# Ordinary Differential Equations: Assignment 2

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May 2020

## Introduction

"*Necessity...the mother of invention*"-Plato

Max and Miu-Miu, two people, decide to swing on a swing, each with different intentions in mind. Max wants to fly into space, and wants his oscillations to grow exponentially. Miu-Miu wants to control her swing and decides that she wants to swing with her oscillations growing linearly so that she can stop the swing before it goes over the top. The equation which dictates their swings is:

$$\frac{d^2\theta}{d\tau^2} + \nu\theta + \epsilon\cos(2\tau)\theta = 0 \tag{1}$$

Where the only difference between Max and Miu-Miu's case is how close they are to resonance of the swinging.
Where:

$$\nu = 4\frac{\omega_0^2}{\omega^2}$$
$$\epsilon = 4\frac{\omega_0^2}{\omega^2}\frac{a}{l_0}$$

## 1 Analysis

### Max's Case

Max is swinging on a swing with a crouching frequency of $\omega$ near $2\omega_0$. in Max's case we have a $\nu$ value of $\nu \approx 1$. He is hoping to achieve exponential growth of $\theta$-oscillations.
Using singular perturbation theory I will find the boundary conditions which satisfy Max's conditions.
Firstly, expanding $\nu$ in powers of $\epsilon$ in the neighbourhood of $\nu \approx 1$:

$$\nu = 1 + \epsilon\nu_1 + \epsilon^2\nu_2 + ... \tag{2}$$

Expanding $\theta$:

$$\theta = \theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + ... \tag{3}$$

And re-scaling time:

$$\omega t = 2\tau$$

and

$$T_0 = \tau; T_1 = \epsilon\tau; T_2 = \epsilon^2\tau; ...; T_n = \epsilon^n\tau \tag{4}$$

and we define $d/d\tau$ as:

$$\frac{d}{d\tau} = D_0 + \epsilon D_1 + \epsilon^2 D_2 + ... \tag{5}$$

which means that $d^2/d\tau^2 = (d/d\tau)(d/d\tau)$

$$\frac{d^2}{d\tau^2} = D_0^2 + 2\epsilon D_0 D_1 + ... \tag{6}$$

Now substituting (2) to (5) into the original equation describing the system:

$$(D_0^2 + 2\epsilon D_0 D_1 + ...)(\theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + ...) + (\theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + ...)\epsilon\cos(2T_0) + \\ (1 + \epsilon\nu_1 + \epsilon^2\nu_2 + ...)(\theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + ...) = 0$$

Set all $\epsilon^0$ terms to zero:

$$D_0^2\theta_0 + \theta_0 = 0 \tag{7}$$

This is a second order homogeneous differential equation, therfore we use the ansatz: $\theta_0 = e^{rt}$

$$r^2 = -1$$
$$\therefore r^2 = i^2$$

where $i^2 = -1$

$$\therefore r = \pm i$$
$$\therefore \theta_0 = Ae^{iT_0} + A^* e^{-iT_0}$$

Where $A$ and $A^*$ are arbitrary constants, with $A^*$ being the complex conjugate of $A$.

Now setting all $\epsilon^1$ terms to zero:

$$D_0^2\theta_1 + 2D_0D_1\theta_0 + \theta_0\cos(2T_0) + \theta_1 + \theta_0\nu_1 = 0$$
$$\therefore D_0^2\theta_1 + \theta_1 = -(2D_0D_1\theta_0 + \theta_0\cos(2T_0) + \theta_0\nu_1)$$

$$\therefore D_0^2\theta_1 + \theta_1 = -2D_0D_1\theta_0 - \theta_0\cos(2T_0) - \theta_0\nu_1 \tag{8}$$

Aside, $\cos(x)$ can be rewritten[1]:

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

For this situation:

$$\cos(2T_0) = \frac{1}{2}(e^{i2T_0} + e^{-i2T_0}) \tag{9}$$

substituting equation (9) into equation (8):

$$D_0^2\theta_1 + \theta_1 = -2D_0D_1\theta_0 - \theta_0\Big(\frac{1}{2}(e^{i2T_0} + e^{-i2T_0})\Big) - \theta_0\nu_1$$

$$\therefore D_0^2\theta_1 + \theta_1 = -2D_1D_0(Ae^{iT_0} + A^*e^{-iT_0}) - (Ae^{iT_0} + A^*e^{-iT_0})\Big(\frac{1}{2}(e^{i2T_0} + e^{-i2T_0})\Big)$$
$$-(Ae^{iT_0} + A^*e^{-iT_0})\nu_1$$

$$\therefore D_0^2\theta_1 + \theta_1 = -2D_1(A(i)e^{iT_0} - A^*(i)e^{-iT_0}) - (Ae^{iT_0} + A^*e^{-iT_0})\Big(\frac{1}{2}(e^{i2T_0} + e^{-i2T_0})\Big)$$
$$-\nu_1Ae^{iT_0} - \nu_1A^*e^{-iT_0}$$

$$\therefore D_0^2\theta_1 + \theta_1 = -2D_1Aie^{iT_0} + 2D_1A^*ie^{-iT_0} - \frac{1}{2}(Ae^{iT_0} + A^*e^{-iT_0})(e^{i2T_0} + e^{-i2T_0})$$
$$-\nu_1Ae^{iT_0} - \nu_1A^*e^{-iT_0}$$

$$\therefore D_0^2\theta_1 + \theta_1 = -2D_1Aie^{iT_0} + 2D_1A^*ie^{-iT_0} - \frac{1}{2}(Ae^{3iT_0} + A^*e^{-3iT_0} + Ae^{iT_0} + A^*e^{-iT_0})$$
$$-\nu_1Ae^{iT_0} - \nu_1A^*e^{-iT_0}$$

$$\therefore D_0^2\theta_1 + \theta_1 = e^{iT_0}\Big( - \nu_1A - 2D_1Ai - \frac{1}{2}A^*\Big) + e^{-iT_0}\Big( - \nu_1A^* + 2D_1A^*i - \frac{1}{2}A\Big)$$
$$-\frac{1}{2}(Ae^{3iT_0} + A^*e^{-3iT_0})$$

