Poisson Statistics Lab

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Abstract

When dealing with data, one needs to understand how to interpret it, how to analyze it, and one way to analyze data is to see if it follows a Poisson distribution. The data being analyzed in this case is the mean count rate of the decay of 60Co. Using 4 mean count rates, we will test and determine that the data for the most part follows a Poisson distribution and what those tests consist of.

Introduction

In physics, the processing of data is crucial to understanding whether it agrees with theory or not. In this case, the data is hypothesized to follow a Poisson distribution as most sets of data follow a Poisson distribution:

$$P(x;\mu) = \frac{\mu^x}{x!}e^{-\mu}$$

Where μ is the mean value of the data. In a Poisson distribution most of the data follows this curve[4].

Aim

To determine whether or not the decay of 60Co follows a Poisson distribution when varying the mean count rate as well as to determine whether background data follows a Poisson distribution using a Geiger counter.

Method

As the data taken is the decay of 60Co, and it decays releasing gamma rays, a Geiger counter was used to take down the counts per 10 seconds of the decay of 60Co. This data was put onto an excel sheet and uploaded to a group google drive. The mean count rates were 4 counts per 10 seconds, 10 counts per 10 seconds, 30 counts per 10 seconds and 100 counts per 10 seconds. Then the background was measured with a 10 second count rate as well. 100 trials of each count rate were done. The data collected for each count rate was from (https://drive.google.com/drive/folders/1p9HmzZ2x0GOKbEH_QAh85PQ5rHzD8DFm) and it was read into a python program using numpy's "genfromtxt" which was used to get the running mean. This was attained using equation (7) in Poisson Statistics manual[4]. Four graphs were outputted using matplotlib's plot, scatter and errorbar functions, one for each of the mean count rates, with the error bars being the uncertainty calculated with the root of the mean divided by the number of trials[4]. Then the sample variance of the data was calculated using equation (8) in the Poisson statistics manual[4]. Then in order to determine the extent that the sample variance agrees with the arithmetic mean[4], the variance over the mean was plotted against the mean and the error bars were the Pythagorean addition of the uncertainty for the variance and the mean. The uncertainty of the variance was given by the square root of equation (9) in the Poisson statistics manual[4]. The square root was taken as it is using the same logic as when calculating the mean. Using the four files once more, Poisson plots were created using the same scatter, plot and errorbar functions as before. The plot was of the Poisson distribution which used the end mean values from the plots of the arithmetic mean as the μ value. The data was added at the end points if they were below the count per trial of 5 and their bins were deleted. In order to make sure that the Poisson spanned the data correctly, the Poisson was plotted against a histogram with 10 bins, where the density was set to true, and another set to false. Then a scaling coefficient was found by dividing a value in the histogram with density set to false by the same value where the density was set to false. This was then used to scale the Poisson. In order to get the tails of the Poisson the scipy function stats.poisson.cdf (cumulative distribution function) to determine the end values for the Poisson. This part of the graph went from the initial point to the next point in the histogram and the same with the second last and last values. When counting the number of bins that lie within the Poisson distribution in order to plot the mean values and whether it produced a Poisson plot, whenever an error bar or point intersected with the Poisson distribution, then it was counted. This was then divided by the total number of points. In order to determine the error bars equation (12) in the Poisson statistics manual[4] was used. This was then divided by the total number of bins to get the uncertainty per bin. This was then plotted using the scatter and errorbar functions and a plot of 68% was plotted in order to determine how many of the trials had bins within 68%.

Proofs

Prove $E[\hat{\mu}] = \mu$:

For a probability distribution the expectation value is:

$$E[f(x)] \equiv \sum_{i=1}^{N} f(x_i)P(x_i)$$

Where $P(x)$ is the probability function. And in this case we have an assumed Poisson distribution, therefore $\hat{\mu} = \bar{x}$. We have that

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i \Rightarrow \hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

$$\Rightarrow E[\hat{\mu}] = E\left[\frac{1}{N}\sum_{i=1}^{N} x_i\right] = \frac{1}{N}\sum_{i=1}^{N} E[x_i]$$

And from the Poisson Lab prep questions addendum and from Cowan[1]:

$$E[x] = \sum_{i=1}^{\infty} x_i P(x_i;\mu) = \sum_{i=1}^{\infty} x_i \frac{\mu^x e^{-\mu}}{x_i!} = \mu$$

$$\Rightarrow E[\hat{\mu}] = \frac{1}{N}\sum_{i=1}^{N} \mu$$

$$\therefore E[\hat{\mu}] = \frac{1}{N} N\mu$$

$$\therefore E[\hat{\mu}] = \mu$$

Prove $E[(\hat{\mu}-\mu)^2] = E[(\bar{x}-\mu)^2] = \frac{\mu}{N} \cong \frac{\bar{x}}{N}$:

Starting with:

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

$$\Rightarrow V[\hat{\mu}] = E\left[\left(\frac{1}{N}\sum_{i=1}^{N} x_i - \mu\right)^2\right]$$

$$\therefore V[\hat{\mu}] = E\left[\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j - \frac{2\mu}{N}\sum_{i=1}^{N} x_i + \mu^2\right] \qquad [1]$$

$$\therefore V[\hat{\mu}] = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} E[x_i x_j] - \frac{2\mu}{N}\sum_{i=1}^{N} E[x_i] + E[\mu^2]$$

Aside:

For $i = j$:

$E[x_i x_j] = E[x^2]$, from the Poisson Lab prep questions addendum, $E[x^2] = \mu(\mu+1)$

For $i \neq j$:

$$E[x_i x_j] = E[x_i]E[x_j] = \mu\mu = \mu^2$$

The Kronecker Delta is now introduced[1]:

$$\delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$$

$$\therefore V[\hat{\mu}] = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} E[x_i]E[x_j] - \frac{2\mu}{N}\sum_{i=1}^{N} E[x_i] + E[\mu^2]$$

$$\therefore V[\hat{\mu}] = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\mu(\mu+1)\delta_{ij} + (1-\delta_{ij})\mu^2\right) - \frac{2\mu}{N}\sum_{i=1}^{N}\mu + \mu^2$$

$$\therefore V[\hat{\mu}] = \frac{1}{N^2}\left(\sum_{i=1}^{N}\sum_{j=1}^{N}\left(\mu(\mu+1)\delta_{ij}\right) + \sum_{i=1}^{N}\sum_{j=1}^{N}\left((1-\delta_{ij})\mu^2\right)\right) - \frac{2\mu}{N}\sum_{i=1}^{N}\mu + \mu^2$$

$$\therefore V[\hat{\mu}] = \frac{1}{N^2}\left(N\left(\mu(\mu+1)\right) + N^2\mu^2 - N\mu^2\right) - 2\mu^2 + \mu^2$$

$$\therefore V[\hat{\mu}] = \frac{\left(\mu(\mu+1)\right)}{N} + \mu^2 - \frac{\mu^2}{N} - 2\mu^2 + \mu^2$$

$$\therefore V[\hat{\mu}] = \frac{\mu}{N}$$

And $\mu \cong \hat{\mu} = \bar{x}$

$$\therefore V[\hat{\mu}] \cong \frac{\bar{x}}{N}$$

$$\therefore E[(\bar{x}-\mu)^2] \cong \frac{\bar{x}}{N}$$

Prove $E[(s^2-\mu)^2]=\frac{2N\mu^2+(N-1)\mu}{N(N-1)}$ using $V[s^2]=\frac{1}{N}\left(m_4-\frac{N-3}{N-1}\,\sigma^4\right)=E[(s^2-\mu)^2]$

