Ordinary Differential Equations: Assignment 1

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1 Introduction

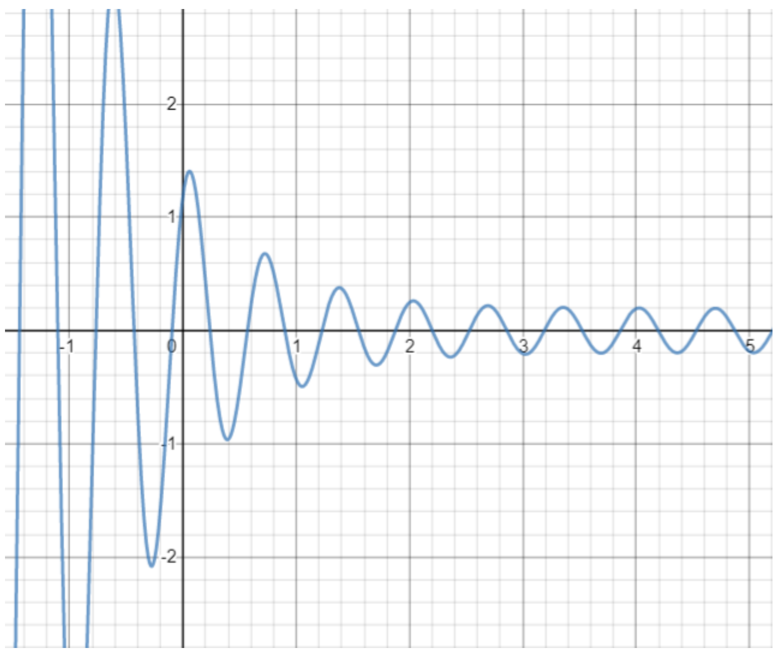

Figure 1: Y(t)

For this project I will illustrate the motion of a car-mass-spring-driving on a "washboard" surface system, using a differential equation. This car travels with an unknown velocity, "v", which will be given different values and graphed versus the maximum amplitude of each solution to the differential equation to find the velocity that will produce the maximum amplitude. This will be done analytically and numerically to see how close the values are, using pen-and-paper methods and computational methods.

2 Modelling

2.1 Question 1

If we look at the rough diagram, Figure 2, in the Appendix below, I have assumed that there is a mass oscillating on top of a car travelling on a "washboard surface" for simplicity's sake. The equation of the washboard surface is given by:

$$y(x) = a \sin\left(\frac{2\pi}{\lambda}x\right) \tag{1}$$

As we are working with $Y(t)$ and $y(t)$ we need to put $y(x)$ in terms of time. We know that in general:

$$v = \frac{x}{t}$$
$$\therefore x = vt$$

Substituting this into equation (1), gives:

$$y(t) = a \sin\left(\frac{2\pi v}{\lambda}t\right) \tag{2}$$

When we use Newton's Second Law on the spring - mass system, we need to consider a restorative force, $F_R = -ks$, where "k" is the spring constant and "s" is the displacement of the spring itself. As $Y(t)$ describes the full motion of the system and $y(t)$ is the "washboard surface" that the car travels on, we see that the displacement of the spring is given by:

$$s = Y(t) - y(t)$$

The restorative force is therefore:

$$F_R = -k(Y(t) - y(t)) \tag{3}$$

There is also damping on the system which we assume is of the form:

$$F_D = -cv$$

Where "c" is the damping constant and "v" is the velocity of the system in the vertical direction (for this situation).
And can be written as:

$$\therefore F_D = -c\dot{y} \tag{4}$$

And here the velocity is governed by the displacement of the spring, therefore the equation is:

$$F_D = -c\frac{d}{dt}(Y(t) - y(t))$$

$$\therefore F_D = -c(\dot{Y}(t) - \dot{y}(t)) \tag{5}$$

Newton's Second Law is:

$$F = \sum F = ma$$

$$\therefore \sum F = m\ddot{Y} \tag{6}$$

Where $\ddot{Y}$ is the acceleration of the entire system. The forces acting on the spring - mass system are the restorative and damping forces. Therefore the differential equation describing the system is:

$$\sum F = F_D + F_R = m\ddot{Y}$$
$$\therefore m\ddot{Y} = -c(\dot{Y}(t) - \dot{y}(t)) - k(Y(t) - y(t))$$
$$\therefore m\ddot{Y} = -c\dot{Y} + c\dot{y} - kY + ky$$

$$\therefore m\ddot{Y} + c\dot{Y} + kY = c\dot{y} + ky \tag{7}$$

Equation (7) describes the upward displacement of the car. Where y is given by equation (2) and v is the velocity of the car (which is constant).

