MATH 6510

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Problem 1:

Claim 0.1 (γ is surjective):

Now $\forall c' \in C'$, δ being an isomorphism implies there exists $d' \in D$ s.t. $\delta(d) = d'$. Now exactness implies that $g_4(d') = g_4 \cdot g_3(c') = 0 \implies$ that $\delta \cdot f_4(b) = 0 \implies$ since ϵ is an isomorphism that $f_4(d) = 0 \implies$ by exactness $d \in img(f_3)$. Let $d = f_3(c) \implies g_3(c') = d' = \delta \cdot f_3(c) = g_3 \cdot \gamma(c)$. Hence $c' - \gamma(c) \in ker(g_3) \implies$ by exactness that $d = f_3(c) \implies$ that $c' - \gamma(c) \in ker(f_3)$ so by exactness $\exists b'$ s.t. $c' - \gamma(c) = g_2 \cdot \beta(b) \implies$ by commutivity that $c' - \gamma(c) = \gamma \cdot f_2(b) \implies c' = \gamma(c + f_2(b))$ Hence surjectivity.

Claim 0.2 (γ is injective):

Now $\forall c \in C \gamma(c) = 0$ implies that $g_3 \cdot \gamma(c) = 0$ which implies by communativity that $\delta \cdot f_3(c) = 0 \implies$ by δ bijective that $f_3(c) = 0 \ldots$
By exactness $\exists b \in B$ s.t. $c = f_2(b) \implies \beta(b') = \gamma \cdot f_2(b) = \gamma(c) = 0$ which implies by exactness that $\exists a' \in A'$ s.t. $b' = f_1(a')$. α bijective implies that $\exists a \in A$ s.t. $a' = \alpha(a) \implies \beta(b) = f_1 \cdot \alpha(a) = \beta \cdot f_1(a)$ so $b - f_1(a) \in ket(\beta)$ which implies since β bijective that $b = f_1(a) \implies$ by exactness that $c = f_2(b) = 0$ hence injectivity.

Hence a structure preserving bijection so isomorphism

Problem 2:

(1) Recall that an n sphere is defined as

$$\mathbb{S}^n := \{x \in \mathbb{R}^n \mid \|x\| = 1\}$$

Claim 0.3 ($\mathbb{S}^n$ is homeomorphic to the $n+1$-simplex):

Let $V = \{0, 1, 2, \cdots, n+1\}$. Then represent each vertex as the standard basis vectors $e_0, \cdots, e_{n+1} \in \mathbb{R}^{n+2}$ so

$$\Delta^{n+1} := \{(x_0 \cdots x_{n+1}) \in \mathbb{R}^{n+2} | x_i \geq 0 | \sum x_i = 1\}$$

.
Then the $\partial\Delta^{n+1}$ consists of all faces of Δ^{n+1}. Now since Δ^{n+1} is a (n+1) manifold with boundary implies that Δ^{n+1} is homeomorphic to D^{n+1}. This implies the boundary is homeomorphic to $\mathbb{S}^n$ and so the boundary of the Δ^{n+1} simplex is homeomorphic to $\mathbb{S}^n$.

This intuitavely makes sense as $\mathbb{S}^1$ requires 3 simplicies to describe and $\mathbb{S}^2$ requires 4 to describe.

(2) For any n points there exists a $n-1$ plane that they all lie on. Therefore we need $n+1$ points at the least to define a simplicial complex that has volume in n dimensions, which is the number of points we used in the previous question.

(3) For $\mathbb{S}^0 = \{-1, 1\}$ we get that it has exactly 2 connected compotnents so $H_0\mathbb{S}^0 = \mathbb{Z}^2, H_n\mathbb{S}^0 = 0, 1 \leq n$.

For $\mathbb{S}^n, 1 \leq n$, we get that $H_0(\mathbb{S}^n = \partial\Delta^{n+1}) = \frac{ker(d_0)}{im(d_1)} = \frac{C_0}{im(d_1)}$ so since the dinnferes for $im(d_1)$ imply there is one conected component, it is equal to $\mathbb{Z}$.

Similarly, for $H_n(\mathbb{S}^n)$ since there are no $n+1$ faces in $\mathbb{S}^n$, we get that $H_n(\mathbb{S}^n)$ is nonzero and therefore $\mathbb{Z}$.

In $1 \leq k < n$, $H_n(\mathbb{S}^n)$, there are no holes in S^n outisde of n so $H_k(\mathbb{S}^n) = 0$.

The pattern then matches to $H_k(\mathbb{S}^n) = \begin{cases} \mathbb{Z} & k = 0 \text{ or } k = n \\ 0 & \text{otherwise} \end{cases}$. Thus for $\mathbb{S}^1$ it is $\mathbb{Z}, \mathbb{Z}, 0, \cdots$, $\mathbb{S}^2$ it is $\mathbb{Z}, 0, \mathbb{Z}, 0, \cdots$ and for $\mathbb{S}^3$ it is $\mathbb{Z}, 0, 0, \mathbb{Z}, 0, \cdots$.

Problem 3:

(1) Torus: The construction of a torus can be created by creating a triangular prisim and then embedding a triangle inside that connects each edge of the traingle to the same edge on the top and bottom of the triangular prisim.

I think you can use less points for a torus by having two triangles one embedded in another and an extrenous point, although I am not sure how to construct it.

(2) $\mathbb{R}P^2$: Create the simplicial complex for $\mathbb{S}^2$ (a tetrahedron). Then embedd a vertex in each face and connect each that vertex to the opposite facing point in the tetrahedron. This is because $\mathbb{R}P^2$ is homeomorphic to $\mathbb{S}^2$ mod the equivlanece relation that identifies opposite points as the same. I think this is the mininum mumber of points needed.

