Project 1 - Laplacian Eigenmodes

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March 2012

Question 1

Solved the problem by hand using separation of variables; the eigenfunction and eigenvalue are parametrised by two integers, here called n and m, which have to be integers to satisfy the boundary conditions.

By writing $u(x, y) = F(x)G(y)$ and then substituting back into the equation $-\Delta u = \lambda u$, we get that $\frac{-F''(x)}{F(x)} = \lambda + \frac{G''(y)}{G(y)}$, and as both the left- and right-hand sides are independent of each other, they most both equal a constant, say, μ.

Then solving the equation for $F(x)$ gives $F(x) = A\cos(\sqrt{\mu}x) + B\sin(\sqrt{\mu}x)$, and to satisfy the boundary conditions $F(0) = F(\pi) = 0$ we need that $A = 0$ and $\sqrt{\mu} = n, n\,\mathrm{is\,an\,integer}$, i.e. $F(x) = B\sin(nx)$.

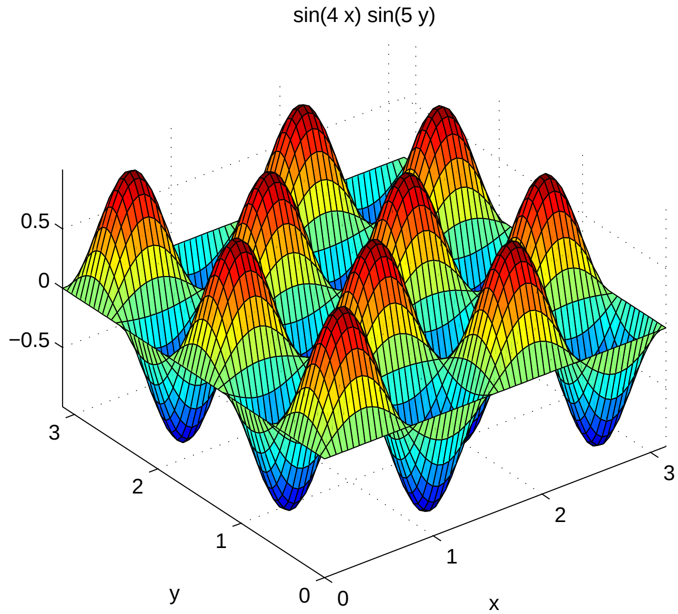
Repeating this method for $G(y)$ we see that either $n^2 < \lambda$ or $n^2 > \lambda$, but the latter leads to a contradiction (namely that $n^2 = \lambda$), so we see that the solution must be in the form $G(y) = b\sin(my)$ for some integer m.

The value of λ can be obtained from these equations as well, but substituting $F(x)$ and $G(y)$ into $u(x, y) = F(x)G(y)$, and then substituting all of this into the original $-\Delta u = \lambda u$ we can easily find the value of λ.

So in summary, the solution and its eigenvalue are given by $u(x, y) = \sin(mx)\sin(ny)$, with $\lambda = m^2 + n^2$ for integer m and n.

```
syms x y;
[ef1, lambda] = eiglap_square(4, 5, x, y) %the example given
ezsurf(ef1, [0 pi 0 pi]);
axis equal;
```

```
ef1 =

sin(4*x)*sin(5*y)


lambda =

    41
```



