A Classification of the Minimal Separating Sets of Low Genus Surfaces

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Abstract

Consider a surface S and let $M \subset S$. If $S \setminus M$ is not connected, then we say M *separates* S, and we refer to M as a *separating set* of S. If M separates S, and no proper subset of M separates S, then we say M is a *minimal separating set* of S. In this paper we classify the minimal separating sets of the orientable surfaces of genus $g \leq 2$.

Surfaces and Separating Sets

We begin with some basic definitions.

Definition 1. *When we say* surface *we are referring to a closed, connected, triangulated, orientable, 2-manifold.*

Let S be a surface and let $M \subset S$.

Definition 2. *If $S \setminus M$ is not connected, then we say M* separates S*, and we refer to M as a* separating set *of S.*

Definition 3. *If M separates S, and no proper subset of M separates S, then we say M is a* minimal separating set *of S.*

In this paper we will not consider pathological separating sets, such as the common boundary of the Lakes of Wada. To that end, when we refer to subsets of a surface it will be understood that we are referring to finitely triangulated subsets. Such subsets may be thought of as finite simplicial subcomplexes of the triangulated surface.

Graphs and Embeddings

Colloquially speaking, when we talk about *graphs* we are referring to undirected multigraphs which may or may not have loops. This is made rigourous as follows.

Definition 4. *A* graph *G is a pair $(V(G), E(G))$ where $V(G)$ is a non-empty finite set of elements called* vertices*, and E is a finite multiset whose elements are unordered multiset pairs of vertices in $V(G)$. Elements of $E(G)$ are referred to as* edges *and the elements of edges are refered to as* edge-ends.

When the graph in question is clear from context, we may use the simpler notation V and E to refer to the vertex set and edge set, respectively, of a given graph.

Definition 5. *A vertex contained in an edge is said to be* incident *to the edge.*

Definition 6. *Vertices $v_1, v_2 \in V$ are said to be* adjacent *if there exists an edge that contains them both.*

Definition 7. *An edge whose edge-ends are identical is called a* loop*. An edge that is not a loop is called a* non-loop*.*

Definition 8. *Let v be a vertex of a graph. Let l and n be the number of loops and non-loops, respectively, incident to v. Then the* degree of v*, denoted $deg(v)$, is given by $n + 2l$.*

Definition 9. *Let G be a graph and let $A \subset V$. Then the* subgraph of G induced by A*, denoted $G[A]$, is the graph whose vertex set is A, and whose edge set is the subset of $E(G)$ whose edge-ends are in A.*

Less formally, the induced subgraph $G[A]$ is the graph obtained from G by removing all vertices from G except those in A. Edges that lose an edge-end during this process are also removed.

We often want to consider what parts of G remain after the removal of some subset of its vertices. For example, let $B \subset V$. Then removal of the vertices in B from the graph G will yield the induced subgraph $G[V \setminus B]$. We may think of the induced subgraph $G[V \setminus B]$ as the part of G that remains when we remove B. Thus, for simplicity, we use the notation $G \setminus B$ to denote the induced subgraph $G[V \setminus B]$.

Definition 10. *Two graphs G and H are said to be* graph-homeomorphic *if there is a graph isomorphism from a subdivision of G to a subdivision of H.*

One can easily verify that graph-homeomorphism is an equivalence relation on the set of all graphs.

Definition 11. *A graph G is said to be* homeomorphically irreducible *if whenever a graph H is homeomorphic to G, either $G = H$ or H is a subdivision of G.*

Behzad and Chartrand show in [1] that in every equivalence class of homeomorphic graphs there exists a unique homeomorphically irreducible graph.

Definition 12. *An* embedding *of a graph $G = (V, E)$ into a surface S is a function ϕ that maps distinct vertices of G to distinct points in S and edges of G to simple arcs (homomorphic images of $[0, 1]$) in S such that:*

1. *for every edge $e = \{v_1, v_2\}$ in E, the endpoints of the simple arc $\phi(e)$ are the points $\phi(v_1)$ and $\phi(v_2)$,*

2. *for all $e \in E$, and for all $v \in V$, $\phi(v)$ is not in the interior of $\phi(e)$,*

3. *for all $e_1, e_2 \in E$ $\phi(e_1)$ and $\phi(e_2)$ do not interesect at a point that is interior to either arc.*

Note that if ϕ is an embedding of a graph G into a surface S, then the $\phi[G]$ is a simplicial 1-subcomplex of S.

