric field $\vec{\mathbf{E}}'$ as measured by an observer moving with velocity $\vec{\mathbf{v}} = v\hat{\mathbf{z}}$?

   ii. What would be the observed magnetic field $\vec{\mathbf{B}}'$ as measured by an observer moving with velocity $\vec{\mathbf{v}} = v\hat{\mathbf{z}}$?

c. An infinitely long wire wire on the z axis is composed of positive charges with linear charge density λ which are at rest, and negative charges with linear charge density $-\lambda$ moving with speed v in the z direction.

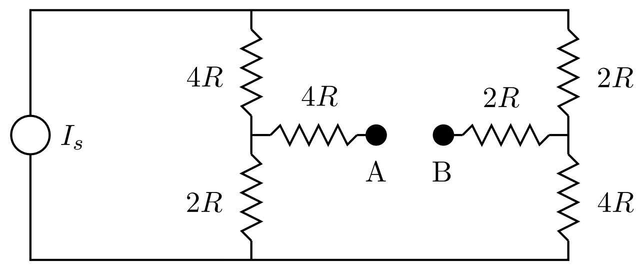
   i. Determine the electric field $\vec{\mathbf{E}}$ (magnitude and direction) at points outside the wire.

   ii. Determine the magnetic field $\vec{\mathbf{B}}$ (magnitude and direction) at points outside the wire.

   iii. Now consider an observer moving with speed v parallel to the z axis so that the negative charges appear to be at rest. There is a symmetry between the electric and magnetic fields such that a variation to your answer to part b can be applied to the magnetic field in this part. You will need to change the multiplicative constant to something dimensionally correct and reverse the sign. Use this fact to find and describe the electric field measured by the moving observer, and comment on your result. (Some familiarity with special relativity can help you verify the direction of your result, but is not necessary to obtain the correct answer.)

Question A2

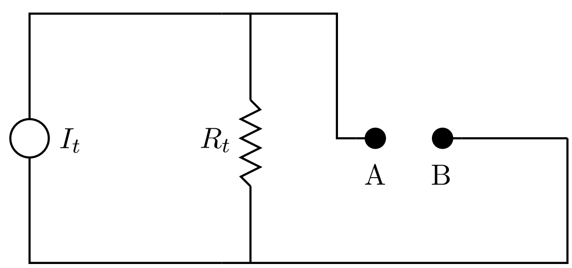
Consider the circuit shown below. I_s is a constant current source, meaning that no matter what device is connected between points A and B, the current provided by the constant current source is the same.



a. Connect an ideal voltmeter between A and B. Determine the voltage reading in terms of any or all of R and I_s.

b. Connect instead an ideal ammeter between A and B. Determine the current in terms of any or all of R and I_s.

c. It turns out that it is possible to replace the above circuit with a new circuit as follows:

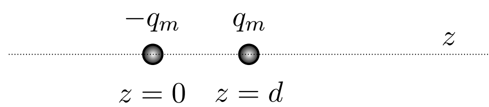


From the point of view of *any* passive resistance that is connected between A and B the circuits are identical. You don't need to prove this statement, but you do need to find I_t and R_t in terms of any or all of R and I_s.

Question B2

The nature of magnetic dipoles.

a. A "Gilbert" dipole consists of a pair of magnetic monopoles each with a magnitude q_m but opposite magnetic charges separated by a distance d, where d is small. In this case, assume that $-q_m$ is located at $z = 0$ and $+q_m$ is located at $z = d$.