3 Analysis

3.1 Solving the Differential Equation

The differential equation can be rewritten as:

$$m\ddot{Y} + c\dot{Y} + kY = c\dot{y} + ky \tag{8}$$

And as $y(t) = a\sin\left(\frac{2\pi v}{\lambda}t\right)$

$$\therefore \dot{y}(t) = a\frac{2\pi v}{\lambda}\cos\left(\frac{2\pi v}{\lambda}t\right)$$

let $\frac{2\pi v}{\lambda} = \omega$

$$\therefore \dot{y}(t) = a\omega\cos(\omega t) \tag{9}$$

and

$$y(t) = a\sin(\omega t) \tag{10}$$

Substituting equations (9) and (10) into equation (8):

$$m\ddot{Y} + c\dot{Y} + kY = c(a\omega\cos(\omega t)) + k(a\sin(\omega t))$$

$$\therefore \ddot{Y} + \gamma\dot{Y} + \omega_0^2 Y = \gamma(a\omega\cos(\omega t)) + \omega_0^2(a\sin(\omega t)) \tag{11}$$

Where $\gamma = c/m$, and, $\omega_0^2 = k/m$. Using the Method of Undetermined Coefficients to solve equation (11)[1]. First the homogeneous solution needs to be found.
For the homogeneous solution, $Y_h = C_1 Y_{h1} + C_2 Y_{h2}$:

$$\ddot{Y} + \gamma \dot{Y} + \omega_0^2 Y = 0$$

Using the ansatz: $Y = e^{rt}$, to get the characteristic equation:

$$r^2 + \gamma r + \omega_0^2 = 0$$
$$\therefore r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2}$$

As $\omega_0^2 = \frac{k}{m} = \frac{7\times10^4}{800} = \frac{175}{2}$ and $\gamma = \frac{c}{m} = \frac{2\times10^3}{800} = \frac{5}{2}$, $\omega_0^2 > \gamma$,
therefore the discriminant will be negative and so $r_{1,2}$ will be complex numbers.

$$\therefore r_{1,2} = -\frac{\gamma}{2} \pm \frac{\sqrt{-(4\omega_0^2 - \gamma^2)}}{2}$$
$$\therefore r_{1,2} = -\frac{\gamma}{2} \pm \frac{\sqrt{4\omega_0^2 - \gamma^2}}{2} i$$

$$\therefore r_{1,2} = -\frac{\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} i \tag{12}$$

where $i^2 = -1$

Let $\omega_1 = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$, therefore:

$$r_{1,2} = -\frac{\gamma}{2} \pm \omega_1 i \tag{13}$$

As the roots are complex, the solution Y_h can be written as:

$$Y_h = e^{-\frac{\gamma}{2}} C_1 \cos(\omega_1 t) + e^{-\frac{\gamma}{2}} C_2 \sin(\omega_1 t) \tag{14}$$

For the non-homogeneous, or particular, solution:

$$\ddot{Y} + \gamma \dot{Y} + \omega_0^2 Y = \gamma a \omega \cos(\omega t) + \omega_0^2 a \sin(\omega t)$$

As the differential equation is equal to "*sine + cosine*", the particular solution is:

$$Y_p = A \cos \omega t + B \sin \omega t \tag{15}$$
$$\dot{Y}_p = -A\omega \sin \omega t + B\omega \cos \omega t \tag{16}$$
$$\ddot{Y}_p = -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t \tag{17}$$

Substituting equations (15), (16) and (17) into equation (11) gives:

[-A$\omega^2\cos\omega t - B\omega^2\sin\omega t] + \gamma[-A\omega\sin\omega t + B\omega\cos\omega t] + \omega_0^2[A\cos\omega t + B\sin\omega t] = \gamma a\omega\cos\omega t + \omega_0^2 a\sin\omega t$

Equating cosines:

$$-A\omega^2\cos\omega t + \gamma B\omega\cos\omega t + \omega_0^2 A\cos\omega t = \gamma a\omega\cos\omega t \tag{18}$$