Definition 13. *The* regions *of an embedding ϕ of a graph G into a surface S are the connected components of $S \setminus \phi[G]$.*

Definition 14. *Every region of an embedding ϕ of a graph G into a surface S is bounded by the image of a closed walk. This closed walk is unique (up to a cyclic permutation) and the ordered list of edges in the walk is refered to as the* boundary walk *of the region.*

Minimal Separating Sets as Graph Embeddings

No 0-complex can separate a surface. Moreover, if M separates a surface S, then so does the 1-skeleton of M and it follows that every minimal separating set of a surface is a 1-complex. As such, a minimal separating set of a surface can be thought of as the image of a graph embedding into the surface.

Definition 15. *If ϕ is an embedding of a graph G into a surface S and the image of ϕ is a minimal separating set of S then we say ϕ is a* minimal separating embedding*.*

Note that if ϕ is an embedding of a graph G into S, and ψ is an embedding of a graph H into S, and if $\phi[G]$ and $\psi[H]$ are homeomorphic (in the topological sense) then the graphs G and H are graph-homeomorphic.

Definition 16. *Suppose M is a minimal separating set of a surface S, and let $\mathcal{C}_M = \{G : \exists$ a graph embedding ϕ of G into S such that $\phi[G] = M\}$. Then $\mathcal{C}_M$ contains a unique homeomorphically irreducible graph G_M. We say G_M is* the graph underlying *M.*

Definition 17. *We refer to a graph G as a* minimal separating graph of *S if G is the graph underlying some minimal separating set of S.*

Definition 18. *Let $\mathbb{G}_g$ denote the set of all minimal separating graphs of an orientable surface of genus g.*

We will classify the minimal separating sets of the orientable surfaces of genus $g \leq 2$ by their underlying graphs.

The Finitude of $\mathbb{G}_g$

By Euler's theorem we know that if ϕ is an embedding of G into S then $|V| - |E| + |R| \geq 2 - 2g$, where V is the vertex set of G, E is the edge set of G, R is the set of regions of the embedding, and g is the genus of S.

Lemma 1. *If ϕ is a minimal separating embedding of G into S then $|R| = 2$.*

Proof. Since $\phi[G]$ separates S, it is clear that $|R| \geq 2$. Now suppose $|R| > 2$. Since S is connected, there exist distinct regions f_1 and f_2 such that $\bar{f}_1 \cap \bar{f}_2 \neq \emptyset$.

Since f_1 and f_2 are distinct regions, $f_1 \cap f_2 = \emptyset$. Thus $bd(\bar{f}_1) \cap bd(\bar{f}_2) \neq \emptyset$. Note that $bd(\bar{f}_1) \cap bd(\bar{f}_2) = \phi[\{e_1\} \cup ... \cup \{e_n\}]$ for some $e_1, ..., e_n \in E$. Let e be one of the $e_1, ..., e_n$ and let $x \in int(\phi(e))$. Then $x \in \phi[G]$, but $S \setminus (\phi[G] \setminus \{x\})$ has exactly one less connected component than $S \setminus \phi[G]$. Since $S \setminus \phi[G]$ has strictly more than 2 connected components, $S \setminus (\phi[G] \setminus \{x\})$ has at least 2 connected components. Thus $\phi[G] \setminus \{x\}$ separates S and $\phi[G]$ is not a minimal separating set. $\square$

Lemma 2. *If ϕ is a minimal separating embedding of G into S and $e \in E$ then $\phi(e)$ is in the boundary of both regions of ϕ.*

Proof. Suppose ϕ is a minimal separating embedding of a graph G into a surface S. It is clear that for every edge e, $\phi(e)$ must be in the boundary of at least one region. Suppose, by way of contradiction, that for some $e \in E$, $\phi(e)$ were in the boundary of exactly one of the two regions of ϕ. Then removal of an interior point x of $\phi(e)$ would not decrease the number of connected components. That is, $S \setminus (\phi[G] \setminus \{x\})$ has at least as many connected components as $S \setminus \phi[G]$. Hence $\phi([G] \setminus \{x\})$ is a separating set of S, and it follows that $\phi[G]$ is not *minimally* separating. $\square$

