Numerical Analysis Assignment 2

CNNAUA001

June 2020

Analytical Problems

1a) Use the Lagrange form of the interpolating polynomial to approximate the function $f(x) = \sin x$ using equally spaced nodes on the interval $[0, \pi/2]$. You may take your nodes to be 0, $\pi/6$, $\pi/3$, and $\pi/2$.

Using the points: $(0,0), (\pi/6, 1/2), (\pi/3, \sqrt{3}/2), (\pi/2, 1)$ I will put this into the Lagrange form of interpolating polynomials.

$$p_3(x) = L_0 y_0 + L_1 y_1 + L_2 y_2 + L_3 y_3 \tag{1}$$

For L_0:

$$\begin{aligned}
L_0 &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \\
\therefore L_0 &= \frac{\left(x-\frac{\pi}{6}\right)\left(x-\frac{\pi}{3}\right)\left(x-\frac{\pi}{2}\right)}{\left(0-\frac{\pi}{6}\right)\left(0-\frac{\pi}{3}\right)\left(0-\frac{\pi}{2}\right)} \\
\therefore L_0 &= \frac{\left(x-\frac{\pi}{6}\right)\left(x^2 - x\left(\frac{\pi}{3}+\frac{\pi}{2}\right)+\frac{\pi^2}{6}\right)}{-\frac{\pi^3}{36}} \\
\therefore L_0 &= -\frac{36}{\pi^3}\left(x^3 - x^2\left(\frac{5\pi}{6}\right) + \frac{\pi^2}{6}x - \frac{\pi}{6}x^2 + \frac{5\pi^2}{36}x - \frac{\pi^3}{36}\right) \\
\therefore L_0 &= -\frac{36}{\pi^3}\left(x^3 - x^2(\pi) + x\left(\frac{11\pi^2}{36}\right) - \frac{\pi^3}{36}\right) \\
\therefore L_0 &= -\frac{36}{\pi^3}x^3 + \frac{36}{\pi^2}x^2 - \frac{11}{\pi}x + 1
\end{aligned}$$

For L_1:

$$
\begin{aligned}
L_1 &= \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \\
\therefore L_1 &= \frac{\left(x-0\right)\left(x-\frac{\pi}{3}\right)\left(x-\frac{\pi}{2}\right)}{\left(\frac{\pi}{6}-0\right)\left(\frac{\pi}{6}-\frac{\pi}{3}\right)\left(\frac{\pi}{6}-\frac{\pi}{2}\right)} \\
\therefore L_1 &= \frac{x\left(x^2-x\left(\frac{\pi}{3}+\frac{\pi}{2}\right)+\frac{\pi^2}{6}\right)}{\frac{\pi}{6}\left(-\frac{\pi}{6}\right)\left(-\frac{\pi}{3}\right)} \\
\therefore L_1 &= \frac{x^3-x^2\left(\frac{5\pi}{6}\right)+\frac{\pi^2}{6}x}{\frac{\pi^3}{108}} \\
\therefore L_1 &= \frac{108}{\pi^3}\left(x^3-x^2\left(\frac{5\pi}{6}\right)+\frac{\pi^2}{6}x\right) \\
\therefore L_1 &= \frac{108}{\pi^3}x^3-\frac{90}{\pi^2}x^2+\frac{18}{\pi}x
\end{aligned}
$$

For L_2:

$$
\begin{aligned}
L_2 &= \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \\
\therefore L_2 &= \frac{\left(x-0\right)\left(x-\frac{\pi}{6}\right)\left(x-\frac{\pi}{2}\right)}{\left(\frac{\pi}{3}-0\right)\left(\frac{\pi}{3}-\frac{\pi}{6}\right)\left(\frac{\pi}{3}-\frac{\pi}{2}\right)} \\
\therefore L_2 &= \frac{x\left(x^2-x\left(\frac{\pi}{6}+\frac{\pi}{2}\right)+\frac{\pi^2}{12}\right)}{\frac{\pi}{3}\left(\frac{\pi}{6}\right)\left(-\frac{\pi}{6}\right)} \\
\therefore L_2 &= -\frac{108}{\pi^3}\left(x^2-x\left(\frac{2\pi}{3}\right)+\frac{\pi^2}{12}\right) \\
\therefore L_2 &= -\frac{108}{\pi^3}x^3+\frac{72}{\pi^2}x^2-\frac{9}{\pi}x
\end{aligned}
$$

For L_3:

$$L_3 = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$
$$\therefore L_3 = \frac{\Big(x-0\Big)\Big(x-\frac{\pi}{6}\Big)\Big(x-\frac{\pi}{3}\Big)}{\Big(\frac{\pi}{2}-0\Big)\Big(\frac{\pi}{2}-\frac{\pi}{6}\Big)\Big(\frac{\pi}{2}-\frac{\pi}{3}\Big)}$$
$$\therefore L_3 = \frac{x\Big(x^2 - x\Big(\frac{\pi}{6}+\frac{\pi}{3}\Big)+\frac{\pi^2}{18}\Big)}{\frac{\pi}{2}\Big(\frac{\pi}{3}\Big)\Big(\frac{\pi}{6}\Big)}$$
$$\therefore L_3 = \frac{36}{\pi^2}\Big(x^3 - x^2\Big(\frac{\pi}{2}\Big)+\frac{\pi^2}{18}x\Big)$$
$$\therefore L_3 = \frac{36}{\pi^3}x^3 - \frac{18}{\pi^2}x^2 + \frac{2}{\pi}x$$