Aside, as we are dividing by cosines, $\cos\omega t \neq 0$, and sines, $\sin\omega t \neq 0$

$$\begin{aligned}\cos\omega t &\neq 0\\ \therefore \omega t &\neq \frac{\pi}{2} + \pi k\\ \therefore t &\neq \frac{\lambda}{2\pi v}(\frac{\pi}{2} + \pi k)\\ \therefore t &\neq \frac{\lambda}{4v} + \frac{\lambda k}{2v}\end{aligned}$$

$$\begin{aligned}\sin\omega t &\neq 0\\ \therefore \omega t &\neq \pi k\\ \therefore t &\neq \frac{\lambda}{2\pi v}(\pi k)\\ \therefore t &\neq \frac{\pi k}{2v}\end{aligned}$$

where $k \in \mathbb{Z}$

Back to equation (18):

$$\begin{aligned}-A\omega^2\cos\omega t + \gamma B\omega\cos\omega t + \omega_0^2 A\cos\omega t &= \gamma a\omega\cos\omega t\\ \therefore -A\omega^2 + \gamma B\omega + \omega_0^2 A &= \gamma a\omega\\ \therefore A(\omega_0^2 - \omega^2) &= \gamma a\omega - \gamma B\omega\\ \therefore A(\omega_0^2 - \omega^2) &= \omega\gamma(a - B)\end{aligned}$$

$$\therefore A = \frac{\omega\gamma(a - B)}{(\omega_0^2 - \omega^2)} \tag{19}$$

Equating sines:

$$\begin{aligned}-B\omega^2\sin\omega t - \gamma A\omega\sin\omega t + \omega_0^2 B\sin\omega t &= \omega_0^2 a\sin\omega t\\ \therefore -B\omega^2 - \gamma A\omega + \omega_0^2 B &= \omega_0^2 a\end{aligned}$$

$$\therefore B(\omega_0^2 - \omega^2) = \omega_0^2 a + A\gamma\omega \tag{20}$$

Now, substituting equation (19) into equation (20):

$$\begin{aligned}B(\omega_0^2 - \omega^2) &= \omega_0^2 a + (\gamma\omega)\Big(\frac{\omega\gamma(a - B)}{(\omega_0^2 - \omega^2)}\Big)\\ \therefore B(\omega_0^2 - \omega^2)^2 &= \omega_0^2 a(\omega_0^2 - \omega^2) + (\omega\gamma)^2(a - B)\\ \therefore B(\omega_0^2 - \omega^2)^2 &= \omega_0^2 a(\omega_0^2 - \omega^2) + a(\omega\gamma)^2 - B(\omega\gamma)^2\\ \therefore B\big[(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2\big] &= a[\omega_0^2(\omega_0^2 - \omega^2) + (\omega\gamma)^2]\\ \therefore B &= \frac{a\omega_0^2(\omega_0^2 - \omega^2) + a(\omega\gamma)^2}{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}\end{aligned}$$

$$\therefore B = \frac{a\omega^2(\gamma^2 - \omega_0^2) + a\omega_0^4}{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2} \tag{21}$$

Substituting equation (21) into equation (19):

$$A = \frac{\omega\gamma\Big(a - \Big(\frac{a\omega^2(\gamma^2-\omega_0^2)+a\omega_0^4}{(\omega_0^2-\omega^2)^2+(\omega\gamma)^2}\Big)\Big)}{(\omega_0^2 - \omega^2)}$$

$$\therefore A(\omega_0^2 - \omega^2) = \omega\gamma a - (\omega\gamma)\frac{\omega_0^2(\omega_0^2 - \omega^2) + (\omega\gamma)^2}{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2}$$

$$\therefore A[(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2](\omega_0^2 - \omega^2) = \omega\gamma a[(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2] - \omega^3\gamma a(\gamma^2 - \omega_0^2) - a\omega_0^4\gamma\omega$$

$$\therefore A[(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2](\omega_0^2 - \omega^2) = \omega\gamma a(\omega_0^2 - \omega^2)^2 + \omega_0^2 a\gamma\omega(\omega^2 - \omega_0^2) - \omega^3\gamma^3 a + \omega^3\gamma^3 a$$

$$\therefore A[(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2] = \omega\gamma a(\omega_0^2 - \omega^2) - \omega_0^2 a\gamma\omega$$

$$\therefore A[(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2] = -\omega^3\gamma a$$

$$\therefore A = \frac{-\omega^3\gamma a}{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2} \tag{22}$$

