

Political Districting

Austin Buchanan

DRAFT: June 20, 2023

Abstract

This mini survey covers optimization, heuristic, and sampling methods for political districting for the Encyclopedia of Optimization (Springer, 3rd edition) edited by Panos M. Pardalos and Oleg A. Prokopyev.

1 Introduction

Political districting is the task of partitioning a state into geographic-based districts for election purposes. These districts may elect a single representative (*single-member district*) or multiple representatives (*multi-member district*). The *voting method* determines how ballots in each district are combined to select representatives (e.g., majority, plurality, and ranked choice voting [97, 108]).

In the United States, districts typically are single-member and use plurality voting (“most votes wins”). This includes all congressional districts (which elect individuals to the US House of Representatives) and most state legislative districts (which elect individuals to the respective state governments). For example, Oklahoma is divided into five congressional districts as depicted in Figure 1 (which send representatives to the US Capitol in Washington, DC), 48 state senate districts (which send state senators to the Oklahoma State Capitol in Oklahoma City, OK), and 101 state house districts (which send state representatives to the Oklahoma State Capitol). The number of congressional seats that each state receives is determined by their respective populations, a process called apportionment [6], while the number of legislative districts is set by state law.

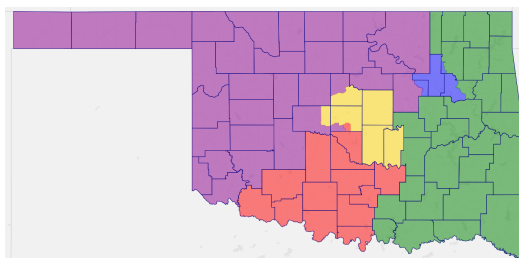


Figure 1: Oklahoma’s five congressional districts, 2022-present

These districts are redrawn every ten years, following the census. A primary motivation is that district populations change over time and must be rebalanced. Current practice in the USA is that most congressional districts satisfy 1-person deviation, e.g., each of Oklahoma’s congressional districts was redrawn in 2022 to have a population of either 791,870 or 791,871 (according to 2020 census counts). Larger deviations of $\pm 5\%$ from the mean are permitted by the courts for legislative districts [32, 61]. These *population balance* constraints are not numerically specified in US federal law, but have emerged from the “one person, one vote” revolution of Supreme Court cases like *Baker v. Carr* (1962), *Wesberry v. Sanders* (1964), and *Reynolds v. Sims* (1964). Federal law in the USA also includes the Voting Rights Act, which requires that districts not dilute the voting power of minority groups [57], see *Thornburg v. Gingles* (1986) and *Allen v. Milligan* (2023). Simultaneously, the Equal Protection Clause of the 14th Amendment to the US Constitution prohibits race from “predominating” the districting process [61], see *Shaw v. Reno* (1993) and *Miller v. Johnson* (1995).

States impose additional laws on redistricting [32]. For example, most states require districts to be contiguous on the map. Other *traditional districting principles* include that districts should be compact and preserve political subdivisions (e.g., counties, cities, towns, wards), communities of interest, and cores of prior districts [82]. Despite these additional “soft” constraints, the set of feasible solutions remains astronomically large, enabling mapmakers to draw plans that favor a political party (*partisan gerrymander*) or incumbent politicians (*incumbent gerrymander*). Given that partisan-controlled state legislatures typically draw the lines, there is incentive to continue gerrymandering, and the Supreme Court has neglected to intervene, see *Rucho v. Common Cause* (2019).

Nevertheless, reform groups have had some success in establishing independent redistricting commissions (IRCs) to draw district lines. IRCs may be instructed to follow traditional redistricting principles, in what might be called procedural fairness [103]. Still, it has been observed that traditional redistricting principles may inadvertently favor one political party over another [26], in ways that are hard to predict [40]. This prompts the explicit consideration of other criteria like partisan fairness, competitiveness, or proportionality [15, 16, 30, 40, 52, 60, 96, 99].

For more information, we refer the reader to the recent book edited by Duchin and Walch [43] (an instant classic), the operations research survey of Ricca et al. [90], and the legal guides by Hebert et al. [61] and Davis et al. [32]. Other related surveys include [20, 55, 58, 59, 69, 75, 92, 105, 111].