Setting the secular terms to zero:

$$-\nu_1A - 2D_1Ai - \frac{1}{2}A^* = 0 \tag{10}$$

and

$$-\nu_1A^* + 2D_1A^*i - \frac{1}{2}A = 0 \tag{11}$$

For equation (10):
Let $A = a + bi$ and $A^* = a - bi$

$$2D_1(a + bi)i + \nu_1(a + bi) + \frac{1}{2}(a - bi) = 0$$
$$\therefore 2iD_1a - 2D_1b + \nu_1a + \nu_1bi + \frac{1}{2}a - \frac{1}{2}bi = 0$$

Equating real and imaginary parts:
Imaginary parts:

$$2D_1a + \nu_1b - \frac{1}{2}b = 0$$

$$D_1a = -\frac{1}{2}b\Big(\nu_1 - \frac{1}{2}\Big) \tag{12}$$

Real parts:

$$-2D_1b + \nu_1 a + \frac{1}{2}a = 0$$

$$D_1 b = \frac{1}{2}a\Big(\nu_1 + \frac{1}{2}\Big) \tag{13}$$

$$D_1 D_1 b = D_1 a \frac{1}{2}\Big(\nu_1 + \frac{1}{2}\Big)$$

And substituting equation (12):

$$D_1^2 b = -\frac{1}{2}b\Big(\nu_1 - \frac{1}{2}\Big)\frac{1}{2}\Big(\nu_1 + \frac{1}{2}\Big)$$
$$\therefore D_1^2 b = -b\frac{1}{4}\Big(\nu_1^2 - \frac{1}{4}\Big)$$
$$\therefore D_1^2 b = i^2 b\frac{1}{4}\Big(\nu_1^2 - \frac{1}{4}\Big)$$

Ansatz: $b = e^{rt}$

$$\therefore r^2 = i^2\frac{1}{4}\Big(\nu_1^2 - \frac{1}{4}\Big)$$
$$\therefore r = \pm i\frac{1}{2}\sqrt{\Big(\nu_1^2 - \frac{1}{4}\Big)}$$

$$b = Me^{i\frac{1}{2}\sqrt{\left(\nu_1^2 - \frac{1}{4}\right)}T_1} + M^* e^{-i\frac{1}{2}\sqrt{\left(\nu_1^2 - \frac{1}{4}\right)}T_1} \tag{14}$$

Where $M$ and $M^*$ are arbitrary constants.
Using equation (12) and substituting equation (14):

$$D_1 a = -\frac{1}{2}\Big(Me^{i\frac{1}{2}\sqrt{\left(\nu_1^2 - \frac{1}{4}\right)}T_1} + M^* e^{-i\frac{1}{2}\sqrt{\left(\nu_1^2 - \frac{1}{4}\right)}T_1}\Big)\Big(\nu_1 - \frac{1}{2}\Big)$$

$$\therefore \int (D_1 a) dT_1 = \int \Big(-\frac{1}{2}\Big(Me^{i\frac{1}{2}\sqrt{\left(\nu_1^2 - \frac{1}{4}\right)}T_1} + M^* e^{-i\frac{1}{2}\sqrt{\left(\nu_1^2 - \frac{1}{4}\right)}T_1}\Big)\Big(\nu_1 - \frac{1}{2}\Big)\Big) dT_1$$

$$a = -\frac{1}{2}\Big(\nu_1 - \frac{1}{2}\Big)\Big(\frac{\tilde{M}}{i\frac{1}{2}\sqrt{\left(\nu_1^2 - \frac{1}{4}\right)}}e^{i\frac{1}{2}\sqrt{\left(\nu_1^2 - \frac{1}{4}\right)}T_1} - \frac{\tilde{M}^*}{i\frac{1}{2}\sqrt{\left(\nu_1^2 - \frac{1}{4}\right)}}e^{-i\frac{1}{2}\sqrt{\left(\nu_1^2 - \frac{1}{4}\right)}T_1}\Big)$$
$$\therefore a = \frac{i\left(\nu_1 - \frac{1}{2}\right)}{\sqrt{\left(\nu_1^2 - \frac{1}{4}\right)}}\Big(\tilde{M}e^{i\frac{1}{2}\sqrt{\left(\nu_1^2 - \frac{1}{4}\right)}T_1} - \tilde{M}^* e^{-i\frac{1}{2}\sqrt{\left(\nu_1^2 - \frac{1}{4}\right)}T_1}\Big)$$

Now putting $a$ and $b$ back into $A$ and $A^*$:

$$A = a + bi = \frac{i\left(\nu_1 - \frac{1}{2}\right)}{\sqrt{\left(\nu_1^2 - \frac{1}{4}\right)}}\left(\tilde{M}e^{i\frac{1}{2}\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}T_1} - \tilde{M}^*e^{-i\frac{1}{2}\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}T_1}\right)$$

$$+i\left(Me^{i\frac{1}{2}\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}T_1} + M^*e^{-i\frac{1}{2}\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}T_1}\right)$$

$$\therefore A = e^{i\frac{1}{2}\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}T_1}\left(\frac{i\left(\nu_1 - \frac{1}{2}\right)}{\sqrt{\left(\nu_1^2 - \frac{1}{4}\right)}}\tilde{M} + iM\right)$$

$$+e^{-i\frac{1}{2}\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}T_1}\left(-\frac{i\left(\nu_1 - \frac{1}{2}\right)}{\sqrt{\left(\nu_1^2 - \frac{1}{4}\right)}}\tilde{M}^* + iM^*\right)$$