Theorem 1. *A graph embedding is a minimal separating embedding if and only if the following conditions are satisfied:*

1. *The graph has no isolated vertices.*
2. *The embedding has exactly two regions.*
3. *The image of each edge is in the boundary of both regions of the embedding.*

Proof. Suppose ϕ is a minimal separating embedding of G into S. Condition 1 is immediate. Lemmas 1 and 2 demonstrate conditions 2 and 3, respectively. So the forward direction is proved.

Now suppose ϕ is an embedding of G into S that satisfies the three conditions. Since condition 2 is satisfied, we know $\phi[G]$ separates S. To show that $\phi[G]$ is a *minimal* separating set, let $x \in \phi[G]$. Condition 1 implies that $x \in \phi[e]$ for some $e \in V$. By condition 3 we know that x is in the boundary of both regions of the embedding. It follows that $S \setminus (\phi[G] \setminus \{x\})$ is connected. $\square$

Lemma 3. *Suppose ϕ is a minimal separating embedding of G into S. Then $|E| - |V| \leq 2g$ where g is the genus of S.*

Proof. By Euler's theorem and lemma 1, $|V| - |E| + 2 \geq 2 - 2g$. Thus $|E| - |V| \leq 2g$ as desired. $\square$

Definition 19. *The graph consisting of one vertex and one edge (the cycle on one vertex) is called an* isolated loop*.*

Definition 20. *A vertex v is a called a* boomerang *if it has degree 2 and is incident to a single edge. That is, a boomerang is the lone vertex in an isolated loop.*

Lemma 4. *For all g, and for all $G \in \mathbb{G}_g$, if v is a vertex of G and $deg(v) = 2$, then v is a boomerang.*

Proof. Suppose $g \geq 0$, and that $G \in \mathbb{G}_g$. Let v be a vertex of G such that $deg(v) = 2$, and suppose that v is not a boomerang. Then v is incident to precisely two edges, and so v can be smoothed. Thus G is not homeomorphically irreducible. Hence $G \notin \mathbb{G}_g$. □

Lemma 5. *For all $g \geq 0$, and for all $G \in \mathbb{G}_g$, G has no vertex of odd degree.*

Proof. Let v be a vertex and consider a neighborhood around $\phi(v)$. The edge-ends emanating from v divide the neighborhood into $deg(v)$ subdivisions. By Lemma 1, there are exactly 2 regions of ϕ and each of these subdivisions must be a subset of exactly one of the two regions. By Lemma 2, each of the edge ends must be in the boundary of both regions. If $deg(v)$ were odd then there would exist an edge end emanating from v that was in the boundary of only one of the two regions. Hence $deg(v)$ is not odd. □

Lemma 6. *For all $g \geq 0$, and for all $G \in \mathbb{G}_g$, if B is the set of boomerangs of G then $2|V(G \setminus B)| \leq |E(G \setminus B)|$, where $V(G \setminus B)$ and $E(G \setminus B)$ denote the vertex set and edge set (respectively) of the induced subgraph $G \setminus B$.*

Proof. Let $g \geq 0$, and let $G \in \mathbb{G}_g$. Let B be the set of boomerangs of G. By Lemmas 4 and 5, and the observation that minimal separating graphs do not contain vertices of degree zero, $deg(v) \geq 4\ \forall v \in V(G \setminus B)$. Thus we have $4|V(G \setminus B)| + 2|B| \leq \left[\sum_{v \in V \setminus B} deg(v)\right] + |B| = 2|E(G \setminus B)| + 2|B|$. Thus $2|V(G \setminus B)| \leq |E(G \setminus B)|$. □

Lemma 7. *Let B be the set of boomerangs of a graph G and suppose G is a minimal separating graph of a surface S of genus g. Either the graph $G \setminus B$ is empty or it is a minimal separating graph of a surface of genus $g - |B|$.*

Proof. Suppose G is a minimal separating graph for a surface S of genus g, and let ϕ be a minimal separating embedding of G into S. It will suffice to show that if L is any isolated loop of G, and $G \setminus L$ is not empty, then there exists an embedding of $G \setminus L$ into a surface of genus $g - 1$.