Outline. Section 2 summarizes some of the criteria and objectives in districting, along with associated complexity results. Section 3 discusses sampling procedures which are used to understand the distribution of possible districting plans. Section 4 covers heuristic approaches to districting, including construction heuristics (which generate districts from scratch) and local search heuristics (which improve an initial plan with respect to given criteria). Section 5 provides mathematical optimization models for districting. We conclude in Section 6.

2 Criteria and Complexity

We can represent each state as a simple (undirected) graph $G = (V, E)$. The vertices V represent the basic geographic units (e.g., counties, census tracts, census blocks, voting precincts) used to construct districts, and the edges E indicate which pairs of geographic units are adjacent on the map. Figure 2 illustrates Oklahoma’s county-level graph. In *rook adjacency*, geographic units u and v must share a border of positive length for the edge $\{u, v\}$ to be in E ; the less common *queen adjacency* only requires u and v to meet at a point. Under rook adjacency, the graph G is planar and very sparse, with the numbers of vertices $n = |V|$ and edges $m = |E|$ satisfying $m \leq 3n - 6$ if $n \geq 3$; queen adjacency permits complete graphs with $m = \binom{n}{2}$ edges on “pizza pie” instances.

The vertices adjacent to vertex $i \in V$ constitute its neighborhood $N(i) = \{j \in V \mid \{i, j\} \in E\}$. The subgraph induced by a subset of vertices $S \subseteq V$ is denoted by $G[S] = (S, E(S))$ where $E(S)$ is the subset of edges with both endpoints in S . We say that a district $D \subseteq V$ is connected in the graph (or contiguous on the map) if its induced subgraph $G[D]$ is connected.

Each geographic unit $i \in V$ has an associated population p_i . Given a subset of geographic units $S \subseteq V$, their combined population is given by the shorthand $p(S) := \sum_{i \in S} p_i$. The ideal district population $p(V)/k$ is the state’s total population $p(V)$ divided by the predefined number of districts k . The smallest and largest populations permitted in a district are denoted by L and U . In 1-person deviation, we have $L = \lfloor p(V)/k \rfloor$ and $U = \lceil p(V)/k \rceil$. In 10% deviation ($\pm 5\%$), we have $L = \lceil 0.95p(V)/k \rceil$ and $U = \lfloor 1.05p(V)/k \rfloor$.

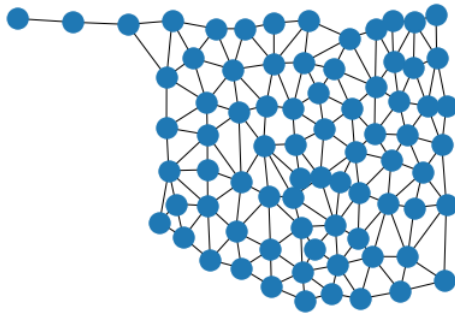


Figure 2: Oklahoma’s county-level graph

The districting problem, in its simplest form, is to partition the vertices V into k districts (D_1, D_2, \dots, D_k) such that each district is contiguous and population-balanced, i.e., each district D_j should be connected and satisfy $L \leq p(D_j) \leq U$. Unfortunately, this problem is already NP-hard for unit populations ($p_i = 1$ for $i \in V$) and $L = U = 3$, see [44]. Even if the contiguity constraints are relaxed, NP-hardness persists as it can express the PARTITION problem [3].

In practice, the population balance constraints are less of an issue than the worst-case complexity would suggest. Intuitively, the reason is that hard

instances of PARTITION involve large integers, while districts are often built from census blocks, which can have single digit (or zero) populations. Contiguity is often blamed for the practical difficulty, with Ricca et al. [90] stating that “it is particularly difficult to deal with and [is often] discarded from [districting] models and considered only *a posteriori*”. We have also found the large size of districting instances to be a challenge, with many states having hundreds of thousands of census blocks. Another challenge is the many criteria that districting plans are supposed to satisfy and the many ways to quantify them.