(3) Klein Bottle: A pyramid with the bottom face removed and a series of traigular faces that wrap around onto the back face.

I am not sure about the mininum number of points needed.

Problem 4:

(1)

Claim 0.4 (Existence of an identity morphism):

For any object m, There exists the identity morphism defined as $id_A := I_m$ where I_m is the identity matrix of size m such that given $f : m \to n$ represented as a $n \times m$ matrix then $f \circ id_m = fI_m = f$ and given $f : l \to m$ represetned as a $m \times l$ matrix, $id_m \circ f = I_m f = f$ Thus there exists an indentity morphism.

Claim 0.5 (Composition is associative):

Given three morphisms between nonnegative objects n, l, k, m we define

$$h : n \to l,\ g : l \to k,\ f : k \to m$$

Which implies that h is a $l \times n$ matrix, g is a $k \times l$ matrix, and f is a $m \times k$ matrix. Thus

$$f \circ (g \circ h) = f(gh)$$
$$(f \circ g) \circ h = (fg)h$$

And since matrix multiplication on the rationals is trivially known to be associative then the composition of morphisms is also associatve.

Thus since we have a collection of objects, a set of morphisms between obejcts (desribed in question), shown existence of an indentity morphism, and that composition of morphisms is associative, this obejct is indeed a category.

(2) Define the functor $F(m) := \mathbb{Q}^m, G(V) := dim(V)$.

Now given $A : m \to n$ and $B : l \to m \implies F(A \circ B) = F(n) = \mathbb{Q}^n$ and $F(A) \circ F(B)$ maps $\mathbb{Q}^l \to \mathbb{Q}^n$ which is equal to $F(A \circ B)$.

Now given G and morphism $f : V \to W$ use axiom of choice to construct an arbitrary basis for V. Then there exists some unique linear map (matrix) that maps $V \to W$ to the respective chosen basis. This implies that $G(f) : G(V) \to G(W)$ is a morphism in C which implies since composition of matrix multiplication responds to the underlying linear maps that G is a functor.

Problem 5:

(1) Suppose that $x \in NP$. Then x is finite with some cardinality n by definition and consits of the elemnts $\{p_0, \cdots, p_n\}$ where there is a ordering $p_1 < p_2 < \cdot < p_n$. Now removing p_1 or p_n gives the set $\{p_2, \cdot, p_n\}$ or $\{p_1, \cdot, p_{n-1}\}$ which trivially has the ordering $p_2 < p_3 < \cdots < p_n$ or $p_1 < p_2 < \cdots < p_{n1}$. Thus they must also be in NP. Now removing and p_i, $1 < i < n$ implies that there is a set $\{p_1, p_2, \cdots p_{i-1}, p_{i+1}, \cdots p_n\}$. Now since it is known that $p_{i-1} < p_i$ and $p_i < p_{i+1}$ then by transitivity $p_{i-1} < p_{i+1}$ which implies there is a ordering $p_1 < \cdots p_{i-1} < p_{i+1} < p_n$. Thus there is a ordering and this set must be in NP. Removing 1 element from a set with cardinality 2 is also solved since subsets with cardinality 1 are trivially ordered. Thus every subset of x containing exactly 1 less element than x is also in NP. Continuting this process recursively results in finite temrination since x is finite and also results in every subset of x being generated by recurisvely removing a single element in x that is not in the subset implying this element x' is also in NP and recursively repeating this process on x' with an element in x' but not in the desired subset of x. Thus every subset of $x \in NP$ is also in $NP \forall x \implies NP$ is a simplicial complex.

(2) Given a morphism $f : P \to Q$ in $PoSet$, define the functor N as $N(f) : N(P) \mapsto N(Q)$.

Now since $id_p : P \to P$ is the identity then $N(id_P)$ is the identitiy on P. Thus $N(id_p) = id_{N(p)}$.

Now since $N(g) \circ N(f)(p) = N(g)(N(f)(p)) = N(g)(f(p)) = g(f(p))$ Hence assosiativity is preserved and it is a functor.

Then $N(f)$ is a simpliial map since $N(g \circ f)(P)$ returns $N(g \circ f)(P) = N(g) \circ N(f)(P)$ which is a set consisiting of the finite sets of Q with ordering $p_0 \leq \cdots \leq p_n$ which implies from the first part of the question it is a.

(3) Since $x \leq y$ becomes $y \leq x$ in $P^{op} \implies P^{op}(A, B) = P(B, A)$. This means there is a isomorphism between $P^{op}(A, B)$ and $P(B, A)$ which implies that the cardinality of $|NP|$ and $|NP^{op}|$ are equivalent.

(4) Define $cf := f := \{f(p_0) \cdots f(p_k)\}$. Then

$cf(x) = f(x) = \{f(p) | p \in x\} \subseteq f(x') - cf(x')$. Thus it is order preserving.

This implies that $c(id_L)(x) = \{id_L(v) | v \in x\} = x$. So idenitiy map is preserved. and

$$(cg)(cf(x)) = (cg)(\{f(p)|p \in x\}) = \{g(f(p))|p \in x\} = (g \circ f)(x) = c(g \circ f)(x)$$

Thus associativity is defined so it is a functor. Since it is order preserving it follows reflexivity and transitive properties so it is a poset.

(5) Now since the property that all subsets of all elements are in a simplex, mapping to the max vertex in the order implies that all subset verticies are also included in the map so it is simplicial.