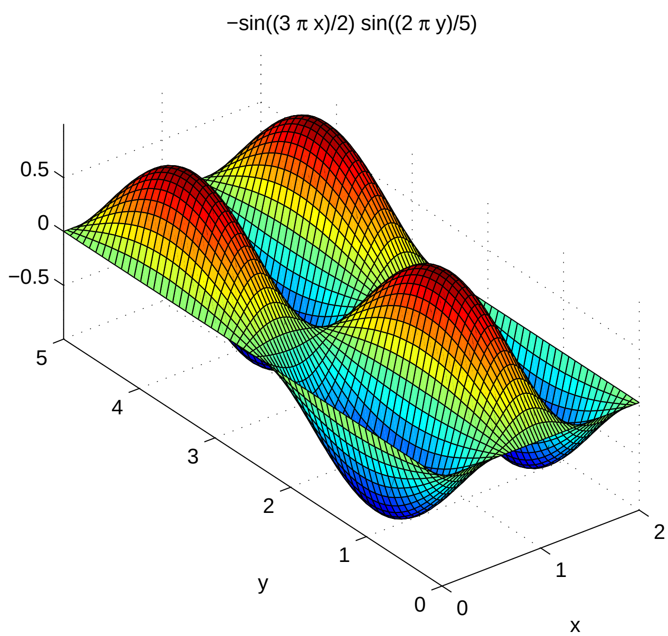
Question 2

Solved the problem again by hand, but the solution is almost identical to the problem posed in question 1, just with a $\frac{\pi}{h}$ and $\frac{\pi}{w}$ so as to still satisfy the boundary conditions. i.e.

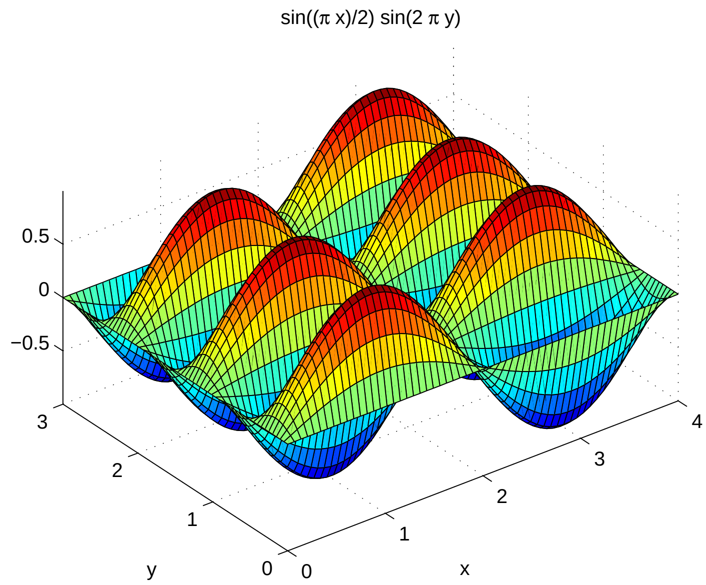
$$u(x,y) = sin(\frac{m\pi x}{w})sin(\frac{n\pi x)}{h})$$
$$\lambda = \pi^2(\frac{m^2}{w^2} + \frac{n^2}{h^2})$$

for some integer m and n.

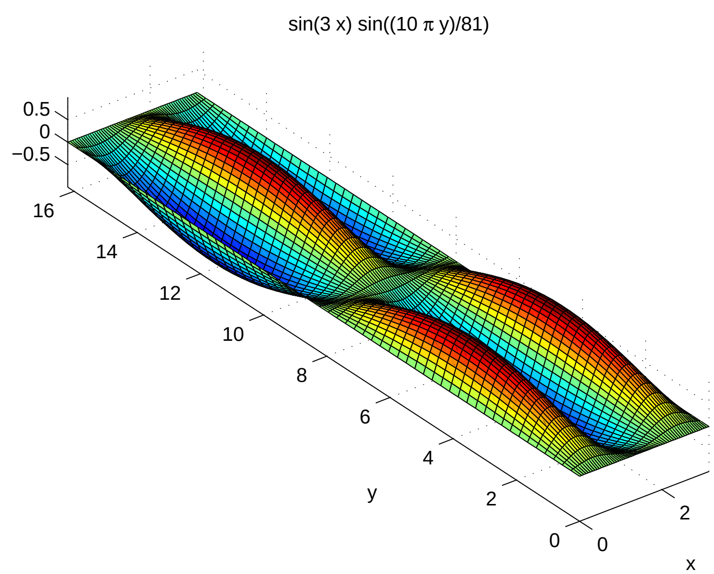
```
ezsurf(eiglap_rect(2, 5, 3, -2, x, y), [0 2 0 5]); %just a random example
axis equal;
```



```
ezsurf(eiglap_rect(4, 3, 2, 6, x, y), [0 4 0 3]);
axis equal;
```




```
ezsurf(eiglap_rect(pi, 16.2, 3, 2, x, y), [0 pi 0 16.2]);
axis equal;
```