As "$(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2$" appears in both denominators, to simplify the equation,

let $(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2 = \beta$

Then A and B become:

$$A = \frac{-\omega^3\gamma a}{\beta} \qquad\qquad B = \frac{a\omega^2(\gamma^2 - \omega_0^2) + a\omega_0^4}{\beta}$$

The particular solution is:

$$Y_p = \Big(\frac{-\omega^3\gamma a}{\beta}\Big)\cos\omega t + \Big(\frac{a\omega^2(\gamma^2 - \omega_0^2) + a\omega_0^4}{\beta}\Big)\sin\omega t \tag{23}$$

Now putting the homogeneous and particular solutions together to get the general solution.

$$Y(t) = Y_h + Y_p$$

$$Y(t) = e^{-\frac{\gamma}{2}}C_1\cos(\omega_1 t) + e^{-\frac{\gamma}{2}}C_2\sin(\omega_1 t) + \Big(\frac{-\omega^3\gamma a}{\beta}\Big)\cos\omega t + \Big(\frac{a\omega^2(\gamma^2 - \omega_0^2) + a\omega_0^4}{\beta}\Big)\sin\omega t \tag{24}$$

As the homogeneous solution decays, the constants C_1 and C_2 do not need to be found through initial conditions and can be assumed to be 1, then equation (24) becomes:

$$Y(t) = e^{-\frac{\gamma}{2}}\cos(\omega_1 t) + e^{-\frac{\gamma}{2}}\sin(\omega_1 t) + \Big(\frac{-\omega^3\gamma a}{\beta}\Big)\cos\omega t + \Big(\frac{a\omega^2(\gamma^2-\omega_0^2)+a\omega_0^4}{\beta}\Big)\sin\omega t \tag{25}$$

3.2 Finding the Amplitude

To find the amplitude of the oscillations we only need to look at the particular solution as, with time, the homogeneous solution decays to zero rapidly as it has an "$e^{-\frac{\gamma}{2}t}$" term.

The amplitude of a function with a sine and cosine graph added together is given by:

$$Amplitude = \sqrt{A^2 + B^2} \tag{26}$$

Where "A" and "B" are the amplitudes of the cosine and sine functions, respectively. [3]
This means that the equation of the amplitude is:
Let $Amplitude = \tilde{A}$

$$\tilde{A}(v) = \sqrt{\Big(\frac{-\omega^3\gamma a}{\beta}\Big)^2 + \Big(\frac{a\omega^2(\gamma^2-\omega_0^2)+a\omega_0^4}{\beta}\Big)^2}$$

Substituting $\omega = \frac{2\pi v}{\lambda}$ and $\beta = (\omega_0^2 - (\frac{2\pi v}{\lambda})^2)^2 + ((\frac{2\pi v}{\lambda})\gamma)^2$:

$$\tilde{A}(v) = \sqrt{\Big(\frac{-(\frac{2\pi v}{\lambda})^3\gamma a}{(\omega_0^2 - (\frac{2\pi v}{\lambda})^2)^2 + ((\frac{2\pi v}{\lambda})\gamma)^2}\Big)^2 + \Big(\frac{a(\frac{2\pi v}{\lambda})^2(\gamma^2-\omega_0^2)+a\omega_0^4}{(\omega_0^2 - (\frac{2\pi v}{\lambda})^2)^2 + ((\frac{2\pi v}{\lambda})\gamma)^2}\Big)^2} \tag{27}$$

Using computer code to find the velocity that will give the equation $Y(t)$ the highest amplitude, I graphed equation (27). (See Figure 3 in the appendix below.)
From Figure 3 it is seen that for the amplitude to be a maximum, the velocity needs to be 14.6077 m/s. Using the sympy function "$solve()$" to solve for the maximum amplitude obtained from the "$amax()$" function, the value of the velocity is 14.6076794657766 m/s, which is approximately 14.6077 m/s as stated above. The velocity in kilometers is:

$$\begin{aligned} v \approx 14.6077m/s &= 14.6077 \times \frac{1m}{1s} \times \frac{1km}{1000m} \times \frac{3600s}{1h} = 14.6077 \times 3.6km/h \\ &= 52.58772km/h \end{aligned}$$