First the fourth central moment (m_4) can be rewritten in terms of μ as $m_4=\mu(1+3\mu)$ [3] and as the distribution is Poisson $\sigma^2=\mu$:

$$\therefore V[s^2]=\frac{1}{N}\left(\left(\mu(1+3\mu)\right)-\frac{N-3}{N-1}\,\mu^2\right)$$

$$\therefore V[s^2]=\frac{1}{N}\left(\frac{\mu(1+3\mu)(N-1)-(N-3)\mu^2}{N-1}\right)$$

$$\therefore V[s^2]=\frac{(\mu+3\mu^2)(N-1)-N\mu^2+3\mu^2}{N(N-1)}$$

$$\therefore V[s^2]=\frac{N\mu-\mu+3\mu^2N-3\mu^2-N\mu^2+3\mu^2}{N(N-1)}$$

$$\therefore V[s^2]=\frac{N\mu-\mu+2\mu^2N}{N(N-1)}$$

$$\therefore V[s^2]=\frac{2N\mu^2+(N-1)\mu}{N(N-1)}$$

$$\therefore E[(s^2-\mu)^2]=\frac{2N\mu^2+(N-1)\mu}{N(N-1)}$$

Data:

Running Mean Data:

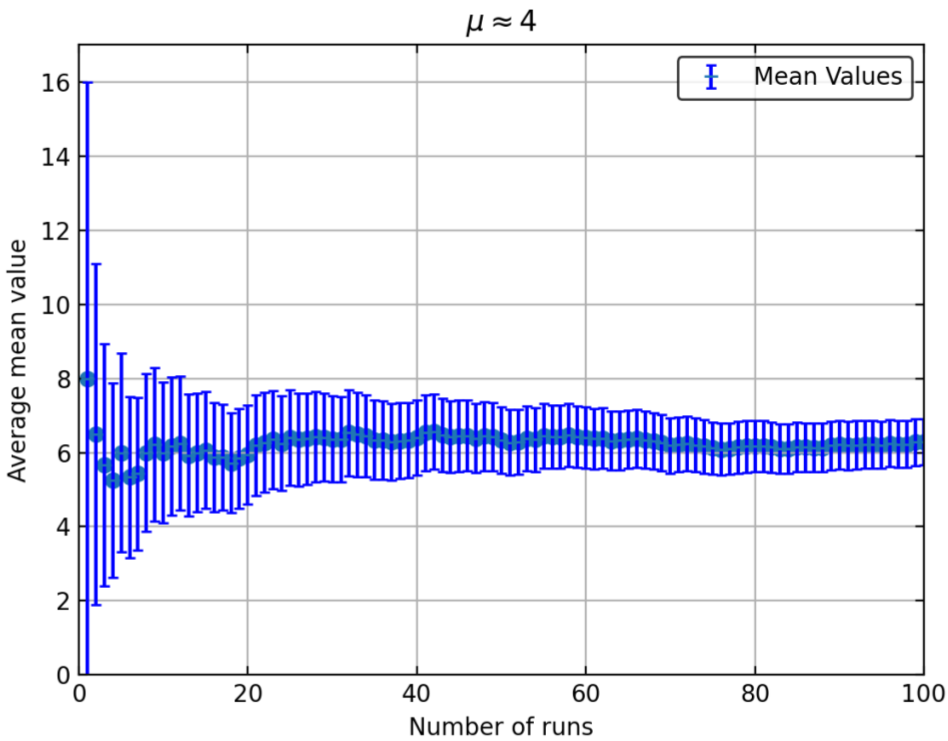

Figure 1. This graph shows the arithmetic mean and how it decays to a fixed value over many trials for $\mu \approx 4$

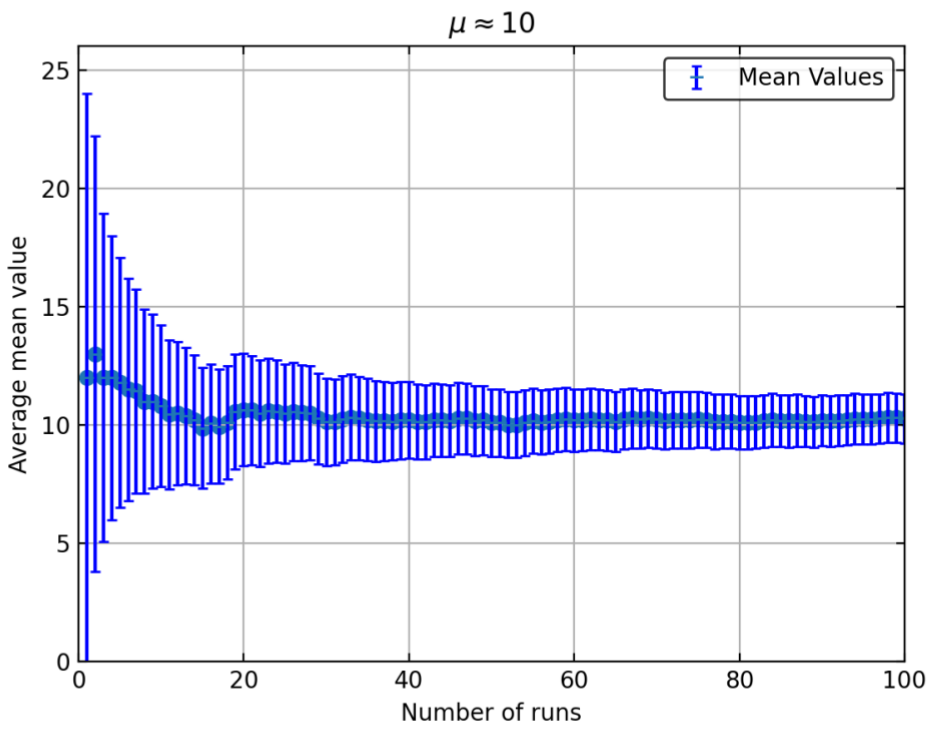

Figure 2. This graph shows the arithmetic mean and how it decays to a fixed value over many trials for $\mu \approx 10$

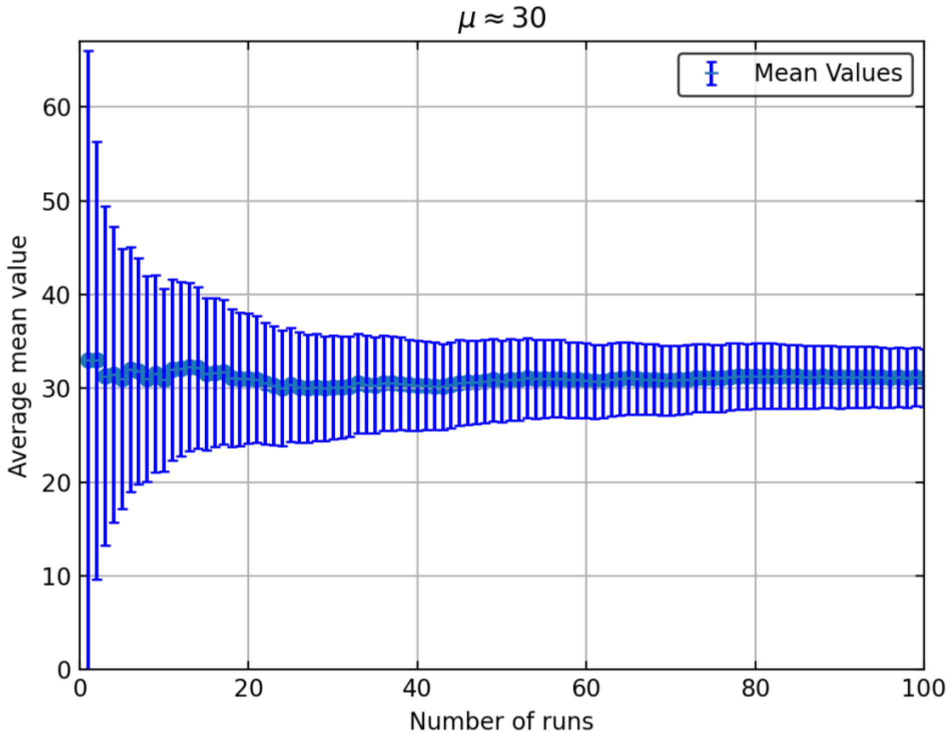

Figure 3. This graph shows the arithmetic mean and how it decays to a fixed value over many trials for $\mu \approx 30$