Assume that magnetic monopoles behave like electric monopoles according to a coulomb-like force

$$F = \frac{\mu_0}{4\pi} \frac{q_{m1} q_{m2}}{r^2}$$

and the magnetic field obeys

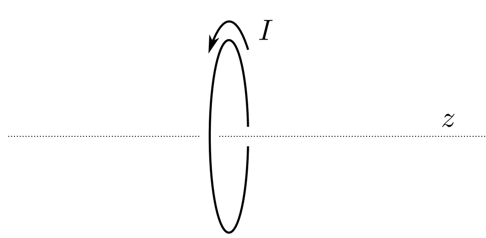
$$B = F/q_m.$$

i. What are the dimensions of the quantity q_m?

ii. Write an exact expression for the magnetic field strength $B(z)$ along the z axis as a function of z for $z > d$. Write your answer in terms of q_m, d, z, and any necessary fundamental constants.

iii. Evaluate this expression in the limit as $d \to 0$, assuming that the product $q_m d = p_m$ is kept constant, keeping only the lowest non-zero term. Write your answer in terms of p_m, z, and any necessary fundamental constants.

b. An "Ampère" dipole is a magnetic dipole produced by a current loop I around a circle of radius r, where r is small. Assume the that the z axis is the axis of rotational symmetry for the circular loop, and the loop lies in the xy plane at $z = 0$.



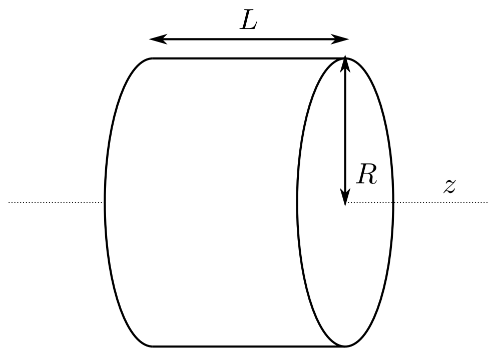
i. Write an exact expression for the magnetic field strength $B(z)$ along the z axis as a function of z for $z > 0$. Write your answer in terms of I, r, z, and any necessary fundamental constants.

ii. Let kIr^γ have dimensions equal to that of the quantity p_m defined above in Part aiii, where k and γ are dimensionless constants. Determine the value of γ.

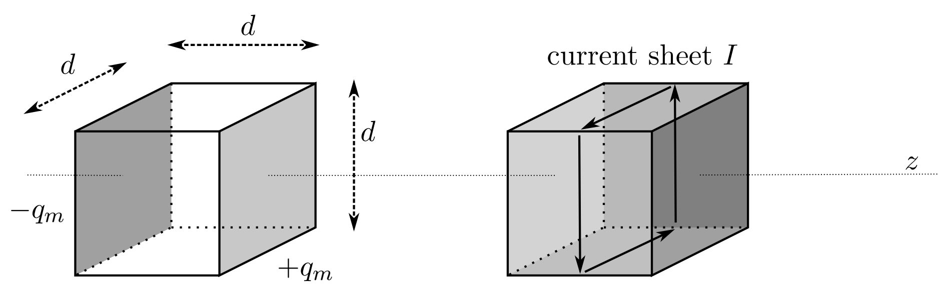
iii. Evaluate the expression in Part bi in the limit as $r \to 0$, assuming that the product $kIr^\gamma = p'_m$ is kept constant, keeping only the lowest non-zero term. Write your answer in terms of k, p'_m, z, and any necessary fundamental constants.

iv. Assuming that the two approaches are equivalent, $p_m = p'_m$. Determine the constant k in Part bii.

c. Now we try to compare the two approaches if we model a physical magnet as being composed of densely packed microscopic dipoles.



A cylinder of this uniform magnetic material has a radius R and a length L. It is composed of N magnetic dipoles that could be either all Ampère type or all Gilbert type. N is a *very* large number. The axis of rotation of the cylinder and all of the dipoles are all aligned with the z axis and all point in the same direction as defined above so that the magnetic field *outside* the cylinder is the same in either dipole case as you previously determined. Below is a picture of the two dipole models; they are cubes of side $d << R$ and $d << L$ with volume $v_m = d^3$.



Gilbert Dipole Ampère Dipole

i. Assume that $R \gg L$ and only Gilbert type dipoles, determine the magnitude and direction of B at the center of the cylinder in terms of any or all of p_m, R, L, v_m, and any necessary fundamental constants.

ii. Assume that $R \ll L$ and only Ampère type dipoles, determine the magnitude and direction of B at the center of the cylinder in terms of any or all of p_m, R, L, v_m, and any necessary fundamental constants.