The velocity needed to create the maximum amplitude of $Y(t)$ is 52.58772 km/h

4 Simulations

4.1 Question 3

To solve equation (11) numerically, I used the Fourth Order Runge-Kutta Method. I changed the second order differential equation into a first order differential equation using the substitution $z = \frac{dy}{dx}$, and this got the differential equation:

$$\dot{z} = \gamma a \omega \cos(\omega t) + \omega_0^2 a \sin(\omega t) - \gamma z - \omega_0^2 y \tag{28}$$

This then gave me the two functions $f(t, y, z)$ and $g(t, y, z)$ which are needed in the Runge-Kutta method[4]. From this I defined a function that took in a v-value and returned an array that contained the values of the steady state part of $Y(t)$ corresponding to the velocity inputted [2]. I then created an array of velocities and an empty array to store the maximum amplitudes of each $Y(t)$. Then I used a for loop, that looped through the array of velocities, entered them into my function and stored the maximum amplitude into the empty array and then plotted that array versus the velocities. Doing this obtained Figure 4.
To make a comparison easier, I plotted both, the graph of the maximum amplitudes versus velocity from the analytical section as well as the graph obtained in this, the numerical, section.(See Figure 5)
The table below shows points on the graphs, including the maximum velocity for the analytical solution as well as the maximum velocity from the numerical solution, as from my code they are slightly different, with a difference of 0.05, as well as the graphs' maximum amplitude values as they also differ by a small amount of 0.0033.

Number	Velocities (m/s)	Analytical (m)	Numerical (m)
	v	$A(v)$	$A(v)$
1	14.6076794657766	0.19515564402464997	0.1917038863851984
2	14.553908355795148	0.19479503690671404	0.19187131613363578
3	15.8576794657766	0.16509950215194757	0.15822189304885326
4	15.803908355795148	0.1672280228841107	0.16033765669281713
5	13.3576794657766	0.16637041466705849	0.16749296087086799
6	13.303908355795148	0.16453345668955843	0.16571841810056337

Number 1's velocity is the velocity that produces the maximum amplitude for the analytical graph. Number 2's velocity is the velocity that produces the maximum amplitude for the numerical graph. The other velocities show the slight differences in the points on the numerical and analytical graphs.
It is seen that there is a percentage error of 0.3681% in the numerical velocity to the analytical velocity and 1.6829% in the numerical amplitude to the analytical amplitude. The reasons for this error could be that there is error in python rounding off numbers at each iteration in my function to determine the maximum amplitude or a rounding error in extracting the maximum amplitudes later in my code.

5 Conclusion

In conclusion, the maximum velocity needed for the car-mass-spring system to have the greatest amplitude is 52.59 km/h, just below the speed limit for residential roads. The difference between the numerical and analytical solutions were very small which shows how accurate the Runge-Kutta method is.

References

[1] Michel van Biezen. *Differential Equation - 2nd Order (39 of 54) Method of Undetermined Coefficient*. URL: https://www.youtube.com/watch?v=moqgrw4KegU&list=PLX2gX-ftPVXVQkHNzmZGsdSaZt7GExpmC&index=39. (accessed: 22.04.2020).

[2] Austin Connolly. "Online interview with Zayd Pandit". In: *Python: Looping through Runge-Kutta* (2020).

[3] Jóhann Ísak. *Trig Rules - Lesson 3*. URL: http://www.rasmus.is/uk/t/F/Su57k03.htm. (accessed:18.04.2020).

[4] Easy Maths Easy Tricks. *Runge kutta method second order differential equation simple example(PART-1)*. URL: https://www.youtube.com/watch?v=Fs1cRieo8XM. (accessed: 22.04.2020).