Now to find the approximation, $p_3(x)$, we substitute L_0, L_1, L_2 and L_3.

$$p_3(x) = (0)\Big(-\frac{36}{\pi^3}x^3+\frac{36}{\pi^2}x^2-\frac{11}{\pi}x+1\Big)+\Big(\frac{1}{2}\Big)\Big(\frac{108}{\pi^3}x^3-\frac{90}{\pi^2}x^2+\frac{18}{\pi}x\Big)$$
$$+\Big(\frac{\sqrt{3}}{2}\Big)\Big(-\frac{108}{\pi^3}x^3+\frac{72}{\pi^2}x^2-\frac{9}{\pi}x\Big)+(1)\Big(\frac{36}{\pi^3}x^3-\frac{18}{\pi^2}x^2+\frac{2}{\pi}x\Big)$$
$$\therefore p_3(x) = \Big(\frac{54}{\pi^3}x^3-\frac{45}{\pi^2}x^2+\frac{9}{\pi}x\Big)+\Big(-\frac{54\sqrt{3}}{\pi^3}x^3+\frac{36\sqrt{3}}{\pi^2}x^2-\frac{9\sqrt{3}}{2\pi}x\Big)$$
$$+\Big(\frac{36}{\pi^3}x^3-\frac{18}{\pi^2}x^2+\frac{2}{\pi}x\Big)$$
$$\therefore p_3(x) = \frac{x^3}{\pi^3}\Big(54-54\sqrt{3}+36\Big)+\frac{x^2}{\pi^2}\Big(-45+36\sqrt{3}-18\Big)+\frac{x}{\pi}\Big(9+2-\frac{9\sqrt{3}}{2}\Big)$$
$$\therefore p_3(x) = \frac{x^3}{\pi^3}\Big(90-54\sqrt{3}\Big)+\frac{x^2}{\pi^2}\Big(-63+36\sqrt{3}\Big)+\frac{x}{\pi}\Big(\frac{22-9\sqrt{3}}{2}\Big) \qquad (2)$$

1b) On the same set of axes, plot the graphs of the four $L_i(x)$ functions from 1a).

Using the code in the appendix in Figure (9) and Figure (10), I plotted $L_0(x), L_1(x), L_2(x)$ and $L_3(x)$ over the interval $[0, \pi/2]$. The Graph produced is seen in Figure (!!!) below.

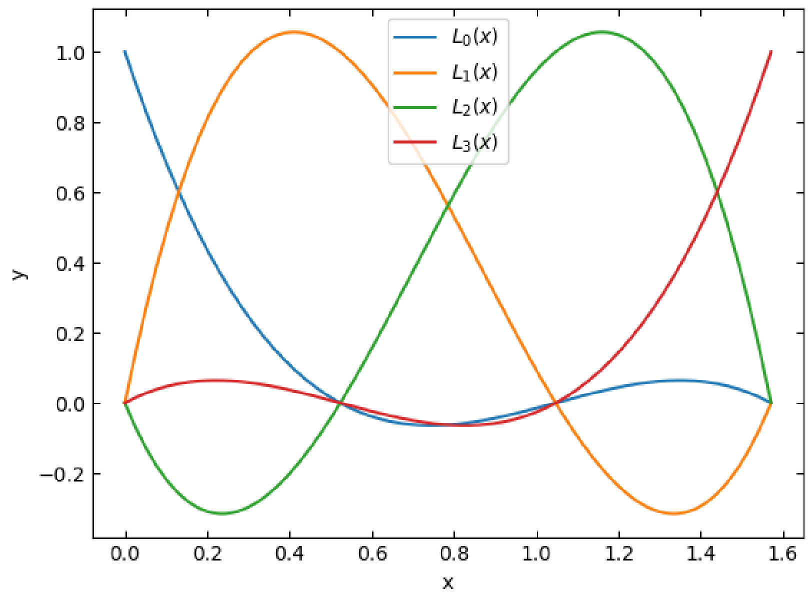

Figure 1: A graph of $L_0(x), L_1(x), L_2(x)$ and $L_3(x)$

1c) Repeat question 1a using the Newton form of the interpolating polynomial and verify agreement with the Lagrange interpolating polynomial.

x_i	x	$f(x)$	1^{st}	2^{nd}	3^{rd}
0	0	0	$\frac{3}{\pi}$	$\frac{9\sqrt{3}-18}{\pi^2}$	$\frac{90-54\sqrt{3}}{\pi^2}$
1	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{3\sqrt{3}-3}{\pi}$	$\frac{27-18\sqrt{3}}{\pi^2}$	
2	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{6-3\sqrt{3}}{\pi}$		
3	$\frac{\pi}{2}$	1			