Let $W_1 = \frac{i\left(\nu_1-\frac{1}{2}\right)}{\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}}\tilde{M} + iM$ and $W_2 = -\frac{i\left(\nu_1-\frac{1}{2}\right)}{\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}}\tilde{M}^* + iM^*$

$$\therefore A = W_1e^{i\frac{1}{2}\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}T_1} + W_2e^{-i\frac{1}{2}\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}T_1} \tag{15}$$

For $A^*$:

$$A^* = a - bi = \frac{i\left(\nu_1 - \frac{1}{2}\right)}{\sqrt{\left(\nu_1^2 - \frac{1}{4}\right)}}\left(\tilde{M}e^{i\frac{1}{2}\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}T_1} - \tilde{M}^*e^{-i\frac{1}{2}\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}T_1}\right)$$

$$-i\left(Me^{i\frac{1}{2}\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}T_1} + M^*e^{-i\frac{1}{2}\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}T_1}\right)$$

$$\therefore A^* = e^{i\frac{1}{2}\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}T_1}\left(\frac{i\left(\nu_1 - \frac{1}{2}\right)}{\sqrt{\left(\nu_1^2 - \frac{1}{4}\right)}}\tilde{M} - iM\right)$$

$$+e^{-i\frac{1}{2}\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}T_1}\left(-\frac{i\left(\nu_1 - \frac{1}{2}\right)}{\sqrt{\left(\nu_1^2 - \frac{1}{4}\right)}}\tilde{M}^* - iM^*\right)$$

Let $W_3 = \frac{i\left(\nu_1-\frac{1}{2}\right)}{\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}}\tilde{M} - iM$ and $W_4 = -\frac{i\left(\nu_1-\frac{1}{2}\right)}{\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}}\tilde{M}^* - iM^*$

$$\therefore A^* = W_3e^{i\frac{1}{2}\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}T_1} + W_4e^{-i\frac{1}{2}\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}T_1} \tag{16}$$

Now substituting equation (15) and (16) into the equation for $\theta_0$:

$$\theta_0 = \Big(W_1 e^{i\frac{1}{2}\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}T_1} + W_2 e^{-i\frac{1}{2}\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}T_1}\Big)e^{iT_0}$$
$$+ \Big(W_3 e^{i\frac{1}{2}\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}T_1} + W_4 e^{-i\frac{1}{2}\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}T_1}\Big)e^{-iT_0}$$
$$\therefore \theta(\tau) = \Big(\Big(\frac{i\left(\nu_1-\frac{1}{2}\right)}{\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}}\tilde{M}+iM\Big)e^{i\frac{1}{2}\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}\epsilon\tau} + \Big(-\frac{i\left(\nu_1-\frac{1}{2}\right)}{\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}}\tilde{M}^*+iM^*\Big)e^{-i\frac{1}{2}\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}\epsilon\tau}\Big)e^{i\tau}$$
$$+\Big(\Big(\frac{i\left(\nu_1-\frac{1}{2}\right)}{\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}}\tilde{M}-iM\Big)e^{i\frac{1}{2}\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}\epsilon\tau} + \Big(-\frac{i\left(\nu_1-\frac{1}{2}\right)}{\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}}\tilde{M}^*-iM^*\Big)e^{-i\frac{1}{2}\sqrt{\left(\nu_1^2-\frac{1}{4}\right)}\epsilon\tau}\Big)e^{-i\tau}$$
$$+ O(\epsilon)$$

This is the leading order solution to the differential equation for Max. In this equation it is seen that if we want exponential growth, the terms under the square root in the exponentials to the power of $T_1$ need to be negative to get rid of the '$i$' terms in the exponentials, so that the $T_1$ terms are no longer oscillatory.[2]

This means that $\left(\nu_1^2 - \frac{1}{4}\right) < 0$:

$$\therefore \nu_1^2 < \frac{1}{4}$$
$$\therefore -\frac{1}{2} > \nu_1 > \frac{1}{2}$$

In order to determine $\epsilon_1 = \epsilon_1(\nu)$ we consider the absolute value of $\nu_1$, i.e. $|nu_1| > \frac{1}{2}$ and use the expansion of $\nu$, $\nu = 1 + \epsilon\nu_1$:

$$\nu = 1 + \epsilon\nu_1$$
$$\therefore \nu_1 = \frac{\nu - 1}{\epsilon}$$
$$\therefore \frac{|\nu - 1|}{|\epsilon|} > \frac{1}{2}$$
$$\therefore 2|\nu - 1| > \epsilon$$
$$\therefore \epsilon < |2\nu - 2|$$
$$\therefore \epsilon_1(\nu) = |2\nu - 2|$$

This means that $\epsilon$ needs to be greater than $\epsilon_1(\nu)$ for the amplitude to grow without bound. Max's growth will be exponential in this case, and if he wants growth he should get $\nu = 1 + \epsilon(0.5)$.