As an example, consider compactness, which is the idea that a district should have a “nice” shape. Dozens of alternative compactness scores have been proposed over the years, with many viewing circles and squares to be the most compact [83]. In the optimization literature, the first was the moment-of-inertia (MOI), proposed by Hess et al. [64]. Taking inspiration from physics, the MOI of a single district $D \subseteq V$ is calculated as $\sum_{i \in D} p_i d_{ij}^2$, where d_{ij} is the Euclidean distance between the (centers of) geographic units i and j on the map and j is the district’s “center”, chosen so as to minimize this sum. A (once) popular score among political scientists is the length-width score, which compares the length and width of the district, with a ratio of one indicating ideal compactness. It is easy to draw districts that look awful but score well according to this score [112]. The Polsby-Popper score [87], which is currently the most popular score in the redistricting literature and in expert testimony [37], is defined as $4\pi A/P^2$ where A is the district’s area and P is the district’s perimeter. It takes values between zero and one, with circular districts achieving a perfect score of one. This score, and others that rely on the district’s perimeter length, are subject to the *Coast-line Paradox*, which roughly states that district perimeters are not well-defined and depend on the choice of map projection and the “size of your ruler” [7, 8]. Recently, the number of *cut edges* between districts has become one of the more prominent compactness scores among mathematicians, computer scientists, statisticians, and operations researchers because it is simple, is less prone to abuse, reasonably agrees with the “eyeball test”, and is well-suited for mathematical, computational, and probabilistic analysis [13, 34, 36, 65, 77, 88, 101].

Likewise, there are many ways to quantify how well political subdivisions, say, counties, are preserved. We could count how many *counties* are divided across districts, how many *times* counties are divided across districts, or examine the *extent* to which they are divided (e.g., a county that is divided 90/10 across two districts may be “less” split than a county that is divided 50/50). For a more involved discussion, we refer the reader to [12, 23, 54, 104]. Similar approaches can be used to quantify the preservation of communities of interest.

We refer the reader to other resources on quantifying or “operationalizing” partisan fairness, proportionality, and competitiveness [16, 30, 40, 52, 60, 96, 99].

3 Sampling Methods

Over the last twenty years, Massachusetts has had either 9 or 10 congressional districts, and all of them have elected Democrats. This is despite the fact that

30% to 40% of the state’s votes went to Republicans. Does this indicate an intent to gerrymander? The answer turns out to be no. Duchin et al. [39] find that it is impossible to draw a Republican-majority district in Massachusetts built from voting precincts. The reason is that Republicans are distributed nearly uniformly throughout the state. Intuitively, if all precincts voted 40% Republican, then all districts built from precincts would likewise vote 40% Republican and thus elect Democrats. So, disproportionate outcomes are not necessarily a sign of intentional gerrymandering.

To arrive at intent, a better approach is to *randomly* draw districting plans to see what outcomes are likely to have occurred by chance. If a proposed or enacted plan is an outlier in this distribution of possible plans (say, with respect to the number of districts won by a particular party), then this may suggest an intent to gerrymander. To make this approach rigorous, we would need to determine which districting plans should not be sampled (e.g., if they violate population balance or contiguity) and what probability we should attach to each remaining plan (e.g., favoring compact plans over snaking, fractal-like districts). This is where *sampling*, *ensemble*, or “simulation” methods enter the picture.

Many ensemble methods take a Markov chain Monte Carlo (MCMC) approach [2, 28, 29, 34, 35, 47, 63, 114]. From an initial districting plan, the approach randomly “walks” to similar, neighboring plans, say, by flipping a vertex from one district to a neighboring district, or swapping two vertices between neighboring districts. These *flip* or *swap* neighborhoods have empirically been found to “mix” slowly, meaning that the approach may not adequately search or sample the solution space in a reasonable number of iterations, cf. [81].

This motivated DeFord et al. [36] to propose a new neighborhood called recombination. In ReCom, two adjacent districts are merged into a double district, a spanning tree is randomly drawn over their nodes, and an edge is deleted to split the tree into two subtrees which are taken as the two new districts (provided that they are feasible). Empirically, this approach gets “lost” quickly with, say, the 10,000th districting plan being nothing like the initial plan. Further, the closely associated *spanning tree distribution* favors compact districts [21, 88]. The GerryChain software package provides an open-source Python implementation [80], cf. a Julia implementation [91]. Despite their practical success, ReCom and other ensemble methods are not guaranteed to mix quickly [25].

Recent works seek to incorporate more “rules of the game” into the approach, like preserving political subdivisions [5, 30, 77] and ensuring minority representation [11, 22], cf. [27, 41, 42]. They may also use an explicit target distribution, like the spanning tree distribution [5, 21, 77], to make the approach more credible. In particular, McCartan and Imai [77] propose an entirely different approach called sequential Monte Carlo (SMC) which avoids the Markov Chain altogether; it builds a batch of districting plans by carving districts off one-by-one, also with random spanning trees and edge deletions. Kenny et al. [70] use the approach to evaluate the nationwide effects of partisan gerrymandering [78].