Let L be any loop of G. Either $\phi[L]$ is simple, or it is not. If $\phi[L]$ is simple, then L separates S, and it follows that $G \setminus L$ is empty.

Suppose $G \setminus L$ is not empty. Then $\phi[L]$ is not simple. So $\phi[L]$ must circumnavigate a handle of S. We will construct a minimal separating embedding of the graph $G \setminus L$ into a surface of genus $g - 1$ using surgery.

Note that ϕ satisfies the three conditions of theorem 1. Remove a closed neighborhood of $\phi[L]$ from S. Note that by condition 3 of theorem 1, this has the result of creating exactly two boundary components (one in each region of ϕ). Glue one disk into each of the two components. What remains is an embedding of the graph $G \setminus L$ into a surface of surface of genus $g - 1$. Note that this embedding is a minimal separating embedding because it satisfies the three requirements of theorem 1 – for it inherited these properties from ϕ. □

Colloquially, the above lemma states that, in minimal separating graphs, the only role played by isolated loops is the role of handle-cutter. The sole exception is for the surface of genus zero, where the isolated loop is the minimal separating graph itself.

Lemma 8. *For all g, and for all $G \in \mathbb{G}_g$, if B is the set of boomerangs of G then $|B| \leq g + 1$.*

Proof. Suppose $G \in \mathbb{G}_g$, and B is the set of boomerangs of G. Let ϕ be a minimal separating embedding of G into S. Consider the case where $G \setminus B$ is not empty. Then by lemma 7, the graph $G \setminus B$ has a minimal separating embedding in a surface of genus $g - |B|$. Thus $|B| \leq g$.

Now consider the case where $G \setminus B$ is empty. Then G consists of $|B|$ isolated loops. If $|B| > 1$ then for each isolated loop L of G, $\phi[L]$ is not simple. That is, $\phi[L]$ circumnavigates at least one handle of S. If two loops circumnavigate the same handle, they separate the surface. Thus, by the pigeon hole principle, any embedding of $g + 1$ isolated loops must separate S. Thus $|B| \leq g + 1$. □

Lemma 9. *For all $g \geq 1$, and for all $G \in \mathbb{G}_g$, if B is the set of boomerangs of G then $0 \leq |V(G \setminus B)| \leq 2g - |B|$.*

Proof. Suppose $g \geq 1$, and let $G \in \mathbb{G}_g$. Let B be the set of boomerangs of G.

If $G \setminus B$ is empty then $|V(G \setminus B)| = 0$, and the lemma follows trivially.

If $G \setminus B$ is not empty then, by lemma 6.5, the graph $G \setminus B$ is in $\mathbb{G}_{\hat{g}}$, where $\hat{g} = g - |B|$.

It follows by lemma 3, $|E(G \setminus B)| - |V(G \setminus B)| \leq 2\hat{g}$. So $|E(G \setminus B)| \leq 2\hat{g} + |V(G \setminus B)|$.

By Lemma 6, $2|V(G\setminus B)| \leq |E(G\setminus B)|$. Thus $2|V(G\setminus B)| \leq 2\hat{g}+|V(G\setminus B)|$. Solving for $|V(G\setminus B)|$ we find $|V(G\setminus B)| \leq 2\hat{g} = 2(g-|B|) = 2g-2|B| \leq 2g-|B|$ as desired. □

Theorem 2. *For all g, $\mathbb{G}_g$ is finite.*

Proof. Fix g, and let $G \in \mathbb{G}_g$. Let B be the set of boomerangs of G. By lemmas 9 and 8, $0 \leq |V(G)| = |V(G \setminus B)| + |B| \leq 2g$. This puts an upper bound on the size of $|V(G)|$ for any graph G in $\mathbb{G}_g$. This, in combination with the statement of Lemma 3, provides an upper bound on $|E|$. In particular, $0 < |E(G)| \leq 2g + |V(G)|$. Thus, $\mathbb{G}_g$ is finite. □