## Miu-Miu's Case

Miu-Miu is swinging on a swing with a crouching frequency of $\omega$ near $\omega_0$. In Miu-Miu's case we have a $\nu$ value of $\nu \approx 4$. She is hoping to achieve linear growth of $\theta$-oscillations.
Using singular perturbation theory I will find the boundary conditions which satisfy Miu-Miu's conditions.
After many hours of calculating using the expansion $\nu = 4 + \epsilon\nu_1 + \epsilon^2\nu_2 + ...$ I did not come up with anything that was useful, even going to powers of $\epsilon^2$, and using unsavoury mathematical methods to get $\theta_1$ did not help. The next expansion that I tried was expanding $\nu$ in powers of $\epsilon$ in the neighbourhood of $\nu \approx 4$ starting with $\epsilon^2$[4]:

$$\nu = 4 + \epsilon^2\nu_1 + \epsilon^3\nu_2 + ... \tag{17}$$

Expanding $\theta$:

$$\theta = \theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + ... \tag{18}$$

And re-scaling time:

$$\omega t = 2\tau$$

and

$$T_0 = \tau; T_1 = \epsilon\tau; T_2 = \epsilon^2\tau; ...; T_n = \epsilon^n\tau \tag{19}$$

and we define $d/d\tau$ as:

$$\frac{d}{d\tau} = D_0 + \epsilon D_1 + \epsilon^2 D_2 + ... \tag{20}$$

which means that $d^2/d\tau^2 = (d/d\tau)(d/d\tau)$

$$\frac{d^2}{d\tau^2} = D_0^2 + 2\epsilon D_0 D_1 + \epsilon^2(D_1^2 + 2D_0D_2) + ... \tag{21}$$

Substituting equations (17) to (21) into equation (1):

$$\begin{aligned}(D_0^2 + 2\epsilon D_0 D_1 + \epsilon^2(D_1^2 + 2D_0D_2) + ...)(\theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + ...)\\ +(4 + \epsilon^2\nu_1 + \epsilon^3\nu_2 + ...)(\theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + ...)\\ +\epsilon\Big(\frac{e^{2iT_0} + e^{-2iT_0}}{2}\Big)(\theta_0 + \epsilon\theta_1 + \epsilon^2\theta_2 + ...) = 0\end{aligned}$$

Setting $\epsilon^0$ terms to zero:

$$D_0^2\theta_0 + 4\theta_0 = 0$$

Ansatz: $\theta_0 = e^{pT_0}$

$$\therefore p^2 = i^2 4$$
$$\therefore p = \pm i2$$

$$\therefore \theta_0 = Ae^{2iT_0} + A^* e^{-2iT_0} \tag{22}$$

Where $A$ and $A^*$ are arbitrary constants, with $A^*$ being the complex conjugate of $A$.
Setting $\epsilon^1$ terms to zero:

$$D_0^2\theta_1 + 2D_0D_1\theta_0 + 4\theta_1 + \frac{1}{2}\Big(e^{2iT_0} + e^{-2iT_0}\Big)\theta_0 = 0$$
$$\therefore D_0^2\theta_1 + 4\theta_1 = -2D_1D_0\theta_0 - \frac{1}{2}\theta_0(e^{2iT_0} + e^{-2iT_0})$$

Now substituting $\theta_0$:

$$D_0^2\theta_1 + 4\theta_1 = -2D_1D_0\Big(Ae^{2iT_0} + A^*e^{-2iT_0}\Big) - \frac{1}{2}\Big(Ae^{2iT_0} + A^*e^{-2iT_0}\Big)(e^{2iT_0} + e^{-2iT_0})$$
$$\therefore D_0^2\theta_1 + 4\theta_1 = -2D_1\Big(A(2i)e^{2iT_0} - A^*(2i)e^{-2iT_0}\Big) - \frac{1}{2}\Big(Ae^{2iT_0} + A^*e^{-2iT_0}\Big)(e^{2iT_0} + e^{-2iT_0})$$
$$\therefore D_0^2\theta_1 + 4\theta_1 = -2D_1A(2i)e^{2iT_0} + 2D_1A^*(2i)e^{-2iT_0} - \frac{1}{2}(Ae^{4iT_0} + A^*e^{-4iT_0}) - \frac{1}{2}A - \frac{1}{2}A^*$$
$$\therefore D_0^2\theta_1 + 4\theta_1 = e^{i2T_0}(-2D_1A(2i)) + e^{-2iT_0}(2D_1A^*(2i)) - \frac{1}{2}(Ae^{4iT_0} + A^*e^{-4iT_0}) - \frac{1}{2}A - \frac{1}{2}A^*$$

Setting secular terms to zero:

$$-2D_1A(2i) = 0$$

and

$$2D_1A^*(2i) = 0$$

$$\therefore A = constant$$

and

$$A^* = constant$$

Let $A = \psi \implies A^* = \psi^*$

$$\therefore \theta_0 = \psi e^{2iT_0} + \psi^* e^{-2iT_0} \tag{23}$$

Going to $\epsilon$ to the power of one, and setting those terms to zero, did not give anything useful. In order to go further, we will need to find $\theta_1$. I will do this using the method of undetermined coefficients and use the superposition principle for second order differential equations.

$$D_0^2\theta_1 + 4\theta_1 = -\frac{1}{2}(\psi e^{4iT_0} + \psi^* e^{-4iT_0}) - \frac{1}{2}\psi - \frac{1}{2}\psi^*$$
$$\therefore D_0^2\theta_1 + 4\theta_1 = -\frac{1}{2}(\psi e^{4iT_0} + \psi^* e^{-4iT_0}) - \frac{1}{2}(\psi + \psi^*)$$

Let $\beta = \psi + \psi^*$

$$\therefore D_0^2\theta_1 + 4\theta_1 = -\frac{1}{2}(\psi e^{4iT_0} + \psi^* e^{-4iT_0}) - \frac{1}{2}\beta \tag{24}$$

For the homogeneous solution:

$$D_0^2\theta_1 + 4\theta_1 = 0$$

Ansatz: $\theta_1 = e^{gT_0}$

$$g^2 = i^2 4$$
$$\therefore g = \pm i2$$

$$\theta_{1homogeneous} = Be^{2iT_0} + B^* e^{-2iT_0} \tag{25}$$

Where $B$ and $B^*$ are arbitrary constants.
For the particular solution:

$$g_1(T_0) = -\frac{1}{2}\psi e^{4iT_0}$$
$$g_2(T_0) = -\frac{1}{2}\psi^* e^{-4iT_0}$$
$$g_3(T_0) = -\frac{1}{2}\beta$$