Putting this into polynomial form using the formula:

$$p_3(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + f[x_0, x_1, x_2, x_3] \tag{3}$$

$$p_3(x) = \frac{3}{\pi}\Big(x-0\Big) + \frac{9\sqrt{3}-18}{\pi^2}\Big(x-0\Big)\Big(x-\frac{\pi}{6}\Big) + \frac{90-54\sqrt{3}}{\pi^3}\Big(x-0\Big)\Big(x-\frac{\pi}{6}\Big)\Big(x-\frac{\pi}{3}\Big)$$
$$\therefore p_3(x) = \frac{3}{\pi}x + \frac{9\sqrt{3}-18}{\pi^2}\Big(x^2-\frac{\pi}{6}x\Big) + \frac{90-54\sqrt{3}}{\pi^3}\Big(x\Big(x^2-x\Big(\frac{\pi}{6}+\frac{\pi}{3}\Big)+\frac{\pi^2}{18}\Big)\Big)$$
$$\therefore p_3(x) = \frac{3}{\pi}x + \frac{9\sqrt{3}-18}{\pi^2}x^2 - \frac{9\sqrt{3}-18}{6\pi}x + \frac{90-54\sqrt{3}}{\pi^3}\Big(x^3-x^2\Big(\frac{\pi}{2}\Big)+\frac{\pi^2}{18}x\Big)$$
$$\therefore p_3(x) = \frac{3}{\pi}x + \frac{9\sqrt{3}-18}{\pi^2}x^2 - \frac{9\sqrt{3}-18}{6\pi}x + \frac{90-54\sqrt{3}}{\pi^3}x^3 - \frac{90-54\sqrt{3}}{2\pi^2}x^2 + \frac{90-54\sqrt{3}}{18\pi}x$$
$$\therefore p_3(x) = \frac{x^3}{\pi^3}\Big(90-54\sqrt{3}\Big) + \frac{x^2}{\pi^2}\Big(-63+36\sqrt{3}\Big) + \frac{x}{\pi}\Big(\frac{22-9\sqrt{3}}{2}\Big) \tag{4}$$

It is seen that equation (4) from Newton's form of interpolating polynomials outputs the same equation as equation (2), therefore equation (2) is verified by equation (4).

2a) Using the error bound formula for polynomial approximation to a function $f(x)$, show that the error in third-degree polynomial approximation to $f(x)$ satisfies:

$$|f(x) - p_3(x)| \leq \frac{h^4}{24}\max_{\xi\in[x_0,x_3]}|f^4(\xi)| \tag{5}$$

if the nodes x_0, x_1, x_2 and x_3 are equally spaced with step-size h.

The error bound formula for the Lagrange Form of Interpolating Polynomials is:

$$f(x) - p_n(x) = \frac{f^{n+1}(\xi)}{(n+1)!}(x-x_0)(x-x_1)(x-x_2)...(x-x_n) \tag{6}$$

And in our case, $n = 3$, which means the formula becomes:

$$f(x) - p_3(x) = \frac{f^4(\xi)}{(4)!}(x-x_0)(x-x_1)(x-x_2)(x-x_3) \tag{7}$$

Let $w(x) = (x-x_0)(x-x_1)(x-x_2)(x-x_3)$ and we change the variables using, $t = x - x_1 - h/2$: Taking into account that all the points are a step size h apart, we can rewrite the points as $x_1 = x_0 + h; x_2 = x_1 + h$ and $x_3 = x_1 + 2h$
For $(x - x_0)$:

$$t = x - x_1 - \frac{h}{2}$$
$$\therefore t = x - (x_0 + h) - \frac{h}{2}$$
$$\therefore t = x - x_0 - \frac{3h}{2}$$
$$\therefore t + \frac{3h}{2} = x - x_0$$

For $(x - x_1)$:

$$t = x - x_1 - \frac{h}{2}$$
$$\therefore t + \frac{h}{2} = x - x_1$$

For $(x - x_2)$:

$$t = x - x_1 - \frac{h}{2}$$
$$\therefore t = x - (x_2 - h) - \frac{h}{2}$$
$$\therefore t = x - x_2 + \frac{h}{2}$$
$$\therefore t - \frac{h}{2} = x - x_2$$

For $(x - x_3)$:

$$t = x - x_1 - \frac{h}{2}$$
$$\therefore t = x - (x_3 - 2h) - \frac{h}{2}$$
$$\therefore t = x - x_3 + \frac{3h}{2}$$
$$\therefore t - \frac{3h}{2} = x - x_3$$

Now we can change $w(x)$ into $w(t)$:

$$w(t) = \left(t + \frac{3h}{2}\right)\left(t + \frac{h}{2}\right)\left(t - \frac{h}{2}\right)\left(t - \frac{3h}{2}\right) \tag{8}$$

If we take equation (7) and find the maximum of it we get:

$$\max_{\xi \in [x_0, x_3]} |f(x) - p_3(x)| = \frac{1}{24} \max_{\xi \in [x_0, x_3]} |f^4(\xi)||w(x)| \tag{9}$$