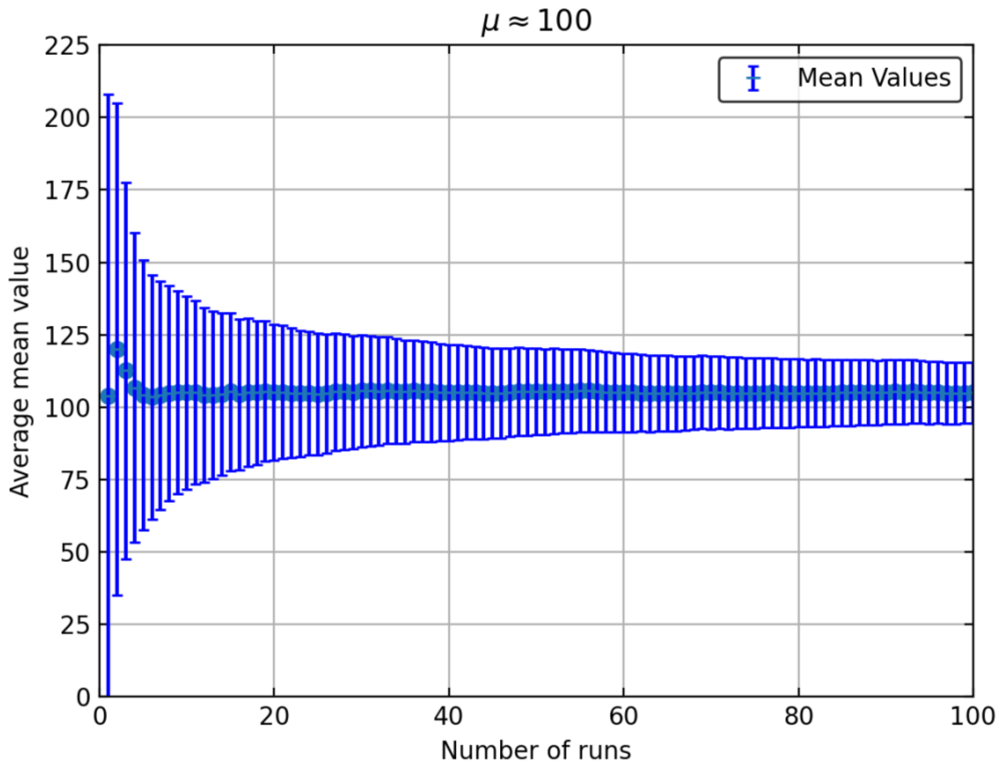

Figure 4. This graph shows the arithmetic mean and how it decays to a fixed value over many trials for $\mu \approx 100$

Arithmetic Mean and Sample Variance of the Data:

Table 1. This table shows the mean, variance and variance over mean as well as their uncertainties

$\bar{x}$	$\Delta\bar{x}$	s^2	Δs^2	$s^2/\bar{x}$	$\Delta s^2/\bar{x}$
6.29	0.25	6.78	0.92	1.08	0.15
10.27	0.32	8.26	1.49	0.80	0.15
31.13	0.56	27.27	4.46	0.88	0.14
105.02	1.02	91.11	14.96	0.87	0.14

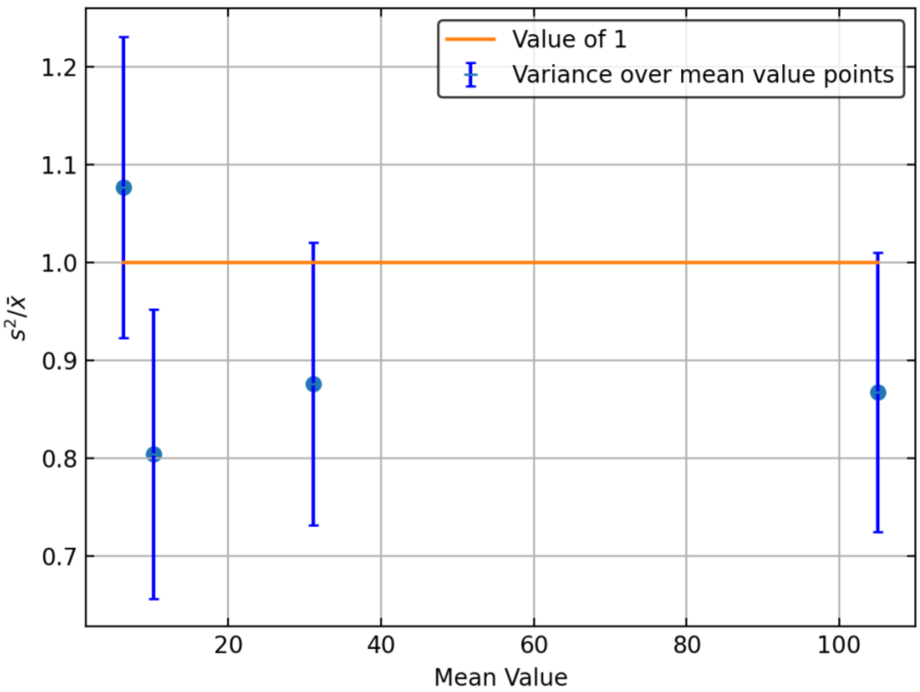

Figure 5. This graph shows the variance over the mean plotted against the mean

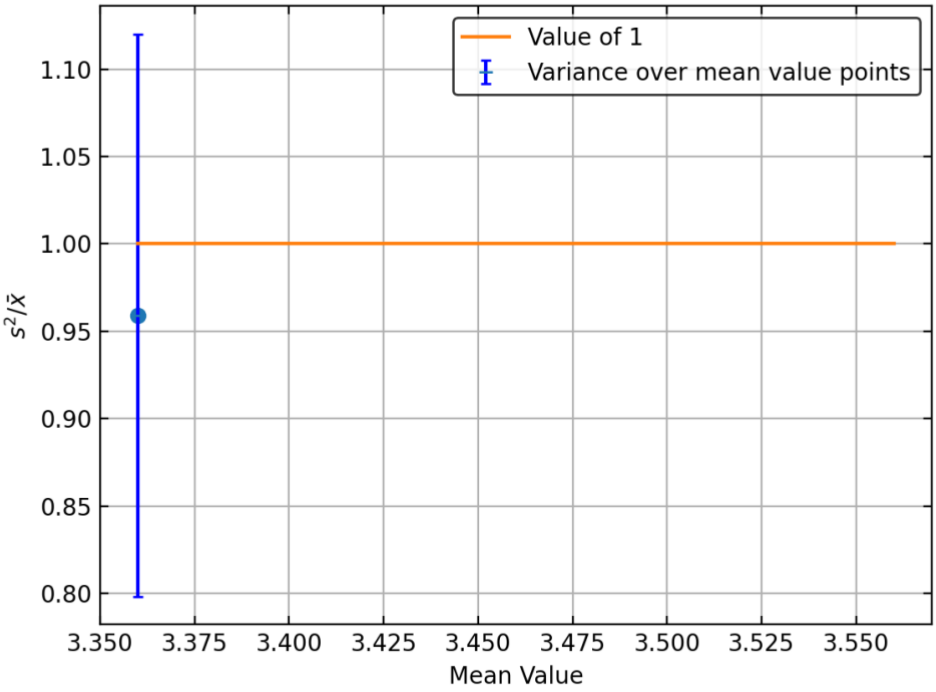

Figure 6. This graph shows the variance over the mean plotted against the mean of the background data