For $g_1$:

$$Y_{p_1} = \alpha_1 e^{4iT_0}$$
$$\therefore \frac{\partial}{\partial T_0}Y_{p_1} = \alpha_1(4i)e^{4iT_0}$$
$$\therefore \frac{\partial^2}{\partial T_0^2}Y_{p_1} = -\alpha_1(16)e^{4iT_0}$$

Substituting this into equation (24):

$$-\alpha_1(16)e^{4iT_0} + 4\alpha_1 = -\frac{1}{2}\psi e^{4iT_0}$$
$$\therefore \alpha_1(4-16) = -\frac{1}{2}\psi$$
$$\therefore \alpha_1 = -\frac{\psi}{2(-12)}$$
$$\therefore \alpha_1 = \frac{\psi}{24}$$

$$\therefore \theta_{1p_1} = \frac{\psi}{24}e^{4iT_0} \tag{26}$$

For $g_2$:

$$Y_{p_2} = \alpha_2 e^{-4iT_0}$$
$$\therefore \frac{\partial}{\partial T_0} Y_{p_2} = \alpha_2(-4i)e^{-4iT_0}$$
$$\therefore \frac{\partial^2}{\partial T_0^2} Y_{p_2} = -\alpha_2(16)e^{-4iT_0}$$

Substituting this into equation (24):

$$-\alpha_2(16)e^{4iT_0} + 4\alpha_2 = -\frac{1}{2}\psi^* e^{4iT_0}$$
$$\therefore \alpha_2(4-16) = -\frac{1}{2}\psi^*$$
$$\therefore \alpha_2 = -\frac{\psi^*}{2(-12)}$$
$$\therefore \alpha_2 = \frac{\psi^*}{24}$$
$$\therefore \theta_{1p_2} = \frac{\psi^*}{24} e^{-4iT_0} \tag{27}$$

For $g_3$:

$$Y_{p_3} = C$$
$$\therefore \frac{\partial}{\partial T_0} Y_{p_3} = 0$$
$$\therefore \frac{\partial^2}{\partial T_0^2} Y_{p_3} = 0$$

Substituting this into equation (24):

$$4C = -\frac{1}{2}\beta$$
$$\therefore C = -\frac{\beta}{8}$$

The particular solution is:

$$\theta_{1p} = \frac{\psi}{24} e^{4iT_0} + \frac{\psi^*}{24} e^{-4iT_0} - \frac{\beta}{8} \tag{28}$$

Now combining the particular and homogeneous solutions gives the solution to $\theta_1$:

$$\theta_1 = Be^{2iT_0} + B^* e^{-2iT_0} + \frac{\psi}{24} e^{4iT_0} + \frac{\psi^*}{24} e^{-4iT_0} - \frac{\beta}{8} \tag{29}$$

Setting $\epsilon^2$ to zero:

$$D_0^2\theta_2 + 2D_0D_1\theta_1 + (D_1^2 + 2D_0D_2)\theta_0 + 4\theta_2 + \nu_1\theta_0 + \frac{1}{2}(e^{2iT_0} + e^{-2iT_0})\theta_1 = 0$$
$$\therefore D_0^2\theta_2 + 4\theta_2 = -2D_1D_0\theta_1 - (D_1^2 + 2D_0D_2)\theta_0 - \nu_1\theta_0 - \frac{1}{2}(e^{2iT_0} + e^{-2iT_0})\theta_1$$

Now substituting $\theta_0$ and $\theta_1$ in:

$$\therefore D_0^2\theta_2 + 4\theta_2 = -2D_1D_0\Big(Be^{2iT_0} + B^*e^{-2iT_0} + \frac{\psi}{24}e^{4iT_0} + \frac{\psi^*}{24}e^{-4iT_0} - \frac{\beta}{8}\Big)$$
$$-(D_1^2 + 2D_0D_2)(\psi e^{2iT_0} + \psi^* e^{-2iT_0}) - \nu_1(\psi e^{2iT_0} + \psi^* e^{-2iT_0})$$
$$-\frac{1}{2}(e^{2iT_0} + e^{-2iT_0})(Be^{2iT_0} + B^*e^{-2iT_0} + \frac{\psi}{24}e^{4iT_0} + \frac{\psi^*}{24}e^{-4iT_0} - \frac{\beta}{8})$$

$$\therefore D_0^2\theta_2 + 4\theta_2 = -2D_1(2i)Be^{2iT_0} + (2i)B^*e^{-2iT_0} - \frac{\psi}{24}(4i)e^{4iT_0} + \frac{\psi^*}{24}(4i)e^{-4iT_0}$$
$$-D_1^2(\psi e^{2iT_0} + \psi^* e^{-2iT_0}) - 2D_2(\psi(2i)e^{2iT_0} - (2i)\psi^* e^{-2iT_0}) - \nu_1\psi e^{2iT_0} - \nu_1\psi^* e^{-2iT_0}$$
$$-\frac{1}{2}(e^{2iT_0} + e^{-2iT_0})\Big(Be^{2iT_0} + B^*e^{-2iT_0} + \frac{\psi}{24}e^{4iT_0} + \frac{\psi^*}{24}e^{-4iT_0} - \frac{\beta}{8}\Big)$$