4 Heuristic Methods

Heuristics are inexact procedures used to find “good enough” solutions to an optimization problem [1]. In particular, construction heuristics build a feasible solution from scratch and have been proposed for political districting since at least 1961 [103]. We refer the reader to the survey of Becker and Solomon [13, Sec. 3.1] for a nice introduction and Ricca et al. [90, Sec. 3.1] for additional references to the literature. One high-level approach is to identify a set of k vertices to “seed” the districts and grow the districts outwards. Another high-level approach is to build districts one-at-a-time by carving them from the state.

Local search heuristics start with a feasible solution and (repeatedly) make small changes to improve its performance with respect to given criteria. Traditionally, local search heuristics for districting have used either a flip neighborhood or a swap neighborhood, sometimes with specialized techniques that exploit planarity to speed up the connectivity checks at each iteration [71, 72, 73]. To escape local optima, many researchers and practitioners adopt metaheuristic techniques such as simulated annealing, tabu search, and genetic algorithms. Again, we refer the reader to [13, 90] for pointers to many papers in this area.

The success of DeFord et al.’s recombination neighborhood [36] has prompted researchers to use it for heuristic optimization purposes. Some researchers apply ReCom in an MCMC fashion and pluck out those that are satisfactory or perform best. Duchin and Schoenbach [40] take this approach to find proportional districting plans. Cannon et al. [22] likewise take an unbiased ReCom walk for a small number of steps, but then restart the walk from the best-performing solution, with the aim of maximizing the number of minority-opportunity districts, cf. [11]. This “short bursts” approach has also been used in the search for plans that are Pareto optimal with respect to compactness and population balance [76]. Geodert et al. [56] use a mix of ReCom and flip moves, with flip moves becoming more prevalent throughout the procedure, to optimize weighted combinations of Black representation and compactness. Swamy et al. [98] also use a mix of ReCom and flip moves to draw districts for Arizona, considering compactness, partisan fairness, competitiveness, and the number of majority-minority districts. Other researchers find the *best* (rather than a random) move in the ReCom neighborhood, merging and redividing up to four districts at a time to minimize the number of county splits [93]. Because the ReCom neighborhood contains exponentially many districting plans—many more than the flip or swap neighborhoods—it is no longer feasible to use brute force enumeration to find the best plan in this neighborhood, thus requiring the solution of an optimization problem at each iteration.

Inspired by work of Henzinger et al. [62] on graph partitioning, Belotti et al. [14] propose another local search neighborhood for political districting called the h -hop neighborhood. The idea is that vertices deep within a district are required to stay in their current district, while vertices near a district border are permitted to switch to a different nearby district. More formally, a vertex $v \in D$ is permitted to switch to a different district D' (in the next iteration) if there is a vertex $v' \in D'$ with hop-based distance $\text{dist}_G(v, v') \leq h$. When

the user-chosen parameter h is relatively small, say $h \in \{1, 2\}$, the associated optimization problem is relatively easy to solve. For example, Belotti et al. [14] apply the approach to optimize the Polsby-Popper compactness score.

5 Optimization Methods

This section covers mathematical optimization models for political districting. We categorize the models based on the their primary decision variables. Example Python codes for several districting models are available on GitHub at <https://github.com/AustinLBuchanan/Districting-Examples-2020>.

5.1 Using Hess Variables

Motivated by the landmark “one person, one vote” Supreme Court cases of the 1960s, Hess et al. [64, 107] proposed the first optimization model for political districting. The binary variable x_{ij} equals one when vertex $i \in V$ is assigned to the district centered at vertex $j \in V$ with $x_{jj} = 1$ indicating that j is a center.

$$\min \sum_{i \in V} \sum_{j \in V} p_i d_{ij}^2 x_{ij} \quad (1a)$$

$$\text{s.t.} \quad \sum_{j \in V} x_{ij} = 1 \quad \forall i \in V \quad (1b)$$

$$\sum_{j \in V} x_{jj} = k \quad (1c)$$

$$Lx_{jj} \leq \sum_{i \in V} p_i x_{ij} \leq Ux_{jj} \quad \forall j \in V \quad (1d)$$

$$x_{ij} \leq x_{jj} \quad \forall i, j \in V \quad (1e)$$

$$x_{ij} \in \{0, 1\} \quad \forall i, j \in V. \quad (1f)$$

The objective (1a) minimizes the moment-of-inertia. Assignment constraints (1b) ensure that each vertex is assigned to one district. Constraint (1c) requires k district centers to be selected. Population balance constraints (1d) require each district to have a population between L and U . The coupling constraints (1e) were not originally imposed by Hess et al. but help to strengthen the linear programming (LP) relaxation. As written, this model lacks contiguity constraints.