As, $w(x) = w(t)$ under the change in variable, we can find the maximum of $w(t)$ and it will be the same as the maximum of $w(x)$. In order to find the maximum we take the derivative of $w(t)$ and set it to zero to find t that would make the function a maximum.

$$\begin{aligned}
w(t) &= \Big(t + \frac{3h}{2}\Big)\Big(t + \frac{h}{2}\Big)\Big(t - \frac{h}{2}\Big)\Big(t - \frac{3h}{2}\Big) \\
\therefore w(t) &= \Big(t^2 - \Big(\frac{3h}{2}\Big)^2\Big)\Big(t^2 - \Big(\frac{h}{2}\Big)^2\Big) \\
\therefore w(t) &= t^4 - t^2\Big(\frac{h}{2}\Big)^2 - t^2\Big(\frac{3h}{2}\Big)^2 + \Big(\frac{3h}{2}\Big)^2\Big(\frac{h}{2}\Big)^2 \\
\therefore w(t) &= t^4 - t^2\Big(\frac{h^2 + 9h^2}{4}\Big) + \frac{9h^4}{16} \\
\therefore w(t) &= t^4 - t^2\Big(\frac{10h^2}{4}\Big) + \frac{9h^4}{16} \\
\therefore w'(t) &= 4t^3 - 2t\Big(\frac{10h^2}{4}\Big) = 0 \\
\therefore 0 &= 2t\Big(2t^2 - \Big(\frac{10h^2}{4}\Big)\Big) \\
\therefore t &= 0 \\
&or \\
\therefore 0 &= 2t^2 - \Big(\frac{10h^2}{4}\Big) \\
\therefore 2t^2 &= \frac{10h^2}{4} \\
\therefore t^2 &= \frac{5}{4}h^2 \\
\therefore t &= \pm\sqrt{\frac{5}{4}}h
\end{aligned}$$

As t is only in powers of 2 and 4, the negative sign will be squared to become a positive sign, for this reason I will only consider the two t-values, $t = 0$ and $t = \sqrt{5/4}h$. For $t = 0$:

$$\begin{aligned}
w(0) &= (0)^4 - (0)^2\Big(\frac{10h^2}{4}\Big) + \frac{9h^4}{16} \\
\therefore w(0) &= \frac{9h^4}{16}
\end{aligned}$$

For $t = \sqrt{5/4}h$:

$$\begin{aligned}
w(\sqrt{5/4}h) &= \Big(\sqrt{\frac{5}{4}}h\Big)^4 - \Big(\sqrt{\frac{5}{4}}h\Big)^2\Big(\frac{10h^2}{4}\Big) + \frac{9h^4}{16} \\
\therefore w(\sqrt{5/4}h) &= \Big(\frac{25h^4}{16}\Big) - \Big(\frac{5h^2}{4}\Big)^2\Big(\frac{10h^2}{4}\Big) + \frac{9h^4}{16} \\
\therefore w(\sqrt{5/4}h) &= \frac{25h^4}{16} - \frac{50h^4}{16} + \frac{9h^4}{16} \\
\therefore w(\sqrt{5/4}h) &= -\frac{16}{16}h^4 \\
\therefore w(\sqrt{5/4}h) &= -h^4
\end{aligned}$$

Now taking the absolute value to get the maximum gives:

$$|w(0)| = |\frac{9h^4}{16}| = \frac{9h^4}{16}$$

And,

$$|w(\sqrt{5/4}h)| = |-h^4| = h^4$$

Therefore it is seen that the maximum of $w(x)$ is h^4. Now putting this into equation (9) gives:

$$\begin{aligned}
\max_{\xi\in[x_0,x_3]} |f(x) - p_3(x)| &= \frac{1}{24}\max_{\xi\in[x_0,x_3]} |f^4(\xi)|h^4 \\
\therefore |f(x) - p_3(x)| &= \frac{h^4}{24}\max_{\xi\in[x_0,x_3]} |f^4(\xi)| \\
\therefore |f(x) - p_3(x)| &\leq \frac{h^4}{24}\max_{\xi\in[x_0,x_3]} |f^4(\xi)|
\end{aligned}$$

Consider the problem of approximating a function $f(x)$ with a cubic Hermite interpolation polynomial with nodes at $x_0 = a$ and $x_1 = b$. Using the error bound formula for cubic Hermite interpolation, show that the error satisfies

$$|f(x) - H_3(x)| \leq \frac{(b-a)^2}{384}\max_{\xi\in[a,b]} |f^4(\xi)| \tag{10}$$

The cubic Hermite formula is:

$$f(x) - H_3(x) = \frac{f^4(\xi)}{24}(x-x_0)^2(x-x_1)^2$$

We can rewrite $(x-x_0)$ and $(x-x_1)$ using the substitution $t = x-((a+b)/2)$[2], with $x_0 = a$ and $x_1 = b$.

$$\therefore (x-x_0)^2(x-x_1)^2 = \Big(t + \frac{a+b}{2} - a\Big)^2\Big(t + \frac{a+b}{2} - b\Big)^2$$