Poisson Plots:

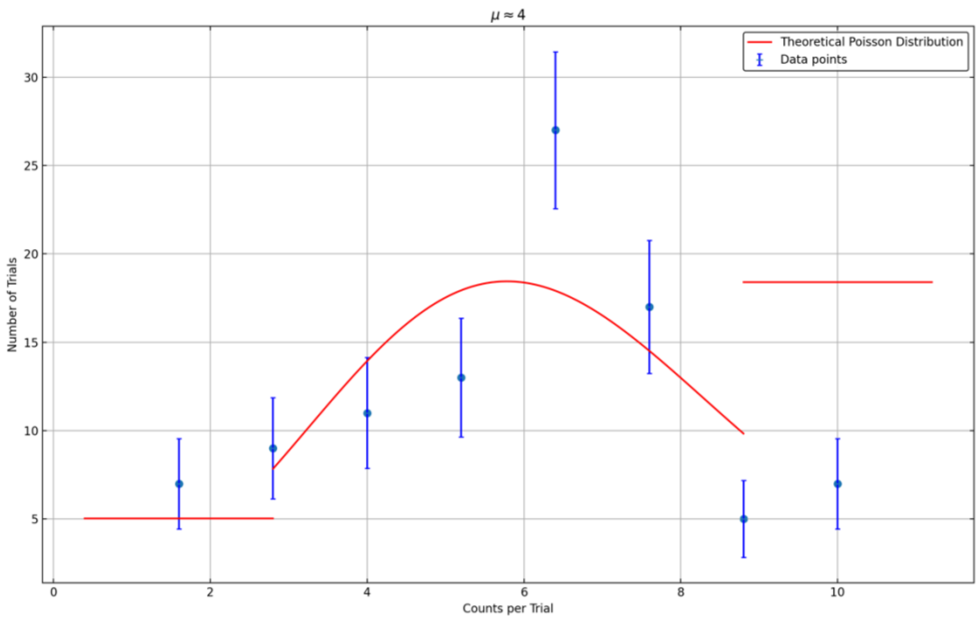

Figure 7. This graph shows the theoretical Poisson distribution vs the bins of the data points collected for $\mu \approx 4$

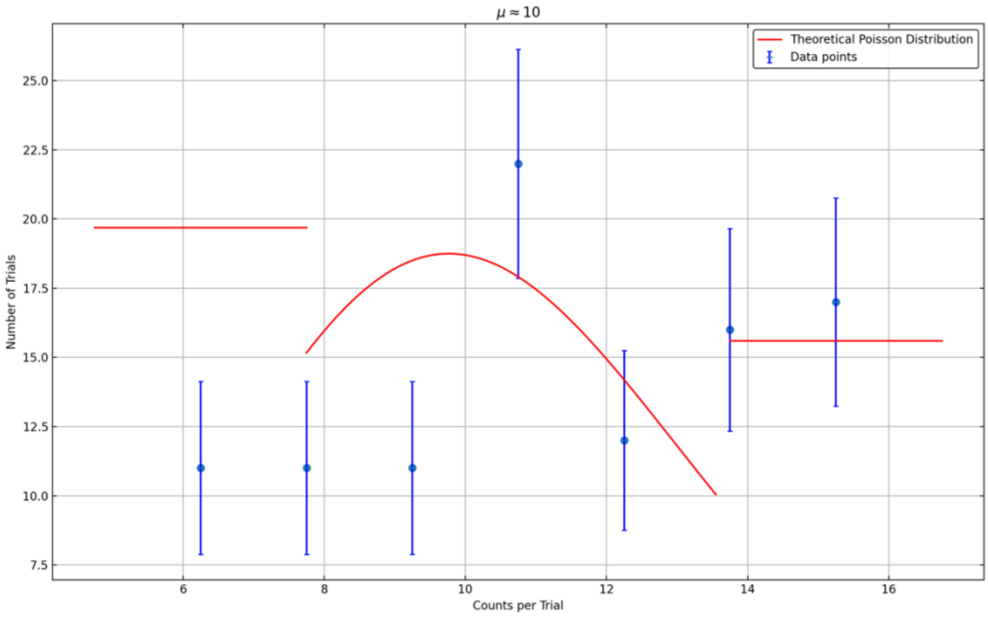

Figure 8. This graph shows the theoretical Poisson distribution vs the bins of the data points collected for $\mu \approx 10$

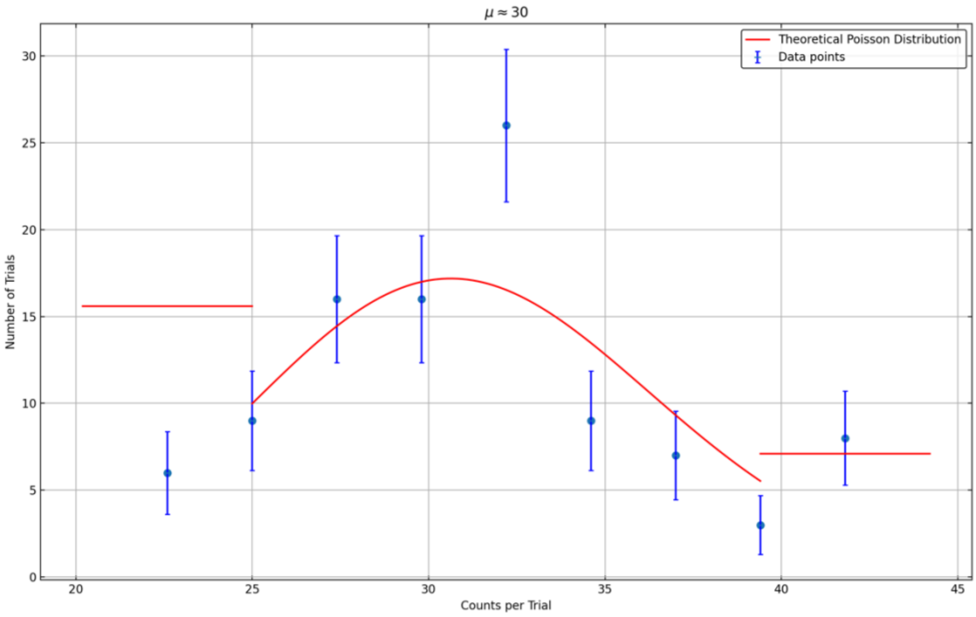

Figure 9. This graph shows the theoretical Poisson distribution vs the bins of the data points collected for $\mu \approx 30$