6 Appendix

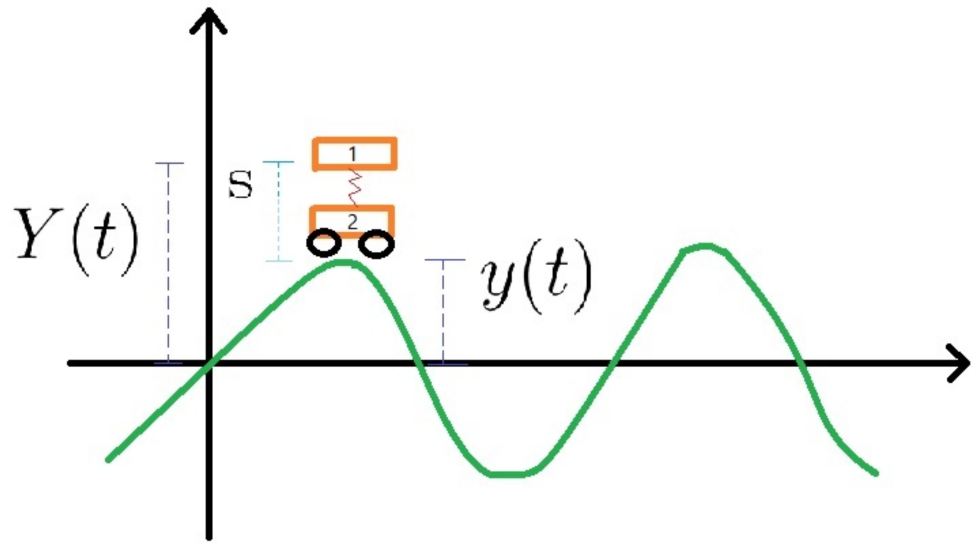

Figure 2: This is a Diagram describing the motion of the oscillating car on a washboard surface