Finding the maximum as we did in question 2a), multiplying out and then taking the derivative with respect to t:

$$\begin{aligned}
w(t) &= \Big(t + \frac{a+b}{2} - a\Big)^2 \Big(t + \frac{a+b}{2} - b\Big)^2 \\
\therefore w(t) &= \Big(\frac{2t+a+b-2a}{2}\Big)^2 \Big(\frac{2t+a+b-2b}{2}\Big)^2 \\
\therefore w(t) &= \frac{1}{16}(2t-a+b)^2(2t+a-b)^2 \\
\therefore 16w(t) &= (4t^2+8(b-a)t+(b-a)^2)(4t^2+8(a-b)t+(a-b)^2) \\
\therefore 16w(t) &= 32t^3(a-b)+32t^3(b-a)+ \\
&4t^2(a-b)^2+4t^2(b-a)^2+64t^2(a-b)(b-a) \\
&+8t(a-b)(b-a)^2+8t(a-b)^2(b-a)+(a-b)^2(b-a)^2+16t^4
\end{aligned}$$

Taking the derivative and setting it to zero gives:

$$\begin{aligned}
0 &= 32(3)t^2(a-b)+32(3)t^2(b-a)+ \\
&4(2)t(a-b)^2+4(2)t(b-a)^2+64(2)t(a-b)(b-a) \\
&+8(a-b)(b-a)^2+8(a-b)^2(b-a)+16(4)t^3 \\
\therefore 0 &= 4(2)t(a-b)^2+4(2)t(b-a)^2+64(2)t(a-b)(b-a)+16(4)t^3 \\
\therefore 0 &= 16t(a-b)^2+64(2)t(a-b)(b-a)+16(4)t^3 \\
\therefore 0 &= 16t((a-b)^2+8(a-b)(b-a)+4t^2) \\
\therefore t &= 0 \\
&or \\
0 &= (a-b)^2+8(a-b)(b-a)+4t^2 \\
\therefore 0 &= 4t^2+(a-b)^2-8(a-b)^2 \\
\therefore 0 &= 4t^2-7(a-b)^2 \\
\therefore 4t^2 &= 7(a-b)^2 \\
\therefore t &= \sqrt{\frac{7}{4}}(a-b)
\end{aligned}$$

Putting this into w(t):

$$\begin{aligned}
16w(0) &= (b-a)^4 \\
\therefore w(0) &= \frac{(b-a)^4}{16}
\end{aligned}$$

To get $w(\sqrt{\frac{7}{4}}(a-b))$, I put the equation into wolfram alpha [1] to obtain:

$$w(\sqrt{\frac{7}{4}}(a-b)) = \frac{1}{16}\Big(\frac{9}{4}(b-a)^4\Big)$$

It is seen that $w(0) > w(\sqrt{\frac{7}{4}}(a-b))$, therefore the maximum of $w(x)$ is $\frac{(b-a)^4}{16}$. Taking the maximum on both sides of cubic Hermite equation gives:

$$\max_{\xi\in[a,b]} |f(x) - H_3(x)| = \max_{\xi\in[a,b]} |\frac{f^4(\xi)}{24}(x-x_0)^2(x-x_1)^2|$$

$$\therefore |f(x) - H_3(x)| = \frac{(b-a)^4}{(16)(24)} \max_{\xi\in[a,b]} |f^4(\xi)|$$

$$\therefore |f(x) - H_3(x)| = \frac{(b-a)^4}{384} \max_{\xi\in[a,b]} |f^4(\xi)|$$

$$\therefore |f(x) - H_3(x)| \leq \frac{(b-a)^4}{384} \max_{\xi\in[a,b]} |f^4(\xi)|$$

Numerical Problems

3a) Write a code that uses your polynomial from 1a to approximate the function $\sin x$. The function should work for any $x \in [0, \infty)$. In particular, do not simply evaluate the polynomial for any input x. Rather, use the fact that the interval $[0, \pi/2]$ is the fundamental Domain of the sine function, i.e any x can be mapped to $[0, \pi/2]$ by using properties of the sine function.

For 3a) I used the code in Figure (8) however I used the array labelled x in my code which goes from 0 to 0.5π and I commented out the other x array as well as the other *plt.plot*'s which produce the other graphs asked for later in this question. The graph outputted under the conditions in 3a) is:

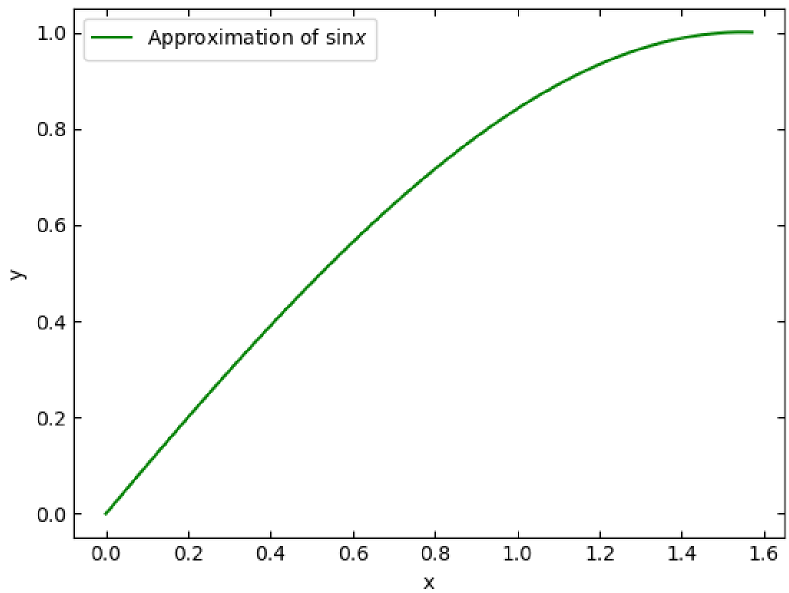

Figure 2: Approximation of sinx from 0 to 0.5π

To see how accurate the approximation was I also plotted $\sin x$ on the same set of axis, this is also seen in my code. This produced Figure(3)

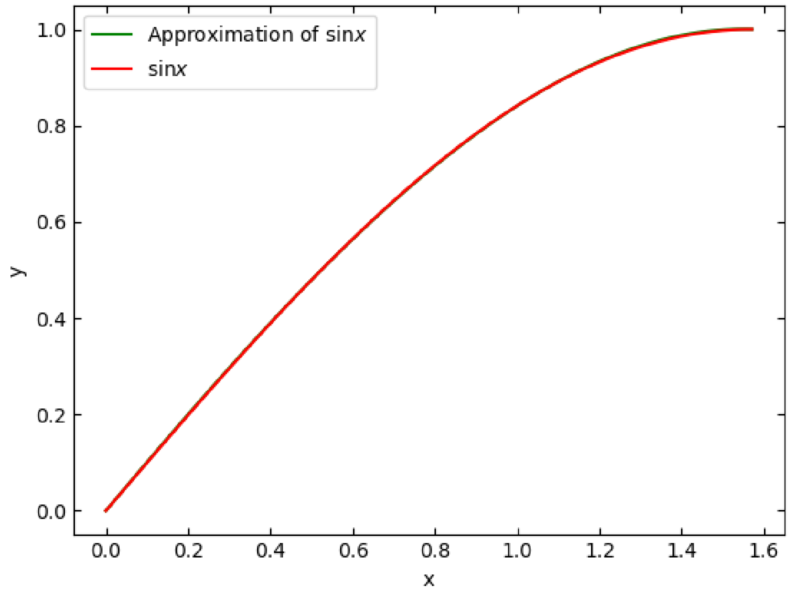

Figure 3: $\sin x$ and $p(x)$ on the same set of axis from 0 to 0.5π

3b) Plot your function on the interval $[0, 2\pi]$.

When plotting my function over the interval $[0, 2\pi]$ I commented out the x array in my code that produced an array from $[0, 0.5\pi]$ and commented out the *plt.plot*'s that did not produce my function $p(x)$ in the code. When plotted it produced Figure (4).

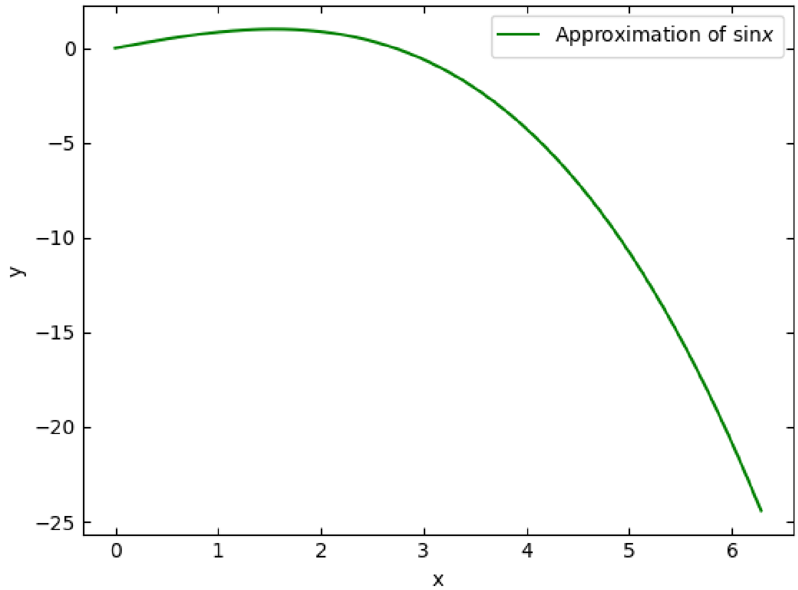

Figure 4: Approximation of $\sin x$ from 0 to 2π

Once more to compare to the actual graph of $\sin x$ to see the difference between the two, I plotted $\sin x$ and $p(x)$ on the same set of axes. This produced Figure (5).