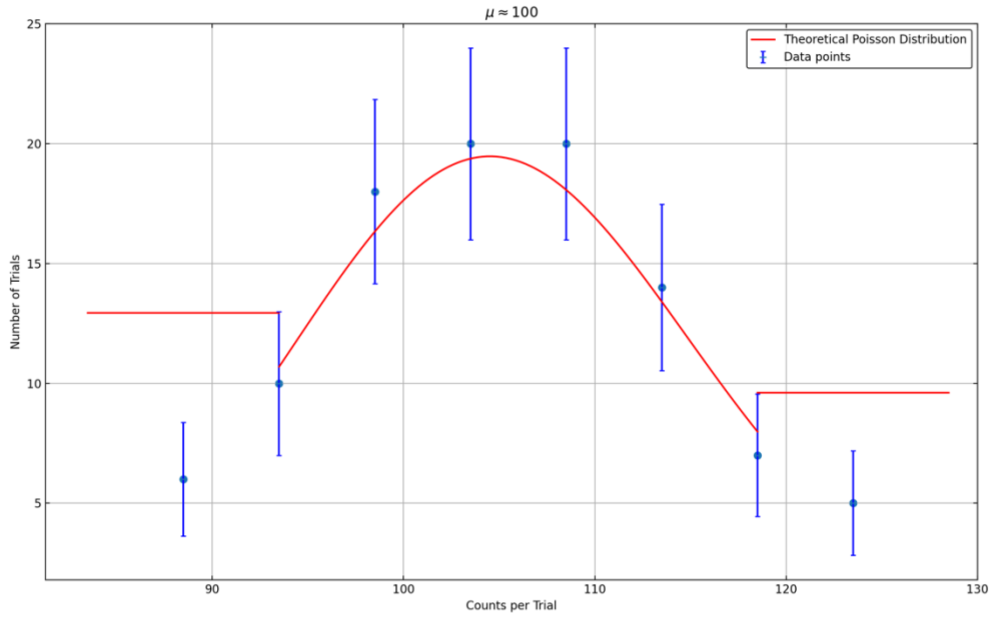

Figure 11. This graph shows the theoretical Poisson distribution vs the bins of the data points collected for $\mu \approx 100$

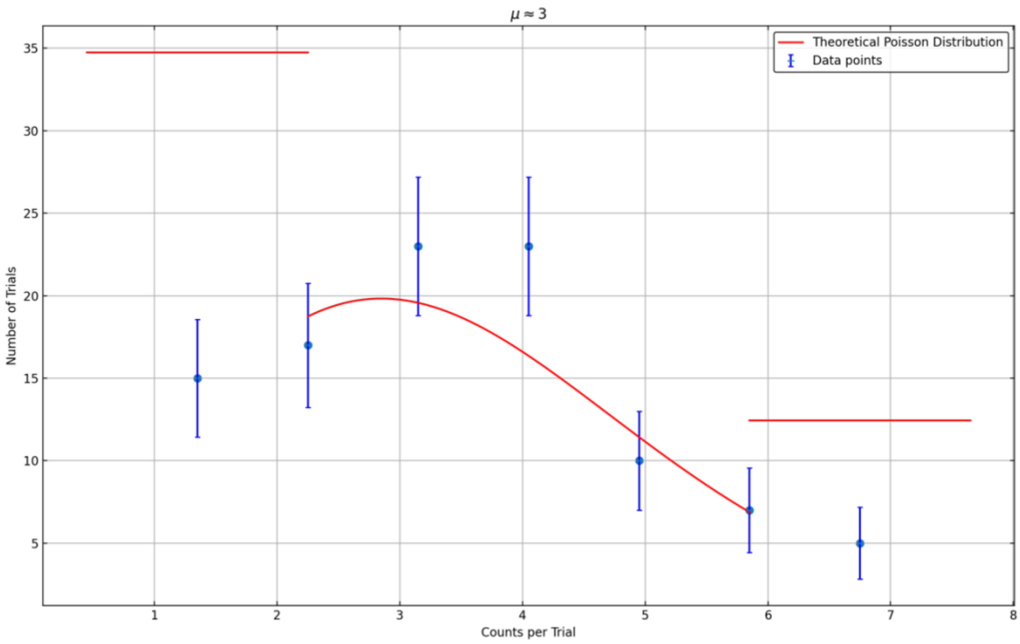

Figure 10. This graph shows the theoretical Poisson distribution vs the bins of the data points collected for $\mu \approx 3$ *for the background data*

Fraction of bins intersecting the Poisson:

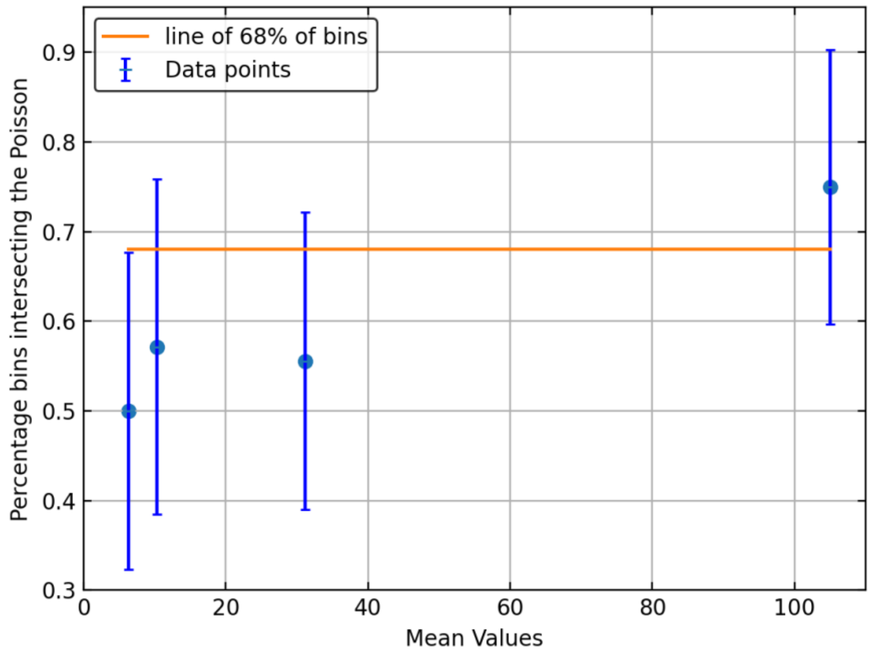

Figure 13. This graph shows the fraction of bins that intersect the Poisson

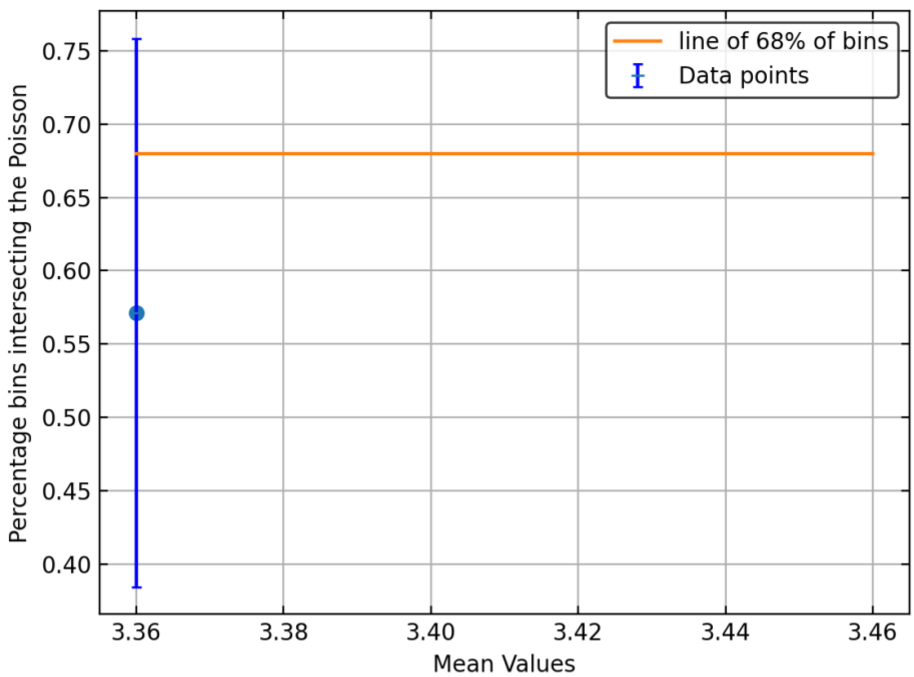

Figure 12. This graph shows the fraction of bins that intersect the Poisson for the background data