Due to computer and software limitations of the time, Hess et al. [64] solved this integer program heuristically. A main insight is that, after the district centers have been selected, the resulting subproblem is essentially a transportation problem, which can be solved efficiently, cf. [49, 53]. Afterwards, we have a partition of the vertices into districts (D_1, D_2, \dots, D_k) . Within each district D , identify the best vertex $j \in D$ to serve as its center, i.e., the vertex j that minimizes $\sum_{i \in D} p_i d_{ij}^2$. With these new district centers, re-solve the transportation problem and repeat until convergence, much like the popular k -means heuristic. See Lawless and Günlük [74] for a recent variant with representation constraints.

To impose contiguity, Shirabe [94, 95] proposes a formulation in which “flow” originates at each district center and is sent across the district’s edges to “fuel” its other nodes, cf. [84, 102]. In particular, introduce the variable f_{uv}^j to indicate how much flow, originating from district center j , is sent across the directed edge (u, v) . To the original n^2 Hess variables, this adds $2nm$ flow variables, which is $O(n^2)$ if G has $m = O(n)$ edges (true for planar G). The constraints below ensure that each non-center vertex is fueled (2a), flow of type j can enter vertex i only if i is assigned to j (2b), and flow cannot re-enter a center (2c).

$$\sum_{u \in N(i)} (f_{ui}^j - f_{iu}^j) = x_{ij} \quad \forall i \in V \setminus \{j\}, \forall j \in V \quad (2a)$$

$$\sum_{u \in N(i)} f_{ui}^j \leq (n-1)x_{ij} \quad \forall i \in V \setminus \{j\}, \forall j \in V \quad (2b)$$

$$\sum_{u \in N(i)} f_{uj}^j = 0 \quad \forall j \in V \quad (2c)$$

$$f_{ij}^v, f_{ji}^v \geq 0 \quad \forall \{i, j\} \in E, \forall v \in V. \quad (2d)$$

These constraints are arguably the most popular contiguity constraints in the literature, e.g., being used by Validi et al [102] for county-level instances of the moment-of-inertia objective and Swamy et al. [99] for objectives relating to partisan symmetry, efficiency gap, and competitiveness. Shahmizad and Buchanan [93] use them to solve the so-called *county clustering problem* which provides a strong lower bound on the minimum number of county splits, cf. [23].

Another approach for imposing contiguity uses separator inequalities [19, 24, 48, 84, 106], cf. [89]. When applied to Hess variables, they take the form

$$(a, b\text{-separator inequality}) \quad x_{aj} + x_{bj} \leq 1 + \sum_{c \in C} x_{cj}$$

where a and b are nonadjacent vertices and $C \subseteq V \setminus \{a, b\}$ is an a, b -separator, i.e., there is no path between a and b in the subgraph $G[V \setminus C]$. Because there are exponentially many of these constraints, they are typically applied in a branch-and-cut fashion. Validi et al. [102] show that, for planar graphs, the associated separation problem is solvable in time $O(n^2 \log n)$ and $O(n^2)$ for fractional and integer x , respectively. In their experiments with the MOI objective, they find that few of these inequalities are necessary, allowing for some instances with up to 1,500 vertices to be solved exactly. By exploiting the population balance constraints, they also write stronger length- U a, b -separator inequalities.

Finally, we consider some contiguity constraints that are fast in practice, but are invalid in the sense that they cut off feasible points. First, we have the *tree-based contiguity constraints* of Zoltners and Sinha [115]; after finding a shortest paths tree rooted at a particular vertex j , impose constraints of the form $x_{ij} \leq x_{vj}$ for $i \in V \setminus \{j\}$ where v is the predecessor of i in the tree. These constraints can be relaxed to *distance-based contiguity constraints* [31, 60, 79, 85], which allow more solutions and take the form $x_{ij} \leq \sum_v x_{vj}$, where the sum is over all

neighbors $v \in N(i)$ that are nearer to j , i.e., $\text{dist}(v, j) < \text{dist}(i, j)$. Önal and Patrick [85] use these constraints to draw plans for Illinois that fare better than the enacted plan with respect to county splits and minority representation. To allow even more solutions, we can use *DAG-based contiguity constraints* [93] in which the edges of G are oriented away from j in an acyclic fashion and impose $x_{ij} \leq \sum_v x_{vj}$, where the sum is over all in-neighbors of i in the orientation.