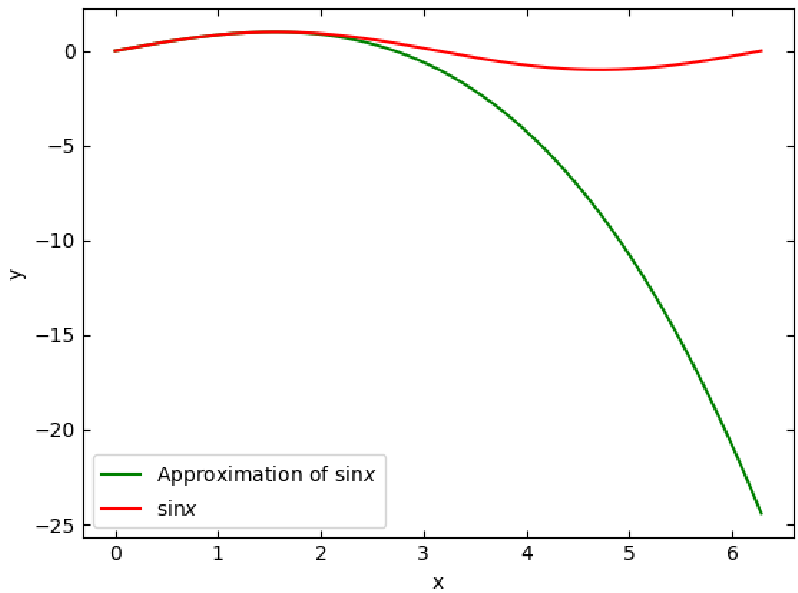

Figure 5: $\sin x$ and $p(x)$ on the same set of axis from 0 to 2π

As one can see, as the value for x increases, the difference between the graphs increase and this is because we did not take into account the nodes up to 2π.

3c) On a separate set of axes, plot the differences $p(x) - \sin(x)$ between your function and the sine function on the interval $[0, 2\pi]$.

When plotting the difference between the two functions, I commented out the *plt.plot*'s that produce $p(x)$ and $\sin x$ as well as the x array that goes from 0 to 0.5π. The graph outputted is Figure (6).

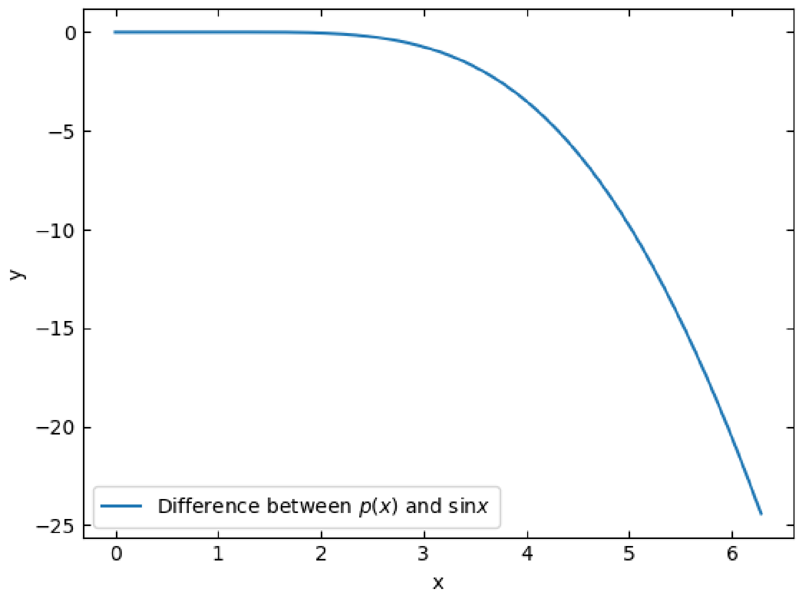

Figure 6: A graph of the difference between $p(x)$ and $\sin x$ over the interval $[0, 2\pi]$

Plotting all three of these graphs illustrates the difference nicely and is seen in Figure (7).

