PSET 0

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1. Q1

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Problem 1:

I understand I read the course website.

2. Q2

Problem 2:

To begin we know that the expectation is

$$\mathbb{E}[x] = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot \frac{1}{2} \cdot 3 + \frac{1}{2} \cdot \frac{1}{2} \cdot \mathbb{E}[\text{die}]$$

and since we know that $\mathbb{E}[\text{die}] = 3.5$ then

$$\mathbb{E}[x] = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot \frac{1}{2} \cdot 3 + \frac{1}{2} \cdot \frac{1}{2} \cdot 3.5$$

and evaluating this gives us

$$\boxed{\mathbb{E}[x] = 2.625}$$

Problem 3:

(1)

$$\mathbb{E}_{x\in\chi}[|f(x)|] \leq \max_{x\in\chi} f(x)$$

Not true by counterexample: Define $\chi := 0, 1$ and $P(0) = \frac{1}{2}$ and $P(1) = \frac{1}{2}$ with $f(x)$ defined as $0 \mapsto -5, 1 \mapsto -4$. Then

$$\mathbb{E}_{x\sim P}[|f(x)|] = \frac{1}{2}|-5| + \frac{1}{2}|-4| = \frac{1}{2}5 + \frac{1}{2}4 = 4.5 \not\leq \max_{x\in\chi} f(x) = \max\{-4, -5\} = -4$$

(2)

$$\mathbb{E}_{x\sim P}[f(x)] \leq \max_{x\in\chi} f(x)$$

Suppose $\forall x \in \chi, f(x) = c, c \in \mathbb{R}$. Then $\max\limits_{x\in\chi} f(x) = c$ and $\mathbb{E}_{x\ P}[f(x)] = \sum_{i=1}^{|X|} P(X[i])c = c$, where $|X|$ is the cardinality of the finite set χ and $X[i]$ is the ith number in a arbitarily ordering of the finite set X. Hence $\mathbb{E}_{x\sim P}[f(x)] = \max\limits_{x\in\chi} f(x)$ which implies if this condition is true then this inequality holds.

Now suppose this condition is false. Then $\exists x_1, x_2 \in \chi$ s.t. $f(x_1) \neq f(x_2)$. Now wlog assume $f(x_1) < f(x_2)$ and $f(x_2) = \max\limits_{x\in\chi} f(x_2)$. Then

$$\mathbb{E}_{x\sim P}[f(x)] = \sum_{i=1}^{|X|} P(X[i])f(X[i]) \leq P(x_1)f(x_1) + (1 - P(x_1)) \max_{x\in\chi} f(x) < \max_{x\in\chi} f(x)$$

Hence this is true in all cases

(3)

$$\mathbb{E}_{x\sim P}[f(x)] \leq \max_{x\in\chi}|f(x)|$$

Now since (b) implies

$$\mathbb{E}_{x\sim P}[f(x)] \leq \max_{x\in\chi} f(x)$$

And since $\forall x \in \chi, f(x) \leq |f(x)|$ implies $\max\limits_{x\in\chi} f(x) \leq \max\limits_{x\in\chi}|f(x)| \implies$

$$\mathbb{E}_{x\sim P}[f(x)] \leq \max_{x\in\chi}|f(x)|$$

Problem 4:

Wlog suppose $\max\limits_{x\in\chi} f_1(x) = c_1, \max\limits_{x\in\chi} f_2(x) = c_2$ and $c_1 \leq c_2$. Then at all $x_i \in \chi$ s.t. $f_2(x_i) = c_2, \quad f_1(x_i) \leq c_1$. Suppose $\exists f_1(x_i)$ s.t. $f_1(x_i) = c_1$. Then $\max\limits_{x_i} |f_2(x_i) - f_1(x_i)|$ is at least equal to $|c_2 - c_1| = |\max\limits_{x\in\chi} f_2(x) - \max\limits_{x\in\chi} f_1(x)|$ if not greater which satisfies the inequality. Suppose $\nexists f_1(x_i)$ s.t. $f_1(x_i) = c_1$. Then let $c_3 = \max\limits_{x\in\chi \text{ s.t. } f_2(x)=c_2} \implies c_3 < c_1 \implies$ that

$$\max_{x_i} |f_2(x_i) - f_1(x_i)| = |c_2 - c_3| \geq |c_2 - c_1| = |\max_{x\in\chi} f_2(x) - \max_{x\in\chi} f_1(x)|$$

Problem 5:

Given that χ finite implies

$$\mathbb{E}_{x\sim P}[x] \coloneqq \sum x_i P(x_i)$$

Implies that by nature of expecation linearity

$$|\mathbb{E}_{x\sim P}[f(x)] - \mathbb{E}_{x\sim P}[g(x)]| = \left|\sum f(x_i)P(x_i) - \sum g(x_i)P(x_i)\right| =$$

$$\left|\sum (f(x_i) - g(x_i))P(x_i)\right| = \mathbb{E}_{x\sim P}[f(x) - g(x)] \leq \mathbb{E}_{x\sim P}[|f(x) - g(x)|]$$

3. Q3

Problem 6:

- True: $Nullity(A) = 0 \implies Rank(A) = n - 0 = n$ by Rank Nullity Theorem. Full rank implies inveritiblity given square matrix.
- False: If such a vector exsists implies $Nullity(A) \neq 0 \implies Rank(A) < n \implies$ Not invertibile.
- True: Range of $A = \mathbb{R}^n \implies$ Full rank which implies inveritiblity.

Problem 7:

- $\|Bv\|_\infty = \max_{1\leq i\leq n} |r_i^T v|$: True: Each element in Bv is created by $r_i^T v$
- $\|Bv\|_\infty = \max_{1\leq i\leq n} |c_i^T v|$ False: Each element in Bv is created by $r_i^T v$ not $c_i^T v$
- $\|Bv\|_\infty = \max_{1\leq i\leq n} |r_i^T c_i|$ False: This definition does not have any notion of v in it.
- $\|Bv\|_\infty = \max_{1\leq i\leq n} |e_i^T Bv|$: True: $e_i^T Bv$ is equivalent to plucking out the ith entry in the new vector Bv.

Problem 8:

$||v||_\infty = \max_{1\leq i\leq n}|v| \implies$
Suppose

$$\forall i, v_i = c \in \mathbb{R} \implies \text{ that the } j\text{th row of } |Bv| = |\sum_{i=0}^{n} B_{ji}c| = |c\sum_{i=0}^{n} B_{ji}| \leq |c\gamma| < |c| \implies$$

$$||Bv||_\infty \leq |\gamma c| < |c| = ||v||_\infty$$

Now suppose the condition is false. Then $\exists v_a, v_b \in v$ s.t. wlog $v_a < v_b$ and $\max|v_i| = |v_b|$. Then the jth row of $|Bv| \leq |v_a B_{ja} + v_b(\gamma - B_{ja})| \leq |\gamma v_b| \leq v_b = \max_{1\leq i\leq n}|v|$ which implies the statment is true.

Thus it is true for both cases.

Problem 9:

Now since we know that the L-infinity norm satisifes the reverse triangle inequality implies that

$$|||Iv||_\infty - ||Bv||_\infty| \leq ||Iv - Bv||_\infty = ||(I-B)v||_\infty$$

Now since we know that $||Iv||_\infty = ||v||_\infty > ||Bv||_\infty$ implies that $|||Iv||_\infty - ||Bv||_\infty| > 0 \implies ||(I-B)v||_\infty > 0$

Problem 10:

Problem 8 implies that

$$||B_v||_\infty \leq |\gamma c| = |\gamma||c| = \gamma||v||_\infty$$

Therefore since $||B||_\infty \leq \gamma||v||_\infty \forall v \in \mathbb{R}^n$ than since $Bv = \lambda v \implies ||Bv||_\infty = ||\lambda v||_\infty$ implies that since $0 < \gamma < 1$ than all eigenvalues $\lambda \leq \gamma < 1$. This implies that the spectral radius of B is less than 1 which implies that the neumann series for operator B, $\sum_{i=0}^{\infty} B^k$ converges which implies $I - B$ is always invertible

4. Q4

Problem 11:

We know that

$$P = \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.6 & 0.1 \end{bmatrix}, \quad p_1 = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix} \implies$$

$$p_2 = \begin{bmatrix} 0.2 & 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.6 & 0.2 & 0.2 \\ 0.3 & 0.6 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.37 & 0.40 & 0.23 \end{bmatrix}$$

Problem 12:

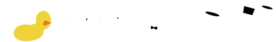
Knowing $p_2 \implies$

$$p_3 = p_2 P = \begin{bmatrix} 0.383 & 0.292 & 0.325 \end{bmatrix}$$

Problem 13:

Given

$$p_3 = p_2 P = p_1 P^2$$



Problem 14:

Given $k \geq 2, \quad p_k = p_1 P^{k-1}$. We can calculate this efficently with diagonalization given known constants for P