Definition 21. *For $g \geq 1$ define $\mathbb{C}_g$ to be the set of all graphs G satisfying the following properties:*

1. $0 \leq |V(G)| \leq 2g$,
2. $0 < |E(G)| \leq 2g + |V(G)|$
3. *G has no vertex of odd degree*
4. *The only vertices of G with degree two are boomerangs*
5. *G has at most $g + 1$ boomerangs*

We refer to $\mathbb{C}_g$ as the set of candidate graphs *for genus g.*

It is clear that for all $g \geq 1$, $\mathbb{C}_g$ is finite and $\mathbb{G}_g \subset \mathbb{C}_g$. For low genus surfaces, $\mathbb{C}_g$ can be generated quite easily using a contemporary computer. To decide which members of $\mathbb{C}_g$ are also in $\mathbb{G}_g$, we provide a method to test whether a given graph in $\mathbb{C}_g$ has a minimal separating embedding in a surface of genus g.

Rotation Systems and Band Decompositions

Definition 22. *A graph embedding induces, for each vertex v, a cyclic ordering of edges incident to v. This ordering is refered to as the* rotation at v, *and is unique up to a cyclic permutation of the edges.*

Definition 23. *The set $\mathcal{R}$ of all cyclic orderings induced by a graph embedding is refered to as a* rotation system.

Embeddings with the same rotation system are considered equivalent. As there are only finitely many possible rotations at each vertex, and only finitely many vertices in a graph, we know there are a finite number of possible rotation systems for a given graph. Thus the infinite set all possible embeddings of a graph into a surface can be partitioned into finitely many cells, with each cell corresponding to a unique rotation system.

For the calculations that follow, it will be useful to consider a closed neigborhood around the image of a graph imbedding.

Definition 24. *Let ϕ be an embedding of a graph G into a surface S. We surround the image of each vertex with a closed disk, refered to as a* 0-band, *and surround the image of each edge with a thin band, referred to as a* 1-band. *The union of 0-bands and 1-bands forms a closed neighborhood in S that preserves the shape of the graph G. We refer to the union of the 0-bands and 1-bands (omitting the rest of the surface) as the* reduced band decomposition *of ϕ.*

For a more thorough discussion of rotation systems and reduced band decompositions see [2].

Identifying Minimal Separating Graphs Using Rotation Systems

Upon fixing a genus g we can create the finite set $\mathbb{C}_g$ of candidate graphs. We know that the set $\mathbb{G}_g$ of minimal separating graphs is contained in $\mathbb{C}_g$. Moreover, for each graph $G \in \mathbb{C}_g$ we can generate the finite set $\mathbf{R}_G$ of all rotation systems on G.

Now suppose $R \in \mathbf{R}_G$. We want a way to test whether there exists a minimal separating embedding ϕ from G into S such that:

1. The rotation system corresponding to ϕ is R and
2. The surface S has genus g.

Let's begin with the following useful observation.

Corollary 1. $\mathbb{G}_0 \subset \mathbb{G}_1 \subset \mathbb{G}_2 \subset ...$

Proof. Let $G \in \mathbb{G}_k$ for some $k \geq 0$. Let S_k be a surface of genus k and let ϕ_k be a minimal separating embedding of G into S_k. If we attach a handle to one of the two regions of ϕ_k, we increase the genus of the surface by one while the three conditions of theorem 1 remain satisfied. The result is a minimal separating embedding ϕ_{k+1} of G into a surface S_{k+1} of genus $k+1$. □

Definition 25. *Let $\mathcal{B}$ be the reduced band decomposition corresponding to a graph embedding ϕ of a graph G into a surface S, and let b be one of the boundary components of $\mathcal{B}$. Then $\langle b \rangle$ denotes the set of edges of G that appear in the boundary walk of b.*

Definition 26. *Let $\mathcal{R}$ be a rotation system for an embedding of a graph G with no isolated vertices. Let b_1, b_2, ..., b_n be the boundary components of the reduced band decomposition corresponding to $\mathcal{R}$. We refer to $\mathcal{R}$ as a* two-sided rotation system *if there exists a partition of the set $\{b_1, b_2, ..., b_n\}$ into two cells, C_1 and C_2, such that for $\forall e \in E, \forall i \in \{1, 2\}, \left(e \in \bigcup_{b \in C_i} \langle b \rangle\right)$.*

Colloquially speaking, a two-sided rotation system $\mathcal{R}$ is a one for which:

1. The underlying graph G has no isolated vertices and
2. There exists an embedding ϕ of G with the following properties:
   (a) The rotation system corresponding to ϕ is $\mathcal{R}$
   (b) The embedding ϕ has exactly two regions (corresponding to the cells C_1 and C_2).
   (c) The image of each edge can be found in the boundary of each region.