$$\therefore D_0^2\theta_2 + 4\theta_2 = -2D_1(2i)Be^{2iT_0} + (2i)B^*e^{-2iT_0} - \frac{\psi}{24}(4i)e^{4iT_0} + \frac{\psi^*}{24}(4i)e^{-4iT_0}$$
$$-D_1^2(\psi e^{2iT_0} + \psi^* e^{-2iT_0}) - 2D_2(\psi(2i)e^{2iT_0} - (2i)\psi^* e^{-2iT_0}) - \nu_1\psi e^{2iT_0} - \nu_1\psi^* e^{-2iT_0}$$
$$-\frac{1}{2}\Big(Be^{4iT_0} + B^* + \frac{\psi}{24}e^{6iT_0} + \frac{\psi^*}{24}e^{-2iT_0} - \frac{\beta}{8}e^{2iT_0} + B + B^*e^{4iT_0} + \frac{\psi}{24}e^{2iT_0} + \frac{\psi^*}{24}e^{-6iT_0} - \frac{1}{8}\beta e^{-2iT_0}\Big)$$

$$\therefore D_0^2\theta_2 + 4\theta_2 = e^{2iT_0}(-2D_1B(2i) - D_1^2\psi - D_2(2i)\psi - \nu_1\psi + \frac{\beta}{16} - \frac{\psi}{48})$$
$$+ e^{-2iT_0}(2D_1B^*(2i) - D_1^2\psi^* + D_2(2i)\psi^* - \nu_1\psi^* - \frac{\psi^*}{48} + \frac{\beta}{16})$$
$$-2D_1\frac{\psi}{24}(4i)e^{4iT_0} + D_1\frac{\psi^*}{24}(4i)e^{-4iT_0} - \frac{1}{2}Be^{4iT_0} - \frac{1}{2}B^* - \frac{\psi}{48}e^{6iT_0} - \frac{1}{2}B - \frac{1}{2}B^*e^{-4iT_0} - \frac{\psi^*}{48}e^{-6iT_0}$$

Setting secular terms to zero:

$$-2D_1B(2i) - D_1^2\psi - D_2(2i)\psi - \nu_1\psi + \frac{\beta}{16} - \frac{\psi}{48} = 0$$

and

$$2D_1B^*(2i) - D_1^2\psi^* + D_2(2i)\psi^* - \nu_1\psi^* - \frac{\psi^*}{48} + \frac{\beta}{16} = 0$$

As shown previously, $\psi$ and $\psi^*$ are constants which means that $D_1^2\psi = D_1^2\psi^* = D_2(2i)\psi = D_2(2i)\psi^* = 0$.

$$\therefore -2D_1B(2i) - \nu_1\psi + \frac{\beta}{16} - \frac{\psi}{48} = 0$$

and

$$\therefore 2D_1B(2i) - \nu_1\psi^* + \frac{\beta}{16} - \frac{\psi^*}{48} = 0$$

Equating real and imaginary parts:
For imaginary parts:

$$-2D_1B(2) = 0 \implies B = constant$$

and

$$2D_1B^*(2) = 0 \implies B^* = constant$$

For real parts:

$$-\nu_1\psi + \frac{\beta}{16} - \frac{\psi}{48} = 0$$
$$\therefore \nu_1\psi = \frac{\beta}{16} - \frac{\psi}{48}$$

$$\therefore \nu_1 = \frac{\beta}{16\psi} - \frac{1}{48} \tag{30}$$

and

$$-\nu_1\psi^* + \frac{\beta}{16} - \frac{\psi^*}{48} = 0$$
$$\therefore \nu_1\psi^* = \frac{\beta}{16} - \frac{\psi^*}{48}$$

$$\therefore \nu_1 = \frac{\beta}{16\psi^*} - \frac{1}{48} \tag{31}$$

equating the $\nu_1$'s:

$$\frac{\beta}{16\psi} - \frac{1}{48} = \frac{\beta}{16\psi^*} - \frac{1}{48}$$
$$\therefore \frac{\beta}{16\psi} = \frac{\beta}{16\psi^*}$$
$$\therefore \psi = \psi^* \tag{32}$$

As $\beta = \psi + \psi^*$, substituting equation (32) gives:

$$\beta = 2\psi \tag{33}$$

Substituting equation (33) into equation (30) gives:

$$\nu_1 = \frac{2\psi}{16\psi} - \frac{1}{48}$$
$$\therefore \nu_1 = \frac{1}{8} - \frac{1}{48}$$
$$\therefore \nu_1 = \frac{5}{48} \tag{34}$$

or

$$\nu_1 \approx 0.10417$$

The leading order solution for Miu-Miu's case is:

$$\theta(\tau) = \psi e^{2i\tau} + \psi^* e^{-2i\tau} + O(\epsilon)$$

In order to determine $\epsilon_4 = \epsilon_4(\nu)$, substitute the value of $\nu_1$ into the expansion of $\nu$, $\nu = 1 + \epsilon^2\nu_1$:

$$\nu = 4 + \epsilon^2\nu_1$$
$$\therefore \nu = 4 + \epsilon^2\left(\frac{5}{48}\right)$$
$$\therefore \epsilon^2\left(\frac{5}{48}\right) = \nu - 4$$
$$\therefore \epsilon = \sqrt{\frac{48}{5}(\nu - 4)}$$

$$\therefore \epsilon_4(\nu) = \sqrt{\frac{48}{5}(\nu - 4)} \tag{35}$$

This means that if $\epsilon \geq \epsilon_4(\nu) \implies \epsilon \geq \sqrt{\frac{48}{5}(\nu - 4)}$ the amplitude of oscillations grow without bound. Miu-Miu's oscillations grow exponentially, which was verified when solving the problem numerically (see Figure (3) in the Appendix), we see that when she is in resonance within the range of 4 and $4 + \epsilon^2(5/48)$, the time needs to be increased greatly before there is any visible sign of growth and one could consider the growth as it is slow, to be linear. We needed 50000 time units to see any kind of growth, which for Miu-Miu means that if the units were seconds and she swung on the swing for roughly 13.8 hours she would experience growth. If she does not want to be captured into resonance she needs a $4 > \nu$ and $\nu > 4 + \epsilon^2(5/48)$. Therefore she is safe from fast growing exponentials and if she wants to be extra careful she needs to stay further than $\epsilon^2(5/48)$ from $\nu = 4$ .