Calculations:

Probability that any nucleus will decay during one trial:

The half-life of 60Co is 5.27 years [5]. Using the equation to find the probability that a nucleus will decay[6]:

$$P = 1 - (1 - p)^n$$

Where p is the probability of decay in a given period, and n is the number of periods.
To calculate n:

$$n = \frac{10}{5.27(365)(24)(60)(60)}$$

The probability of decay in a given period is $p = 0.5$
Therefore the probability of decaying in a single trial is:

$$P = 1 - (1 - 0.5)^{\frac{1}{5.27(365)(24)(60)(6)}} = 4.17 \times 10^{-8}$$

Therefore it is seen that one can make the approximation that for the binomial distribution as p which in this case is P, is very small and N the number of trials is big, that $\mu = pN$.

Discussion

The Running Mean:

It is seen in Figure 1-Figure 4, when using the running mean a better estimate for the mean is attained as it is seen that in the first run, the uncertainty, given by the error bars are very large, and the last value, at the "x"-value of 100, the uncertainty is very small, therefore these values for the mean are good estimates for the value of μ.

Arithmetic Mean and Sample Variance:

It is seen in Figure 5 that for most of the values, that their uncertainties put the value of $s^2/\bar{x}$ at 1. This is because we want the variance to equal to the mean to show that it is Poisson, therefore if $s^2/\bar{x} = 1$ we actually have that $s^2 = \bar{x}$. The value of the mean equal to 10.27, is seen to disagree with this, and lies a bit away from the value. This means that it is suspected that the data does not follow a Poisson distribution. And the values that possibly follow a Poisson distribution are the means equation are for the mean equal to 6.29, 31.13 and 105.02. In Figure 6 it is seen when using the background data, that the value $s^2/\bar{x}$ intersects with the line $s^2/\bar{x} = 1$, which means that the background data most likely follows a Poisson distribution. For this fact the background data does not need to be subtracted from the actual data collected as it would not affect the overall shape of the data.

Plotting the Poisson

When plotting the actual data against a theoretical Poisson distribution using the mean from The Running Mean: it can be better seen which sets of data follows a Poisson distribution and which do not. It is seen that in Figure 7 the data does not seem to follow a Poisson distribution due to the number of error bars that intersect with the Poisson distribution as it is 4/8 bins that intersect which is 50% of them that intersect, the required amount is 68% in order for it to be considered Poissonian, however the uncertainties have not been considered, therefore one cannot say yet. Figure 8, Figure 9 and Figure 11 all have bins that intersect with the Poisson at a percentage higher than 50%. Figure 10 shows the background data and it is seen that the number of bins that intersect here are also higher than 50%. Looking at the next plots more closely can better determine whether the data follows a Poisson distribution.

Fraction of Bins described by a Poisson:

It is seen in Figure 13 that all of the means that has data that follows a Poisson distribution has an uncertainty that intersects with the line of 68%. Specifically the means of 10.27, 31.13 and 105.02 all have data that follows a Poisson distribution and the mean of 6.29 does not follow a Poisson distribution as was discussed in the previous section. It is seen that the background data given by Figure 12 follows a Poisson Distribution. Therefore the statement made in Arithmetic Mean and Sample Variance: can be stated with more surety that the background data follows a Poisson distribution and therefore would not affect the overall data.

Considering both tests:

When considering both tests, it is seen that the mean values of 6.29 and 10.27 are the only that have conflicting interpretations. From the arithmetic mean test it is seen that for mean 6.29 the data is Poisson distributed and the mean of 10.27 is not, and when looking at the fraction of bins test it is seen that the mean of 6.29 does not follow a Poisson distribution and the mean of 10.27 does. Therefore these could be Poissonian however it cannot be as conclusively stated as that of means of 31.13 and 105.02.

Conclusion

In conclusion based on the two tests discussed above the data for the mean of 6.29 and 10.27 are almost described by a Poisson distribution and the means of 31.13 and 105.02 are described by a Poisson distribution, therefore the data from a decaying radioactive nucleus of 60Co follows a Poisson distribution. It is also concluded that even data from outside sources, are Poisson distributed.

Bibliography

[1] Cowan, G., 1998. *Chapter 5.* [Online]
Available at: https://vula.uct.ac.za/access/content/group/275677d7-258c-4507-b368-7cb674c91ca0/Laboratory/Lab%203%20Poisson%20statistics/Cowan/CowanCh5.pdf

[2] jbstatistics, 2014. *Proof that the Sample Variance is an Unbiased Estimator of the Population Variance.* [Online]
Available at: https://youtu.be/D1hgiAla3KI

[3] Weisstein, E., 2021. Poisson Distribution -- from Wolfram MathWorld. [online] Mathworld.wolfram.com. Available at: <https://mathworld.wolfram.com/PoissonDistribution.html> [Accessed 28 May 2021].

[4] Horowitz, W., 2021. *Poisson Statistics*. [ebook] Cape Town: UCT, pp.1-7. Available at: <https://vula.uct.ac.za/access/content/group/275677d7-258c-4507-b368-7cb674c91ca0/Laboratory/Lab%203%20Poisson%20statistics/Poisson.pdf> [Accessed 28 May 2021].

[5] En.wikipedia.org. 2021. *Cobalt-60 - Wikipedia*. [online] Available at: <https://en.wikipedia.org/wiki/Cobalt-60> [Accessed 28 May 2021].

[6] Physics Stack Exchange. 2021. Calculate the probability that a radioactive nucleus will have decayed after the passage of three half-lives. [online] Available at: <https://physics.stackexchange.com/questions/229632/calculate-the-probability-that-a-radioactive-nucleus-will-have-decayed-after-the#:~:text=3%20Answers&text=It%20means%20that%20after%20every,particular%20nucleus%20will%20have%20decayed.> [Accessed 29 May 2021].

Poisson Lab Prep Questions
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2.1) A Geiger counter consists of a Geiger-Müller tube and a counter. Inside the tube there is a gas that when hit by a high energy photon ionises the gas. In the middle of the tube there is a positively charged rod, and a negatively charged outer wall. When the gas is ionised the positive nuclei go towards the outer wall, and the electrons cause other electrons to be ionised, this process is called a Townsend avalanche and the electrons flow towards the positive rod which then causes a current which then causes the counter to go off and the speaker to make a crackling sound.[1] When the positive nuclei hit the walls they gain electrons and the gas becomes neutral once more until the process is repeated by another photon.