Thus, in light of theorem 1, it follows that minimal separating graph embeddings are exactly those with two-sided rotation systems. This is made rigorous with the following theorem.

Theorem 3. *There exists a minimal separating embedding ϕ with corresponding rotation system $\mathcal{R}$ if and only if $\mathcal{R}$ is a two-sided rotation system .*

Proof. Let $\mathcal{R}$ be a rotation system for an embedding of a graph G with no isolated vertices. Let b_1, b_2, ..., b_n be the boundary components of the reduced band decomposition $\mathcal{B}$ corresponding to $\mathcal{R}$.

Suppose there exists a minimal separating embedding ϕ with corresponding rotation system $\mathcal{R}$. Then by condition 2 of theorem 1, there are exactly two regions of ϕ. Let r_1 and r_2 be the regions of ϕ. Note that every boundary component of $\mathcal{B}$ lies in exactly one of the two regions of ϕ. So form the desired partition of $\{b_1, b_2, ..., b_n\}$ by defining C_1 to be the set of boundary components of $\mathcal{B}$ that lie in region $r_1$and defining C_2 analogously. Then condition 3 of theorem 1 tell us that $\forall e \in E, \forall i \in \{1, 2\}, \left(e \in \bigcup_{b \in C_i} \langle b \rangle\right)$, and the forward direction is proved.

Now suppose that the set $\{b_1, b_2, ..., b_n\}$ can be partitioned into two cells, C_1 and C_2, such that for $\forall e \in E, \forall i \in \{1, 2\}, \left(e \in \bigcup_{b \in C_i} \langle b \rangle\right)$. We then construct a minimal separating embedding ϕ with corresponding rotation system $\mathcal{R}$ using surgery.

First consider two spheres, A_1 and A_2. Remove $|C_1|$ open discs from A_1 and name the result $\bar{A}_1$. Construct $\bar{A}_2$ similarly. We will identify the boundary components of $\mathcal{B}$ to the boundary components of $\bar{A}_1$ and $\bar{A}_2$ to create a surface in the following way.

For each $i \in \{1,2\}$ let $BOUNDARIES(\bar{A}_i)$ be the set of boundary components of $\bar{A}_i$, and let $\psi_i : BOUNDARIES(\bar{A}_i) \to C_i$ be a bijection. Next, for each $i \in \{1,2\}$, and for each $j \in BOUNDARIES(\bar{A}_i)$, identify the boundary component j of $\bar{A}_i$ with the boundary component $\psi_i(j)$ of $\mathcal{B}$. The result is a band decomposition of a graph imbedding.

Finally, contract the 0-bands and 1-bands of $\mathcal{B}$ to points and lines, respectively, and we get a graph embedding with corresponding rotation system $\mathcal{R}$. By construction, there are exactly two regions of the embedding (the regions corresponding to $\bar{A}_1$ and $\bar{A}_2$), and every edge is on the boundary of both regions. So by theorem 1, the embedding is a minimal separating embedding. □

The previous theorem can tell us whether there exists a minimal separating embedding that corresponds to a given rotation system, but it doesn't tell us anything about the *genus* of the codomain of such an embedding. The following theorem helps us with this.

Theorem 4. *Let $\mathcal{R}$ be a rotation system for a minimal separating embedding of a graph G. Let n be the number of boundary components of the reduced band decomposition corresponding to $\mathcal{R}$. Then* $min\{g \in \mathbb{N} : G \in \mathbb{G}_g\} = \frac{|E|-|V|+n}{2} - 1$.