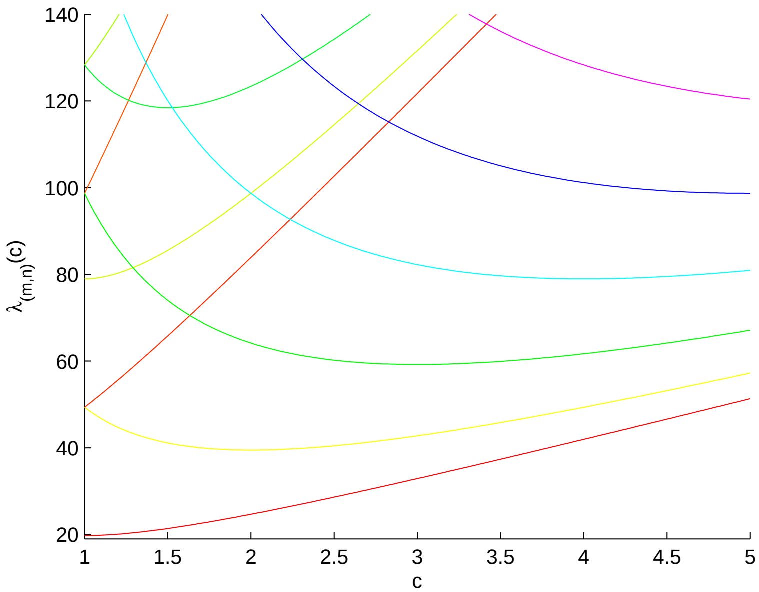


Question 3

It would have been preferable to use ezplot and plot c continuously from 1 to 5, but as the accuracy has no visible improvement compared to the method below and question 6 uses even less accurate numerical approximations, I stuck with the discrete method. Also, if c is plotted continuously then question 5 proves much more troublesome, whereas with discrete values of c it is almost trivial.

```
clear all;
syms x y;
c = 1:.02:5; %generate equally spaced values of c from 1 to 5
w = sqrt(c); %w and h are define like so to make sure the area is 1
h = 1./sqrt(c);
```

```
m = 1:6;
n = 1:6;
[mi, ni] = meshgrid(m, n);
lambda =[];

for i = 1:201
  %find the values for each value of c
    [~, l] = eiglap_rect(w(i), h(i), mi, ni, x, y);
    %make the matrix look nicer so we can plot across the rows
    l = reshape(l, 36, 1);
%append the eigenvalues just found to the ones previous
    lambda = [lambda l];
end
%generate a colour map to plot the eigenvalues all with different colours
cmap = hsv(36);
clf;
hold on
for i = 1:36
%plot each eigenvalue against c
    plot(c, lambda(i, :), '-', 'Color', cmap(i,:));
end
hold off;
axis([1 5 19 140]); %as required in the question
xlabel('c'); %label the axes
ylabel('\lambda_{(m,n)}(c)');
```



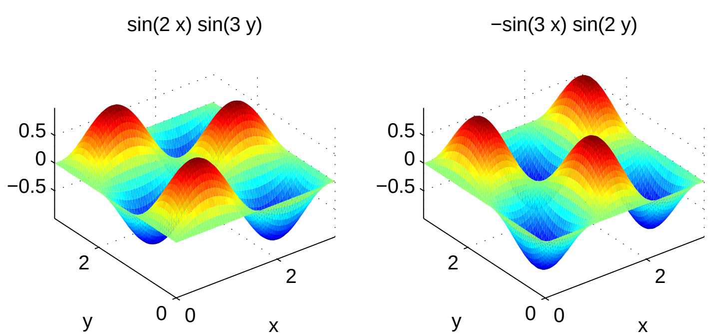
Question 4

From the equation for the eigenvalue ($\lambda = m^2 + n^2$) for a square, we can see that, given m and n, letting $m = n$ and $n = -m$, or swapping them over and changing their signs in any way, will not affect the value of λ.

```
[f1, l1] = eiglap_square(2, 3, x, y)
[f2, l2] = eiglap_square(-3, 2, x, y)
subplot(1,2,1), ezsurf(f1, [0 pi 0 pi]);
axis equal;
shading flat;
subplot(1,2,2), ezsurf(f2, [0 pi 0 pi]);
```

```
axis equal;
shading flat;
```

```
f1 =

sin(2*x)*sin(3*y)


l1 =

    13


f2 =

-sin(3*x)*sin(2*y)


l2 =

    13
```