## 2 Simulations

In order to find the solution to the differential equation (equation (1)) I used the fourth order Runge-Kutta Method. This took in an initial angle ($\theta_0 = 0.03$), an initial time ($T_0 = 0$), an initial velocity ($z = 0$) where $z = d\theta/d\tau$, an increment ($h = 0.01$) and the time it would go up to ($N = 50$) for finding $\epsilon_1(\nu)$ and $\epsilon_4(\nu)$ and up to $N = 10^4$ when plotting the solutions to the differential equation to tell whether the graph was oscillating or growing and outputs a graph of $\theta$ vs $\tau$. The reason for choosing a small angle was to maintain an accurate solution when using the Runge-Kutta Method as well as the small angle approximation at the beginning of the assignment. When plotting the solution curves, to be very accurate I went up to a large time in order to make sure that there was no growth, this provided useful in Miu-Miu's case as there were values of $\nu$ which produced very slow growth.The reason that I went up to 50 seconds for finding $\epsilon_1(\nu)$ and $\epsilon_4(\nu)$ was because I had a small increment (time step) of 0.05 which means that I have a more accurate picture.

### Max's Case

For this case I had a $\nu$-array of values and an $\epsilon$-array of values that I looped through. To find the condition that would consider an $\epsilon$ and a $\nu$ that produces exponential growth, I took different values of $\nu$ around the boundary condition found in the analytical section and found the maximum amplitude that was produced from one of the first non-oscillatory graphs. This was then the point at which if the maximum value of a graph was greater than this point, I appended the value of $\nu$ and $\epsilon$ into two empty arrays, one that stored $\nu$-values and the other that stored $\epsilon$-values. I then plotted these along with the analytical solution, that was graphed, to get Figure (1). The Table below shows different values of $\nu$ and their corresponding $\epsilon$ values:

| Number | $\nu$ | $\epsilon_1(\nu)$ |
|---|---|---|
| 1 | 0.8010204081632653 | 0.4742211055276382 |
| 2 | 0.9030612244897959 | 0.3394472361809045 |
| 3 | 0.9540816326530612 | 0.2945226130653266 |
| 4 | 1.0051020408163265 | 0.2720603015075377 |
| 5 | 1.0561224489795917 | 0.27954773869346733 |
| 6 | 1.1071428571428572 | 0.3244723618090452 |
| 7 | 1.2091836734693877 | 0.4742211055276382 |

It is seen here that the values seem parabolic, as Number 1's $\epsilon_1(\nu)$-value is the same as Number 7's $\epsilon_1(\nu)$-value, and it is verified by Figure (1).

### Miu-Miu's Case

Miu-Miu's case was handled in the same way as Max's case, however had a different boundary condition as it had a different maximum value of the first non-oscillatory graph. The values were stored in the two arrays as before and some of the points are outputted in the table below:

| Number | $\nu$ | $\epsilon_4(\nu)$ |
|---|---|---|
| 1 | 3.952828282828283 | 0.2661654135338346 |
| 2 | 3.9895959595959596 | 0.39323308270676693 |
| 3 | 3.9952525252525253 | 0.412781954887218 |
| 4 | 4.000909090909091 | 0.4518796992481203 |
| 5 | 4.006565656565657 | 0.48120300751879697 |
| 6 | 4.012222222222222 | 0.5105263157894737 |
| 7 | 4.051818181818182 | 0.7646616541353384 |

The values for $\epsilon_4(\nu)$ are increasing as $\nu$ increases. The increase appears to be linear, and when looking at Figure (2) one can see that it is almost linear.

## Graph of $\epsilon_1(\nu)$ vs $\nu$ for Max's Case

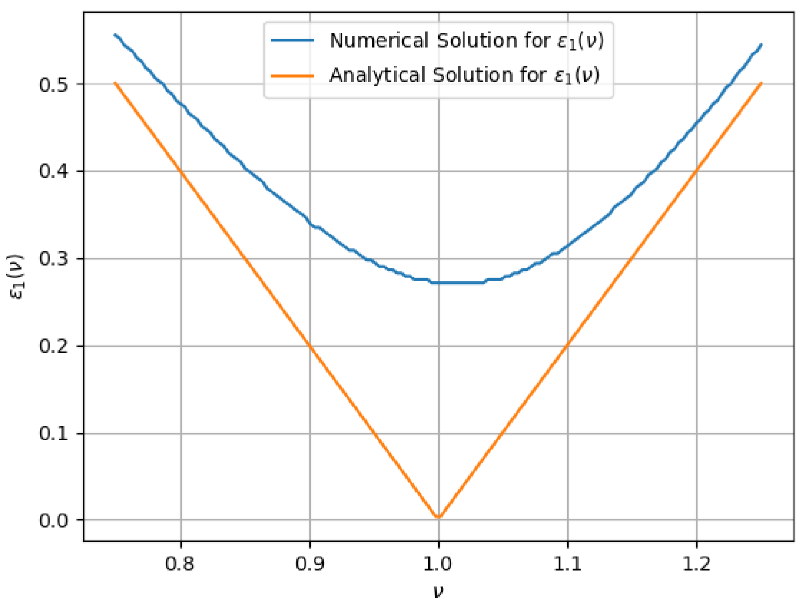


Figure 1: This graph shows the analytical and numerical solution for $\epsilon_1(\nu)$

As seen in Figure (1) the numerical solution almost matches the analytical solution.