Proof. Let $\mathcal{R}$ be a rotation system for a minimal separating embedding of a graph G into a surface S of genus g_S. Let $\mathcal{D}$ be the reduced band decomposition corresponding to $\mathcal{R}$, and let n be the number of boundary components of $\mathcal{D}$.

By theorem 2 we know the embedding has exactly two regions, and that each of the n boundary components of $\mathcal{D}$ lies in exactly one of these two regions. We will refer to the two regions of the embedding as A and B. Let n_A denote the number of boundary components of $\mathcal{D}$ found in A, and define n_B similarly. Note that $n_A + n_B = n$.

Note that region A must have exactly n_A boundary components and region B must have exactly n_B boundary components. Let $\overline{A}$ be the surface obtained by gluing n_A discs into the n_A boundary components of A. Define the surface $\overline{B}$ analogously. Let $g_{\overline{A}}$ denote the genus of the surface $\overline{A}$, and let $g_{\overline{B}}$ denote the genus of the surface $\overline{B}$.

The Euler characteristic of $\overline{A}$, denoted $\chi(\overline{A})$, is given by $\chi(\overline{A}) = 2 - 2g_{\overline{A}}$. Since the surface A is obtained from $\overline{A}$ by removing n_A discs, we know $\chi(A) = 2 - 2g_{\overline{A}} - n_A$. Similarly, $\chi(\overline{B}) = 2 - 2g_{\overline{B}}$, and $\chi(B) = 2 - 2g_{\overline{B}} - n_B$. The Euler characteristic of the reduced band decomposition is given by $\chi(D) = |V| - |E|$. Thus the Euler characteristic of S can be expressed as follows:

$$2-2g_S = \chi(S) = \chi(A)+\chi(B)+\chi(D) = (2-2g_{\overline{A}}-n_A)+(2-2g_{\overline{B}}-n_B)+(|V|-|E|)$$

Solving for g_S, we obtain:

$$g_S = \frac{|E| - |V| + n}{2} + (g_{\overline{A}} + g_{\overline{B}}) - 1$$

This expression is minimized when $g_{\overline{A}} = g_{\overline{B}} = 0$. That is, g_S is as small as possible when the regions A and B are spheres with n_A and n_B discs removed, respectively. In this case we have $g_S = \frac{|E|-|V|+n}{2} - 1$ as desired. □

Corollary 2. *A graph $G \in \mathbb{G}_g$ if and only if there exists a two-sided rotation system $\mathcal{R}$ on G such that $g \geq \frac{|E|-|V|+n}{2} - 1$ (where n is the number of boundary components of the reduced band decomposition corresponding to $\mathcal{R}$).*

Proof. The result follows immediately from theorem 4 and corollary 1. $\square$

An Algorithm for Finding Minimal Separating Graphs

Given a genus g, we compute the set $\mathbb{G}_g$ as follows.

1. Generate the finite set $\mathbb{C}_g$ of candidate graphs for genus g.

2. For each graph G in $\mathbb{C}_g$ do the following:

    (a) Generate the finite set $\mathbf{R}_G$ of rotation systems on G.

    (b) Search $\mathbf{R}_G$ for a two-sided rotation system $\mathcal{R}$ for which $g \geq \frac{|E|-|V|+n}{2} - 1$ (where n is the number of boundary components of the reduced band decomposition corresponding to $\mathcal{R}$). G is in $\mathbb{G}_g$ if and only if such a rotation system is found.

Though this algorithm is super-exponential in g, it is sufficient for finding minimal separating graphs for surfaces of genus $g \leq 2$ using a contemporary desktop computer.

Results

Using the algorithm in the previous section we find all graphs in $\mathbb{G}_1$ and $\mathbb{G}_2$. These graphs are listed and numbered in the Appendix along with thier two-sided rotation systems. The set $\mathbb{G}_1$ consists of graphs 1, 2, 3, 6, and 10. The set $\mathbb{G}_2$ consists of graphs 1 through 26.

Final Thoughts

[Optional: explain that min sep set classification can be made finer than classifying by underlying graph. We can do it up to surface homeomorphism. Fix any min sep embedding. Handles of the surface fully contained in one of the two regions can be moved to the other region. The location of these handles doesn't affect the underlying graph (or two-sided rotation system) but may affect whether two min sep set are equivalent up to surface homeomorphism.]

