MATH 135 - Algebra for Honours Mathematics

Winter 2014

Contents

Chapter 1

Injective, Surjective and Bijections

1.1 Injective(One-to-One)

1.1.1 Definition

Injective: Let S and T be two sets. A function $f : S \to T$ is **one-to-one**(or **injective**) iff for every $x_1 \in S$, $f(x_1) = f(x_2)$ implies that $x_1 = x_2$ and $|S| \leq |T|$. When trying to prove that a function is one-to-one, start off with $f(x_1) = f(x_2)$ and try to use algebraic manipulation to obtain $x_1 = x_2$.

1.1.2 Simple Example

Proposition: Let $m \neq 0$ and b be fixed real numbers. The function $f : \Re \to \Re$ defined by $f(x) = mx + b$ is one to one
Proof: Let $x_1, x_2 \in S$. Suppose that $f(x_1) = f(x_2)$. Now we show that $x_1 = x_2$. Since $f(x_1) = f(x_2)$, $mx_1 + b = mx_2 + b$. Subtracting b from both sides and dividing by m gives $x_1 = x_2$ as required.

1.1.3 Hard Example

Proposition: Let $f : T \to U$ and $g : S \to T$ be one-to-one functions. Then $f \circ g$ is a one-to-one function.
Proof: Let $x_1, x_2 \in S$. Suppose that $(f \circ g)(x_1) = (f \circ g)(x_2)$. Since $(f \circ g)(x_1) = (f \circ g)(x_2)$, we know that $f(g(x_1)) = f(g(x_2))$. Since f is one-to-one, we know that $g(x_1) = g(x_2)$. And since g is one-to-one, $x_1 = x_2$ as required.

1.2 Surjective

1.2.1 Definition

Surjective: A function $f : S \to T$ is **surjective**(or **onto**) if and only if for every $y \in T$ there exists an $x \in S$ so that $f(x) = y$. This implies that $|S| \geq |T|$.

When trying to prove that a function is onto, try to find a function $g(x)$ such that $f(g(x)) = y$ to prove that each y in the codomain is mapped to.

1.3 Bijections

1.3.1 Definition

Bijection: A function $f : S \rightarrow T$ is **bijective** iff f is both surjective and injective.

1.3.2 Simple Example

We have already shown that for $m \neq 0$ and b a fixed real number, the function $f : \Re \rightarrow \Re$ defined by $f(x) = mx + b$ is both surjective and injective. Hence, f is bijective.

1.4 Summary

- $f : S \rightarrow T$ is a function iff $\forall s \in S \exists! t \in T, f(s) = t$ where ! means unique

- $f : S \rightarrow T$ is surjective iff $\forall t \in T \exists s \in S, f(s) = t$, meaning for each element $t \in T$, there is at least one element $s \in S$ so that $f(s) = t$

- $f : S \rightarrow T$ is injective iff $\forall x_1 \in S \forall x_2 \in S, f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ or $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, meaning for each element $t \in T$, there is at most one element $s \in S$ so that $f(s) = t$

1.4.1 Frequently Asked Questions

Questions to be added

Chapter 2

Counting

2.1 Bijection and Cardinality

2.1.1 Definition

Cardinality: If there exists a bijection between the sets S and T, we say that the sets have the same and we write $|S| = |T|$.

Number of Elements, Finite, Infinite: If there exists a bijection between a set S and $\natural_n$, we say that the **number of elements** in S is n and we write $|S| = n$. Moreover, we also say that S is a **finite set**. If no bijection exists between a set S and $\natural_n$ for any n, we say that S is an **infinite set**.

Countable: A set S is **countable** if there exists an injective function f from S to the natural numbers $\natural$

2.1.2 Guidelines

Proposition: Let $S = ...$ Let $T = ...$ Then there exists a bijection $f : S \to T$. Hence, $|S| = |T|$.

To do this, we must prove that f is both surjective and injective.

Consider the function $f : S \to T$ defined by $f(s) = ...$ We show that f is surjective. Let $t \in T$. Consider $s = ...$ We show that $s \in S....$ Now we show that $f(s) = t$.