In the spirit of Beasley [10], Hojati [66] proposes a Lagrangian relaxation of the Hess model in which population balance (1d) is first imposed strictly ($L = U$). Then, the population balance constraints and assignment constraints (1b) are relaxed, with their violation penalized in the objective function with respect to the Lagrange multipliers. This results in a Lagrangian relaxation model that is easy to solve combinatorially and that satisfies the “integrality property”, implying that the Lagrangian relaxation bound coming from the best Lagrange multipliers equals that of the LP relaxation. Motivation for using Lagrangian relaxation instead of LP relaxation included that the Lagrangian was easier to solve and less memory-intensive, especially using LP software of the day. Later, Validi et al. [102] propose a similar Lagrangian relaxation model where $L < U$, and use it to fix $x_{ij} = 0$ when it can be deduced that $x_{ij} = 1$ would be suboptimal, making rigorous earlier approaches that heuristically fix such variables [85]. They also exploit the contiguity constraints in the Lagrangian relaxation, using it to solve instances of the Hess model with 1,500 vertices.

5.2 Using Labeling Variables

Another class of integer programming models uses labeling or assignment variables of the form x_{ij} which equal one when vertex $i \in V$ is assigned to district $j \in [k] := \{1, 2, \dots, k\}$. Because the number of districts is usually much smaller than the number of vertices, labeling models are typically smaller than Hess models. For example, if applied to Oklahoma’s $|V| = 1,205$ census tracts and $k = 5$ congressional districts, there would be 6,025 labeling variables or 1,452,025 Hess variables. A typical use of labeling models is to minimize the number of cut edges between districts [13, 17, 46, 68, 93, 101]. So, introduce a binary variable y_e for each edge $e \in E$ indicating whether it is cut and write:

$$\min \sum_{e \in E} y_e \quad (3a)$$

$$\text{s.t. } x_{uj} - x_{vj} \leq y_e \quad \forall e = \{u, v\} \in E, \forall j \in [k] \quad (3b)$$

$$\sum_{j=1}^k x_{ij} = 1 \quad \forall i \in V \quad (3c)$$

$$L \leq \sum_{i \in V} p_i x_{ij} \leq U \quad \forall j \in [k] \quad (3d)$$

$$x_{ij} \in \{0, 1\} \quad \forall i \in V, \forall j \in [k] \quad (3e)$$

$$y_e \in \{0, 1\} \quad \forall e \in E. \quad (3f)$$

One notable drawback of labeling formulations is symmetry; the same districting plan can be represented in $k!$ different ways by permuting the district labels [68]. Consequently, even if we added valid inequalities to recover the convex hull in the x -space of variables, the LP relaxation would still allow the point (\bar{x}, \bar{y}) where $\bar{y} = \mathbf{0}$ by setting $\bar{x}_{ij} = 1/k$ for all $i \in V$ and $j \in [k]$, see [101]. One way to avoid this symmetry is to apply the extended formulation for partitioning orbitopes due to Faenza and Kaibel [45], cf. [101], which forces the districts to be sorted lexicographically. Another computational speedup comes by exploiting the population balance constraints, specifically the population lower bounds L , in a procedure called L -fixing [101]. Many forms of the contiguity constraints that were previously discussed for Hess models can be adapted to the labeling context, such as the Shirabe model [94, 95, 101], separator constraints [101], as well as a notable single-commodity flow formulation from Hojny et al. [67], cf. [18]. Ferreira et al. [46] propose valid inequalities and an extended formulation to strengthen the cut edge LP relaxation which are very helpful [101].

Labeling models can also be applied to minimize the sum of district perimeters [101], as this amounts to a *weighted* cut edges objective. Belotti et al. [14] extend the model to handle the Polsby-Popper score in a mixed-integer second-order cone program (MISOCP). They apply the MISOCPs to draw compact majority-minority districts, including a case study motivated by the Supreme Court case *Allen v. Milligan* (2023) in which they draw a compact plan for Alabama that has two Black-majority districts. Fravel et al. [50] extend the labeling model to handle or estimate nonconvex objectives relating to the Polsby-Popper score, Black representation, and partisan outcomes. Arredondo et al. [4] use a labeling model focused on Indigenous representation in Mexico.