[Optional conjecture: We noticed the followng curious fact when searching for min sep graphs: It turns out that for each min sep graph in genus 0, 1, and 2, there is – up to a relabeling of the vertices and edges – exactly one two-sided rotation system for that graph. Is this the case for all min sep graphs of every genus? We suspect so.]

References

[1] Behzad, Mehdi, and Gary Chartrand. *Introduction to the Theory of Graphs*. Boston: Allyn and Bacon, 1972. 91-92. Print.

[2] Gross, Jonathan L., and Thomas W. Tucker. "SURFACES AND GRAPH IMBEDDINGS." *Topological Graph Theory*. New York: Wiley, 1987. Print.

Appendix

#	Rotation System
1	$v_1 : e_1$
2	$v_1 : e_1e_1e_2e_2$
3	$v_1 : e_1e_1e_2e_2e_3e_3$
4	$v_1 : e_1e_1e_2e_2e_3e_3e_4e_4$
5	$v_1 : e_1e_1e_2e_2e_3e_3e_4e_4e_5e_5$
6	$v_1 : e_1e_1$ $v_2 : e_2e_2$
7	$v_1 : e_1e_1e_2e_2$ $v_2 : e_3e_3$
8	$v_1 : e_1e_2e_3e_1e_2e_3$ $v_2 : e_4e_4$
9	$v_1 : e_1e_1e_2e_3$ $v_2 : e_2e_3e_4e_4$
10	$v_1 : e_1e_2e_3e_4$ $v_2 : e_1e_2e_3e_4$
11	$v_1 : e_1e_2e_3e_1e_2e_4$ $v_2 : e_3e_4e_5e_5$
12	$v_1 : e_1e_1e_2e_3e_4e_5$ $v_2 : e_2e_3e_4e_5$
13	$v_1 : e_1e_2e_3e_1e_2e_4e_5e_6$ $v_2 : e_3e_4e_5e_6$
14	$v_1 : e_1e_2e_3e_1e_2e_4$ $v_2 : e_3e_5e_6e_4e_5e_6$
15	$v_1 : e_1e_2e_3e_1e_4e_5$ $v_2 : e_2e_4e_6e_3e_5e_6$
16	$v_1 : e_1e_2e_3e_4e_5e_6$ $v_2 : e_1e_2e_3e_4e_5e_6$
17	$v_1 : e_1e_1$ $v_2 : e_2e_2$ $v_3 : e_3e_3$
18	$v_1 : e_1e_1$ $v_2 : e_2e_3e_4e_5$ $v_3 : e_2e_3e_4e_5$

#	Rotation System
19	$v_1 : e_1e_1e_2e_3$ $v_2 : e_2e_4e_5e_6$ $v_3 : e_3e_4e_5e_6$
20	$v_1 : e_1e_2e_3e_4$ $v_2 : e_1e_2e_5e_6$ $v_3 : e_3e_4e_6e_5$
21	$v_1 : e_1e_2e_3e_1e_2e_4$ $v_2 : e_3e_5e_6e_7$ $v_3 : e_4e_5e_6e_7$
22	$v_1 : e_1e_2e_4e_1e_3e_5$ $v_2 : e_2e_3e_6e_7$ $v_3 : e_4e_5e_6e_7$
23	$v_1 : e_1e_2e_3e_4e_5e_6$ $v_2 : e_1e_2e_3e_7$ $v_3 : e_4e_5e_6e_7$
24	$v_1 : e_1e_2e_3e_4$ $v_2 : e_1e_2e_3e_5$ $v_3 : e_4e_6e_7e_8$ $v_4 : e_5e_6e_7e_8$
25	$v_1 : e_1e_3e_2e_4$ $v_2 : e_1e_5e_2e_6$ $v_3 : e_3e_7e_4e_8$ $v_4 : e_5e_8e_6e_7$
26	$v_1 : e_1e_2e_3e_4$ $v_2 : e_1e_2e_5e_6$ $v_3 : e_3e_5e_7e_8$ $v_4 : e_4e_6e_7e_8$