We then show that f is injective. Let $s_1, s_2 \in S$ and suppose that $f(s_1) = f(s_2)$. Now we show that $s_1 = s_2$.

Hence, $f : S \to T$ is a bijection and $|S| = |T|$.

2.2 Finite Sets

2.2.1 Definitions

Disjoint: Set S and T are **disjoint** if $S \cap T = \emptyset$

2.2.2   Propositions

Cardinality of Intersecting Sets(CIS): If S and T are any finite sets, then
$$|S \cup T| = |S| + |T| - |S \cap T|$$
Cardinality of Disjoint Sets(CDS): If S and T are disjoint finite sets, then
$|S \cup T| = |S| + |T|$

2.2.3   Example

Proof of CDS:

1. Since S is a finite set, there exists a bijection $f : S \to \natural_m$ for some non negative integer m, and $|S| = m$

2. Since T is a finite set, there exists a bijection $f : T \to \natural_n$ for some non negative integer m, and $|T| = n$

3. Construct function $h : S \cup T \to \natural_{m+n}$ as follows:
   $h(x) = $ f(x) if $x \in S$ else $g(x) + m$ if $x \in T$

4. To show that h is surjective, let $y \in \natural_{m+n}$. If $y \le m$, then because f is surjectivethere exists an element $x \in S$ so that $f(x) = y$, hence $h(x) = y$. If $m + 1 \le y \le m + n$, then because g is surjective, there exists an element $x \in T$ so that $g(x) = y - m$ and so $h(x) = (y - m) + m = y$.

5. To show that h is injective, let $x_1, x_2 \in S \cup T$ and suppose that $h(x_1) = h(x_2)$. If $h(x) \le m$ then $h(x) = f(x)$ so if $h(x_1) \le m$ we have
   $$h(x_1) = h(x_2) \Rightarrow f(x_1) = f(x_2)$$
   But since f is a bijection $f(x_1) = f(x_2)$ implies $x_1 = x_2$ as needed. If $h(x) > m$ then $h(x) = g(x)$ so if $h(x_1) > m$ we have
   $$h(x_1) = h(x_2) \Rightarrow g(x_1) + m = g(x_2) + m \Rightarrow g(x_1) = g(x_2)$$
   But since g is a bijectoin $g(x_2) = g(x_2)$ implies $x_1 = x_2$ as needed. Since h is a function which is both injective and surjective,h is bijective.

6. Thus
   $$|S \cup T| = |\natural_{m+n}| = m + n = |\natural_m| + |\natural_n| = |S| + |T|$$

If it wasn't clear, f(x) is mapped to 1,2..m and $g(x) + m$ is mapped to $m + 1, m + 2,...m + n$.

2.3   Infinite Sets

2.3.1   Propositions

Cardinality of Subsets of Finite Sets(CSFS): If S and T are finite sets, and $S \subset T$, then $|S| < |T|$
$|\natural| = |2\natural|$: Let $2\natural$ be the set of positive even natural numbers. Then $|\natural| = |2\natural|$
$|\natural x \natural| = |\natural|$

Even-Odd Factorization of Natural Numbers(EOFNN): Any natural number n can be written uniquely as $n = 2^i q$ where i is a non-negative integer and q is an odd natural number. Note: use EOFNN to prove $|♮x♮| = |♮|$.
Note: Not all infinite sets have the same size

2.3.2 Example

Proof of $|♮| = |2♮|$
We want to prove that there's a bijection between both sets

1. Consider the function $f : ♮ \to 2♮$ defined by $f(s) = 2s$

2. We show that f is surjective. Let $t \in 2♮$. Consider $s = \frac{1}{2}t$. We show that $s \in ♮$ since $f(\frac{1}{2}t) = t$ and therefore is surjective

3. We show that f is injective. Let $s_1, s_2 \in ♮$ and suppose that $f(s_1) = f(s_2)$. Now we show that $s_1 = s_2$. Since $f(s_1) = 2s_1$ and $f(s_2) = 2s_2$, $s_1 = s_2$.

4. Hence, $f : ♮ \to 2♮$ is a bijection and $|♮| = |2♮|$.