On a rectangle it is slightly more complicated, but

$$m = 2, n = 2 \Rightarrow (\frac{2}{\sqrt{2}})^2 + (2\sqrt{2})^2 = \frac{16}{2} + 4 * 2 = \frac{4 + 2 * 4 * 2}{2}$$

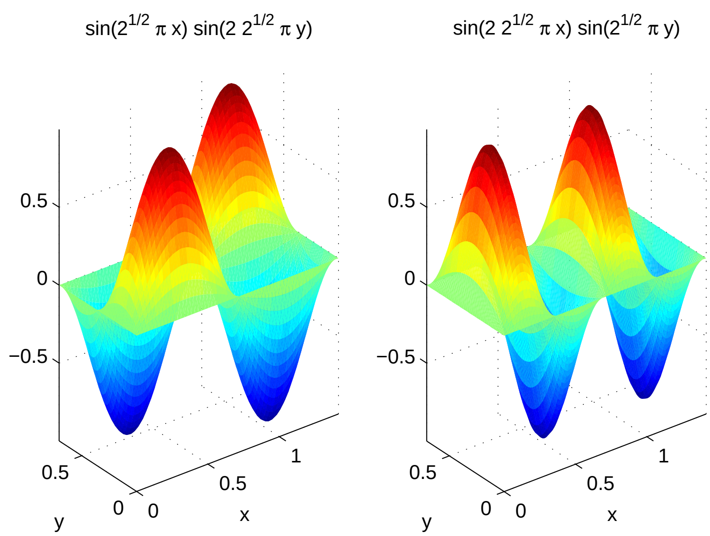
$$= \frac{2 * 4 * 2 + 2 * 2}{2} = \frac{16}{2} + 2 = (\frac{4}{\sqrt{2}})^2 + (\sqrt{2})^2 \Leftarrow m = 4, n = 1$$

i.e., $m = 2, n = 2$ and $m = 4, n = 1$ give the same eigenvalue for $w = \sqrt{2}, h = \frac{1}{\sqrt{2}}$.

In practice:

```
[f3, l3] = eiglap_rect(sqrt(2), 1/sqrt(2), 2, 2, x, y)
[f4, l4] = eiglap_rect(sqrt(2), 1/sqrt(2), 4, 1, x, y)
subplot(1,2,1), ezsurf(f3, [0 sqrt(2) 0 1/sqrt(2)]);
axis equal;
```

```
shading flat;
subplot(1,2,2), ezsurf(f4, [0 sqrt(2) 0 1/sqrt(2)]);
axis equal;
shading flat;


f3 =

sin(pi*2^(1/2)*x)*sin(2*pi*2^(1/2)*y)


l3 =

   98.6960


f4 =

sin(2*pi*2^(1/2)*x)*sin(pi*2^(1/2)*y)


l4 =

   98.6960
```