## Graph of $\epsilon_4(\nu)$ vs $\nu$ for Miu-Miu's Case

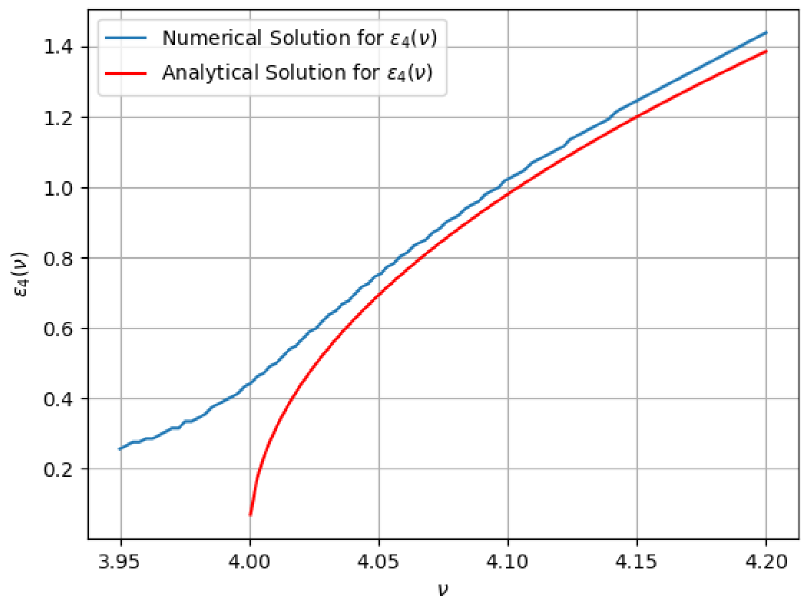


Figure 2: This graph shows the analytical and numerical solution for $\epsilon_4(\nu)$

As seen in Figure (2) the numerical solution almost matches the analytical solution, with some deviation of the numerical part, this could be rectified if higher powers of epsilon were considered.

## 3 Conclusion

In conclusion, Max got what he wanted as his growth was exponential with a fast increase, as seen in Figure (4) where his oscillations grow to 397, and Miu-Miu's growth was also exponential, however it was very slow to 0.034, and I needed to increase the time to 50000 time units in order to see this exponential growth, which means that for small times, it appears as if the swing is oscillating, however has slight growth. Therefore Miu-Miu is safe on the swing, whereas if Max is not careful, he could fly off.


## Acknowledgements

I would like to thank the student with the EmplId 1716621 for aiding me in searching for a way to determine an analytical solution for Miu-Miu's case in terms of different expansions of $\nu$ and how useful they are.

# References

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# Appendix

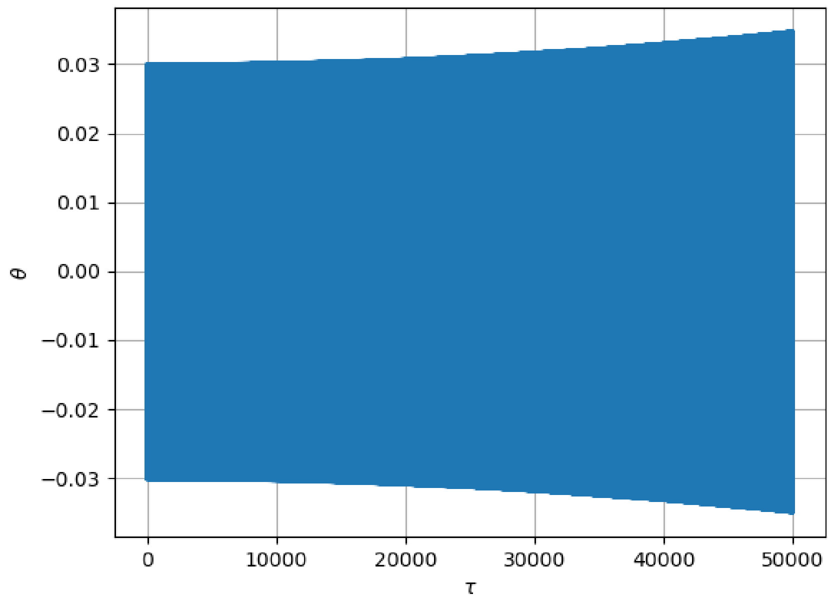


Figure 3: Graph of Miu-Miu's oscillations near 4.0002

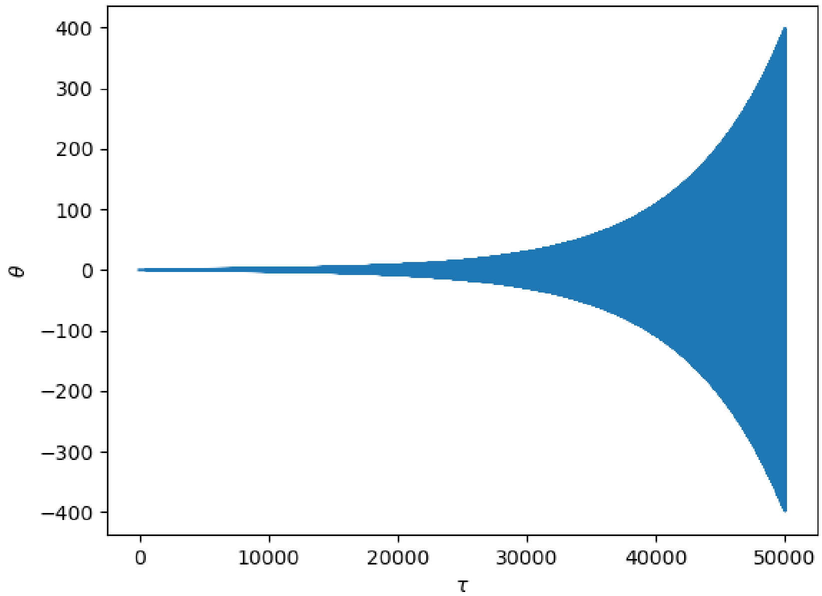


Figure 4: Graph of Max's oscillations near 1.005