2) Starting with the Binomial Distribution:

$$P(x; p, z) \equiv \frac{z!}{x!(z-x)!} p^x (1-p)^{z-x}$$

As $pz \equiv \mu$, $\frac{\mu}{z} = p$:

$$\therefore P(x; p, z) = \frac{z!}{x!(z-x)!} \left(\frac{\mu}{z}\right)^x \left(1 - \cancel{p}\left(\frac{\mu}{z}\right)\right)^{z-x}$$

With some rearranging we get:

$$\therefore \cancel{P(x,} \frac{z!}{x!(z-x)!} \left(\frac{\mu}{z}\right)^x \left(1 - \left(\frac{\mu}{z}\right)\right)^{z-x}$$

$$= \frac{\mu^x}{x!} \frac{z!}{(z-x)!} \left(\frac{1}{z}\right)^x \left(1 - \frac{\mu}{z}\right)^z \left(1 - \frac{\mu}{z}\right)^{-x}$$

Now expanding $z!$:

$$z! = z(z-1)(z-2)\cdots(z-x+1)(z-x)!$$

$$\therefore \frac{z!}{(z-x)!} = \frac{z(z-1)(z-2)\cdots(z-x+1)\cancel{(z-x)!}}{\cancel{(z-x)!}} \quad [2]$$

$$= z(z-1)(z-2)\cdots(z-x+1) \quad \ldots ①$$

$$\therefore P(x; \mu, z) = \frac{\mu^x}{x!}\left[z(z-1)(z-2)\cdots(z-x+1)\right]\frac{1}{z^x}\left(1-\frac{\mu}{z}\right)^z\left(1-\frac{\mu}{z}\right)^{-x}$$

It is seen that in fact there are x terms in ①, therefore we can split up the middle:

$$\frac{z(z-1)(z-2)\cdots(z-x+1)}{z^x} = \left(\frac{z}{z}\right)\left(\frac{z-1}{z}\right)\left(\frac{z-2}{z}\right)\cdots\left(\frac{z-x+1}{z}\right)$$

$$= 1\left(1-\frac{1}{z}\right)\left(1-\frac{2}{z}\right)\cdots\left(1-\frac{x+1}{z}\right)$$

As z is very large, to divide by z ~~its~~ makes the term approximately zero, similar to taking the limit as $z \to \infty$:

$$1\left(1-\frac{1}{z}\right)\left(1-\frac{2}{z}\right)\cdots\left(1-\frac{x+1}{z}\right) \approx 1(1-0)(1-0)\cdots(1-0) = 1 \quad [3]$$

$$\therefore P(x;\mu,z) \approx \frac{\mu^x}{x!}\left(1-\frac{\mu}{z}\right)^z\left(1-\frac{\mu}{z}\right)^{-x}$$

For the last term in the expression for P, we use the same argument as before, namely:

$$\left(1-\frac{\mu}{z}\right)^{-x} \approx (1-0)^{-x} = 1^{-x} = 1$$

$$\therefore P(x;\mu,z) \approx \frac{\mu^x}{x!}\left(1-\frac{\mu}{z}\right)^z$$

Using a very useful identity: $e^x = \lim_{z\to\infty}\left(1+\frac{x}{z}\right)^z$ [2]

In our case we can approximate $e^{-\mu}$ as: $e^{-\mu} \approx \left(1-\frac{\mu}{z}\right)^z$

$$\therefore P(x;\mu) \approx \frac{\mu^x}{x!}e^{-\mu}$$

which is the Poisson distribution.

a) $$\langle x \rangle = \sum_{x=1}^{\infty} x P$$

$$= \sum_{x=1}^{\infty} x\left(\frac{\mu^x}{x!}e^{-\mu}\right)$$

$$= e^{-\mu}\sum_{x=1}^{\infty}\frac{\mu^x}{(x-1)!}$$

$$\therefore \langle x \rangle = e^{-\mu} \sum_{x=1}^{\infty} \frac{\mu \mu^{x-1}}{(x-1)!}$$

$$= \mu e^{-\mu} \sum_{x=1}^{\infty} \frac{\mu^{x-1}}{(x-1)!}$$

Expanding $\sum_{x=1}^{\infty} \frac{\mu^{x-1}}{(x-1)!}$:

$$\sum_{x=1}^{\infty} \frac{\mu^{x-1}}{(x-1)!} = 1 + \mu + \frac{\mu^2}{2!} + \frac{\mu^3}{3!} + \frac{\mu^4}{4!} + \dots$$

It is seen that this is the Taylor Series of e^{μ}

$$\therefore \langle x \rangle = \mu e^{-\mu} e^{\mu}$$

$$\therefore \langle x \rangle = \mu$$

b) $\langle x^2 \rangle = \sum_{x=0}^{\infty} x^2 P$, however as the first term is zero, we can start with $x = 1$

$$\langle x^2 \rangle = \sum_{x=1}^{\infty} x^2 P$$

$$= \sum_{x=1}^{\infty} x^2 \frac{\mu^x}{x!} e^{-\mu}$$

$$= e^{-\mu} \sum_{x=1}^{\infty} x \frac{\mu^x}{(x-1)!}$$

$$= e^{-\mu} \sum_{x=1}^{\infty} x \frac{\mu \mu^{x-1}}{(x-1)!}$$

$$= \mu e^{-\mu} \sum_{x=1}^{\infty} x \frac{\mu^{x-1}}{(x-1)!}$$

$$\therefore \langle x^2 \rangle = \mu e^{-\mu} \sum_{x=1}^{\infty} x \frac{\mu^{x-1}}{(x-1)!}$$

Let $n = x-1$:

$$\therefore \langle x^2 \rangle = \mu e^{-\mu} \sum_{n=0}^{\infty} (n+1) \frac{\mu^n}{n!}$$

$$= \mu e^{-\mu} \left[\sum_{n=0}^{\infty} n \frac{\mu^n}{n!} + \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \right]$$

As the first term $\sum_{n=0}^{\infty} n \frac{\mu^n}{n!}$ is zero, we can start with $n=1$:

$$\therefore \langle x^2 \rangle = \mu e^{-\mu} \left[\sum_{n=1}^{\infty} n \frac{\mu^n}{n!} + \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \right]$$

$$= \mu e^{-\mu} \left[\sum_{n=1}^{\infty} \frac{\mu^n}{(n-1)!} + \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \right]$$

$$= \mu e^{-\mu} \left[\sum_{n=1}^{\infty} \frac{\mu \mu^{n-1}}{(n-1)!} + \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \right]$$

$$= \mu e^{-\mu} \left[\mu \sum_{n=1}^{\infty} \frac{\mu^{n-1}}{(n-1)!} + \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \right]$$

And it is seen that $\sum_{n=1}^{\infty} \frac{\mu^{n-1}}{(n-1)!} = e^{\mu}$ and $\sum_{n=0}^{\infty} \frac{\mu^n}{n!} = e^{\mu}$

$$\therefore \langle x^2 \rangle = \mu e^{-\mu} \left[\mu e^{\mu} + e^{\mu} \right]$$

$$= \mu^2 + \mu$$

$$= \mu(\mu+1)$$

c) $\langle (x-\mu)^2 \rangle = \sum_{x=0}^{\infty} (x-\mu)^2 P$

$$\therefore \quad \langle (x-\mu)^2 \rangle = \sum_{x=0}^{\infty} (x-\mu)(x-\mu) \frac{\mu^x}{x!} e^{-\mu}$$

$$= \sum_{x=0}^{\infty} (x^2 - 2\mu x + \mu^2) \frac{\mu^x}{x!} e^{-\mu}$$

$$= \sum_{x=0}^{\infty} x^2 \frac{\mu^x}{x!} e^{-\mu} - 2\mu \sum_{x=0}^{\infty} x \frac{\mu^x}{x!} e^{-\mu} + \mu^2 \sum_{x=0}^{\infty} \frac{\mu^x}{x!} e^{-\mu}$$

$$= \sum_{x=1}^{\infty} x^2 \frac{\mu^x}{x!} e^{-\mu} - 2\mu \sum_{x=1}^{\infty} x \frac{\mu^x}{x!} e^{-\mu} + \mu^2 \sum_{x=0}^{\infty} \frac{\mu^x}{x!} e^{-\mu}$$