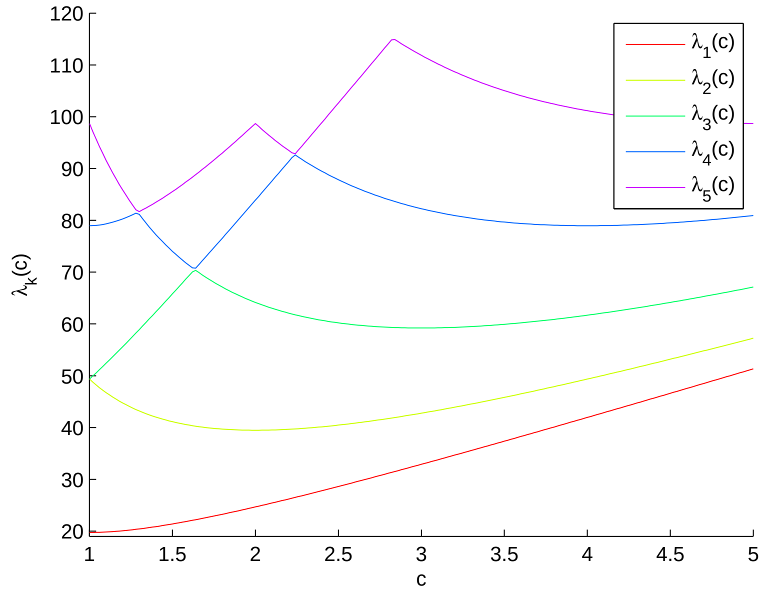


Question 5

We can just sort the previous matrix so that, for each value of c, the eigenvalues are sorted in increasing order

```
clambda = sort(lambda, 1);
cmap2 = hsv(5);
clf;
hold on
for i = 1:5
    plot(c, clambda(i, :), '-', 'Color', cmap2(i,:));
end
hold off;
axis([1 5 19 120]);
```

```
xlabel('c');
ylabel('\lambda_k(c)');
legend('\lambda_1(c)', '\lambda_2(c)',
'\lambda_3(c)', '\lambda_4(c)', '\lambda_5(c)');
```

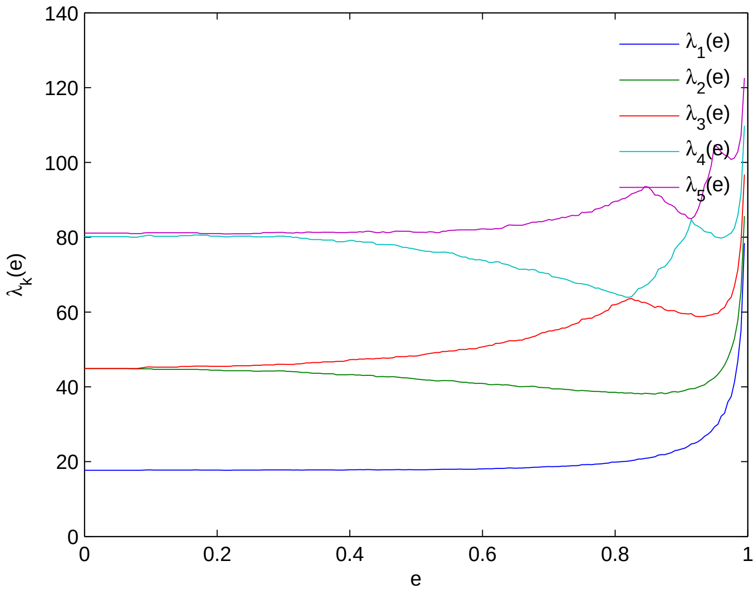


Question 6

Without loss of generality we can assume that $a > b$, and for an ellipse the eccentricity ranges such that $0 \leq e < 1$. Using $b = a\sqrt{1-e^2}$, $\pi ab = 1$, and that $a, b > 0$ we can get that:

$$b = \sqrt{\frac{1}{\pi}\sqrt{1-e^2}}$$
$$a = \frac{1}{\pi b}$$

```
e = 0:.005:.995; %generate values for e
b = sqrt((1/pi).*sqrt(1-e.^2)); %define a and b
a = 1./(pi.*b);
t = [];
ews = [];
for i = 1:200
    [t] = ellipse(a(i),b(i)); %cycle through all values of e
    [ews] = [ews t]; %append eigenvalues found to those previous
end
ews = ews'; %just so we can plot along rows again, as before
plot(e, ews);
xlabel('e');
ylabel('\lambda_k(e)');
legend('\lambda_1(e)', '\lambda_2(e)',
'\lambda_3(e)', '\lambda_4(e)', '\lambda_5(e)');
legend boxoff;
```



Question 7

By sorting the matrix found in the last question we can find for which value of e the largest/smallest value of the eigenvalue is obtained

```
%use I as an index array so we can see where the values came from originally
[ews_s, I] = sort(ews, 1);
min2e = e(I(1, 2))
max3e = e(I(end, 3))
min1e = e(I(1, 1))
```

```
min2e =

    0.8600


max3e =

    0.9950


min1e =

    0.0800
```