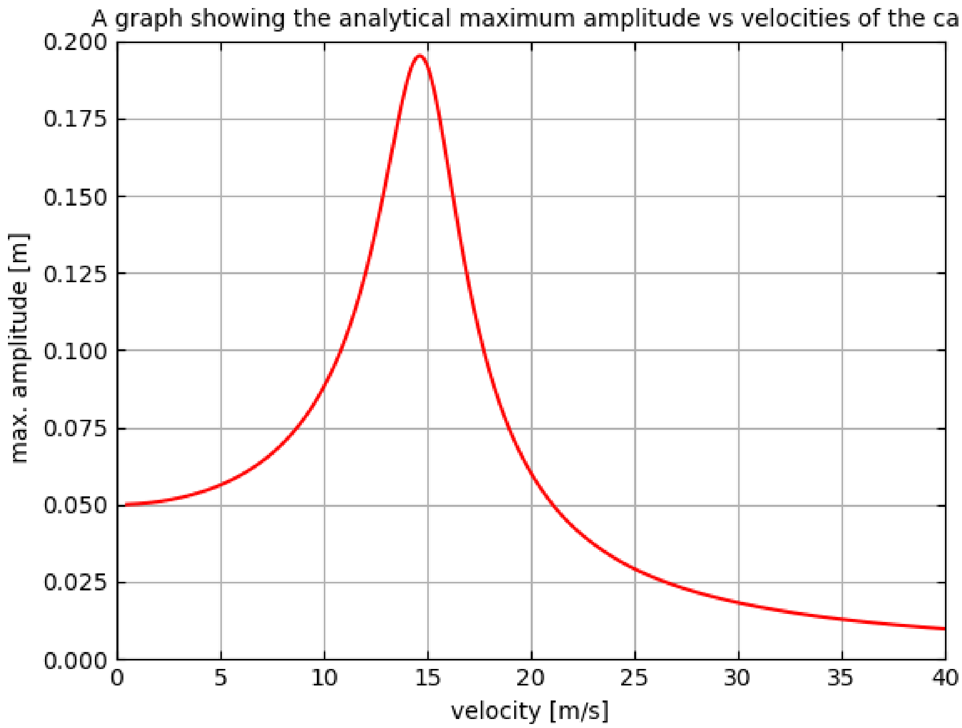

Figure 3: A graph showing the analytical solution of amplitude

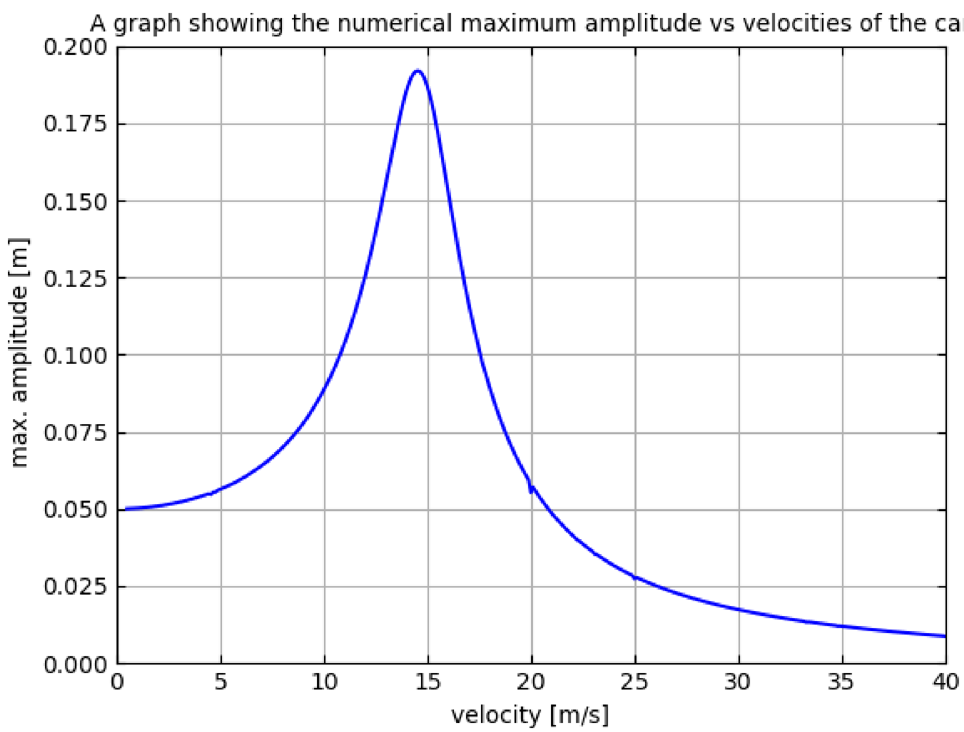

Figure 4: A graph showing the numerical solution of amplitude

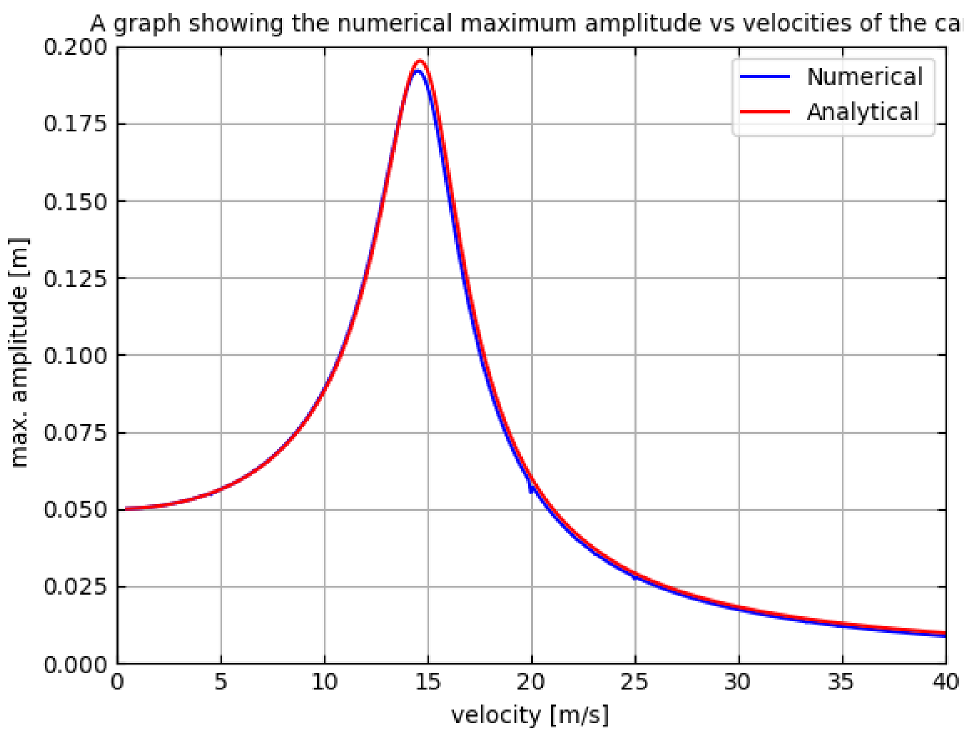

Figure 5: A comparison of both analytical and numerical graphs