We have seen the solution to all 3 summations, therefore:

$$\langle (x-\mu)^2 \rangle = [\mu(\mu+1)] - 2\mu[\{\mu\}] + \mu^2[1]$$

$$= \mu^2 + \mu - 2\mu^2 + \mu^2$$

$$= \mu$$

4) $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} \, dx$

$$\Gamma(1) = \int_0^\infty x^0 e^{-x} \, dx$$

$$= \int_0^\infty e^{-x} \, dx$$

$$= \left[-e^{-x}\right]_0^\infty$$

$$= \left[-e^{-\infty}\right] - \left[-e^{-0}\right]$$

$$= 1$$

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} \, dx$$

Using IBP: $u = x^{n-1}$ $\quad dv = e^{-x} dx$

$du = (n-1) x^{n-2} dx$ $\quad v = -e^{-x}$

$$\therefore \int_0^\infty x^{n-1} e^{-x} \, dx = \left[-x^{n-1} e^{-x}\right]_0^\infty - \int_0^\infty (n-1) x^{n-2} (-e^{-x}) \, dx$$

$$= \left[-x^{n-1} e^{-x}\right]_0^\infty + \int_0^\infty (n-1) x^{n-2} e^{-x} \, dx$$

$$= \lim_{k \to \infty} \left[-x^{n-1} e^{-x}\right]_0^k + \int_0^\infty (n-1) x^{n-2} e^{-x} \, dx$$

$$= \lim_{k \to \infty} \left(-k^{n-1} e^{-k}\right) + \int_0^\infty (n-1) x^{n-2} e^{-x} \, dx$$

$$= 0 + (n-1) \int_0^\infty x^{n-2} e^{-x} \, dx$$

$$\therefore \Gamma(n) = (n-1) \int_0^\infty x^{n-2} e^{-x} \, dx$$

$$\Gamma(n-1) = \int_0^\infty x^{(n-1)-1} e^{-x} \, dx$$

$$= \int_0^\infty x^{n-2} e^{-x} \, dx$$

Using IBP:

$$\int_0^\infty x^{n-2} e^{-x} \, dx = \left[-x^{n-2} e^{-x} \right]_0^\infty - \int_0^\infty -(n-2) x^{n-3} e^{-x} \, dx$$

$$= 0 + \int_0 (n-2) x^{n-3} e^{-x} \, dx$$

~~It~~ $$= (n-2) \int_0^\infty x^{n-3} e^{-x} \, dx$$

It is seen that $\Gamma(n) = (n-1) \int_0 x^{n-2} e^{-x} \, dx$

$$= (n-1) \Gamma(n-1)$$

And furthermore $\Gamma(n-1) = (n-2) \int_0^\infty x^{n-3} e^{-x} \, dx$

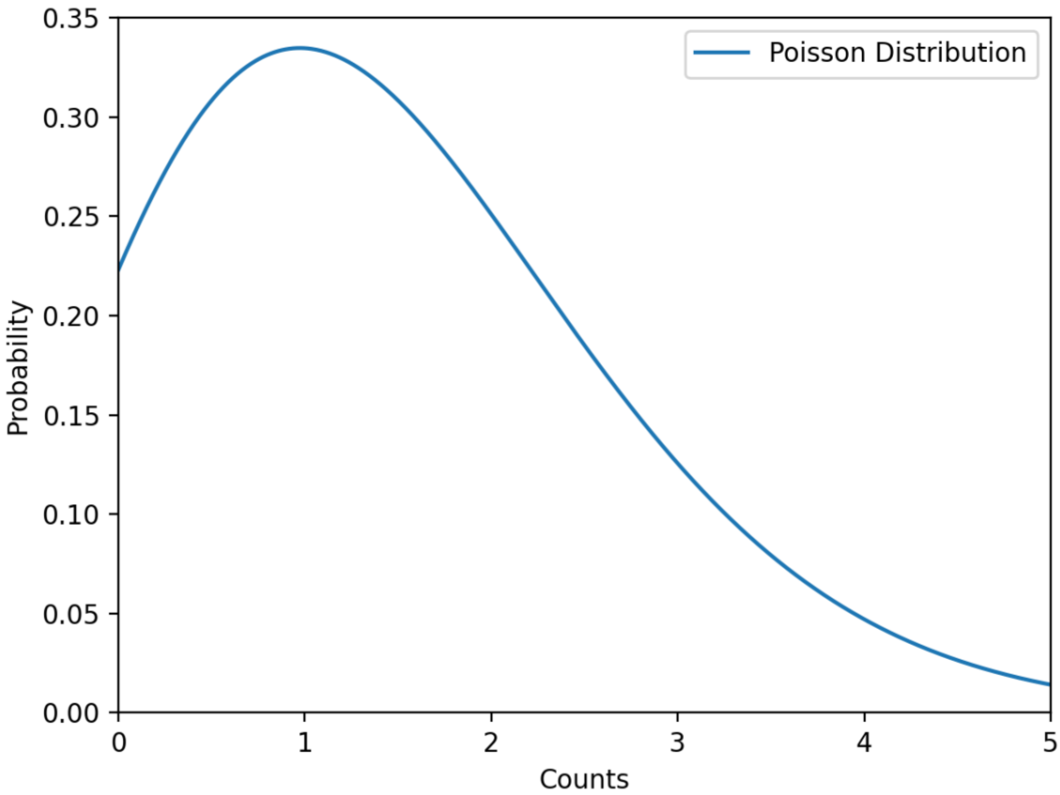
and $\Gamma(n-2) = \int_0^\infty x^{n-3} e^{-x} \, dx$

Therefore it can be seen that $\Gamma(n) = (n-1)(n-2)\Gamma(n-2)$

is actually $\Gamma(n) = (n-1)!$

5)

This graph shows the frequency distribution of counts for the value $\mu = 1.5$



6) Using the Poisson Distribution:

$$P(x;\mu) = \frac{\mu^x}{x!} e^{-\mu}$$

$$P(0;2.1) = \frac{(2.1)^0}{0!} e^{-2.1}$$

$$= e^{-2.1}$$

This is the probability obtaining zero from a mean counting rate of 2.1

References:

[1] SciShow, 2018, Why Do Geiger Counters Make That Clicking Sound?. [video] Available at: <https://youtu.be/qtvz8IH5zhk> [Accessed 13 May 2021]

[2] jbstatistics, 2021. Proof that the Binomial Distribution tends to the Poisson Distribution. [video] Available at: <https://youtu.be/ceOw1HnVCqo> [Accessed 13 May 2021]

[3] V Software, 2021. ~~Vose~~ Deriving the Poisson Distribution from the Binomial. [online] Vosesoftware.com Available at: <https://www.vosesoftware.com/riskwiki/Derivingthe Poissondistribution from theBinomial.php