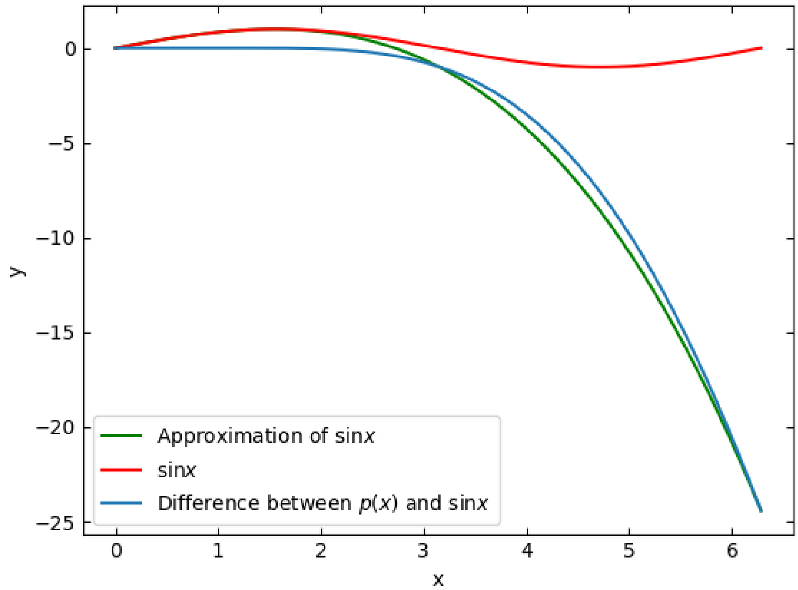

Figure 7: A nice illustration of the difference between $p(x)$ and $\sin x$

References

[1] Wolfram Alpha. URL: `https://www.wolframalpha.com/input/?i=y%3D%5Cleft%28%28%5Csqrt%7B%5Cfrac%7B7%7D%7B4%7D%7D%28a-b%29%29%2B%5Cfrac%7Ba%2Bb%7D%7B2%7D-a%5Cright%29%5E2%5Cleft%28%28%5Csqrt%7B%5Cfrac%7B7%7D%7B4%7D%7D%28a-b%29%29%2B%5Cfrac%7Ba%2Bb%7D%7B2%7D-b%5Cright%29%5E2`. (accessed: 04.06.2020).

[2] Austin Connolly. "Online interview with Zayd Pandit". In: *Cubic Hermite change of variables* (2020).

[3] AF Math Engineering. *Newton's Divided Differences Interpolation Polynomial Example*. URL: `https://youtu.be/hcsBjizQ9X8`. (accessed: 04.06.2020).

Appendix

```python
# Numerical Problems in assignmnent 2 Question 3
# CNNAUA001
# 4/6/2020

## Importing Libraries ----------------------------------------------------------------------------------------------
import numpy as np
import scipy as sc
import matplotlib.pyplot  as plt
from matplotlib import rc
##------------------------------------------------------------------------------------------------------------------

## Defining Interpolating Polynomial --------------------------------------------------------------------------------
def p(x):
    y = ((x**3)/np.pi**3)*(90-54*np.sqrt(3)) + ((x**2)/np.pi**2)*(-63+36*np.sqrt(3)) + ((x)/np.pi)*((22-9*np.sqrt(3))/2)
    return y
##------------------------------------------------------------------------------------------------------------------

## Initialising Variables and Arrays --------------------------------------------------------------------------------
x=np.linspace(0,2*np.pi,1000) # Array of x-values for 0 to 2\pi
x=np.linspace(0,0.5*np.pi,1000) # Array of x-values for 0 to 0.5*\pi
##------------------------------------------------------------------------------------------------------------------

## Plotting ---------------------------------------------------------------------------------------------------------
plt.plot(x,p(x),'-g',label='Approximation of $\sin x$') # Plots approximation p(x)
plt.plot(x,np.sin(x),'-r',label='$\sin x$') # Plots sin(x)
plt.plot(x,p(x)-np.sin(x),label='Difference between $p(x)$ and $\sinx$') # Plots difference between p(x) and sin(x)
plt.tick_params(direction='in',top=True,right=True)
plt.xlabel('x')
plt.ylabel('y')
plt.legend()
##------------------------------------------------------------------------------------------------------------------
```

Figure 8: Code for Question 3

```python
# Numerical Problems in assignmnent 2 Question 3
# CNNAUA001
# 4/6/2020

## Importing Libraries ---------------------------------------------------------------------------------------
import numpy as np
import scipy as sc
import matplotlib.pyplot  as plt
from matplotlib import rc
##-----------------------------------------------------------------------------------------------------------

## Defining L0(x) --------------------------------------------------------------------------------------------
def L0(x):
    y = -(36/np.pi**3)*x**3 + (36/np.pi**2)*x**2-(11/np.pi)*x +1
    return y
##-----------------------------------------------------------------------------------------------------------

## Defining L1(x) --------------------------------------------------------------------------------------------
def L1(x):
    y = (108/np.pi**3)*x**3 - (90/np.pi**2)*x**2+(18/np.pi)*x
    return y
##-----------------------------------------------------------------------------------------------------------

## Defining L2(x) --------------------------------------------------------------------------------------------
def L2(x):
    y = -(108/np.pi**3)*x**3 + (72/np.pi**2)*x**2-(9/np.pi)*x
    return y
##-----------------------------------------------------------------------------------------------------------

## Defining L3(x) --------------------------------------------------------------------------------------------
def L3(x):
    y = (36/np.pi**3)*x**3 - (18/np.pi**2)*x**2 + (2/np.pi)*x
    return y
##-----------------------------------------------------------------------------------------------------------
```

Figure 9: Code for 1b)

```python

## Initialising Variables and Arrays -------------------------------------------------------------------------
x=np.linspace(0,0.5*np.pi,1000) # Array of x-values for 0 to 0.5*\pi
##-----------------------------------------------------------------------------------------------------------

## Plotting --------------------------------------------------------------------------------------------------
plt.plot(x,L0(x),label='$L_0(x)$') # Plots L1(x)
plt.plot(x,L1(x),label='$L_1(x)$') # Plots L2(x)
plt.plot(x,L2(x),label='$L_2(x)$') # Plots L3(x)
plt.plot(x,L3(x),label='$L_3(x)$') # Plots L4(x)
plt.tick_params(direction='in',top=True,right=True)
plt.xlabel('x')
plt.ylabel('y')
plt.legend()
##-----------------------------------------------------------------------------------------------------------
```

Figure 10: Code for 1b) continued