5.3 Using District Variables

In a completely different class of optimization models, introduce a binary variable x_D and a cost c_D for each possible district $D \in \mathcal{D}$. Then, a set partitioning model over the district variables can be written as follows.

$$\min \sum_{D \in \mathcal{D}} c_D x_D \quad (4a)$$

$$\text{s.t.} \quad \sum_{D \in \mathcal{D}: i \in D} x_D = 1 \quad \forall i \in V \quad (4b)$$

$$\sum_{D \in \mathcal{D}} x_D = k \quad (4c)$$

$$x_D \in \{0, 1\} \quad \forall D \in \mathcal{D}. \quad (4d)$$

When Garfinkel and Nemhauser [51] first introduced this model for districting, they took a two-step approach. The first step was to enumerate all suitable districts $D \in \mathcal{D}$, and the second step was to solve the resulting set partitioning model (4). Nowadays, a typical strategy for “solving” models like these (with exponentially many variables) is to first solve the LP relaxation using column generation and then solve the associated integer program over the generated

columns. This optimization-based heuristic was taken by Mehrotra et al. [79]. More recently, Gurnee and Shmoys [60] take a column generation approach, using stochastic hierarchical partitioning to quickly generate many columns, with an eye towards fairness. To truly *solve* the set partitioning IP, one would likely need to take a branch-and-price approach [9], which is essentially branch-and-bound where the LP relaxations are solved using column generation. However, to the best of our knowledge, the political districting literature does not contain any true branch-and-price implementations, although Borndörfer et al. [18] test their approach on related commercial territory design instances.

5.4 Using Spanning Tree Edge Variables

Recognizing that connected districts admit spanning trees, we could define a variable x_e that equals one if the endpoints of edge e belong to the same district and if this edge is selected as part of the district’s spanning tree. By adding $k - 1$ other edges to these spanning trees’ edges, we can obtain a spanning tree for the entire graph. Indeed, there is a linear-size extended formulation for spanning trees in planar graphs due to Williams [109, 110]; see [86, 100] for corrections to this model. With this modeling primitive, we can write a *linear-size* formulation for partitioning the vertices of a planar graph into k components *that is integral*. Unfortunately, Zhang et al. [113] find that, when the population balance is imposed, the integrality of the formulation is destroyed, and it performs worse than the Hess model. It is an open question whether this spanning tree model can be redeemed with alternative population balance constraints.

6 Conclusion

Political districting remains a challenging problem for optimization methods. This is partially due to the large size of districting instances and the many objectives and criteria that one must deal with. This is not to say that optimization cannot have an impact, but rather to emphasize the mathematical and computational ingenuity that is required, as well as the familiarity with the entirety of the districting literature, including political science, computer science, mathematics, and litigation [38]. Optimization *can* be a powerful tool for districting, to illuminate tradeoffs between districting criteria and to show the limits of what is possible. Indeed, in an amicus brief cited by the Supreme Court in *Allen v. Milligan* (2023), a team of computational redistricting experts wrote that “optimization algorithms are well-suited to the task of generating [remedial plans in VRA litigation]. . . as they can identify innovative combinations of geography that better comply with multiple traditional redistricting principles than any individual mapmaker is likely to find manually through trial and error” [33].

Acknowledgements

This material is based upon work supported by the National Science Foundation under Grant No. 1942065.

References

- [1] Emile HL Aarts and Jan Karel Lenstra. *Local search in combinatorial optimization*. Princeton University Press, 2003.
- [2] William T Adler and Samuel S-H Wang. Response to Cho and Liu, “Sampling from complicated and unknown distributions: Monte Carlo and Markov chain Monte Carlo methods for redistricting”. *Physica A: Statistical Mechanics and its Applications*, 516:591–593, 2019.
- [3] Micah Altman. The computational complexity of automated redistricting: Is automation the answer? *Rutgers Computer & Tech. LJ*, 23:81, 1997.
- [4] Verónica Arredondo, Miguel Martínez-Panero, Teresa Peña, and Federica Ricca. Mathematical political districting taking care of minority groups. *Annals of Operations Research*, 305(1):375–402, 2021.
- [5] Eric A Autry, Daniel Carter, Gregory J Herschlag, Zach Hunter, and Jonathan C Mattingly. Metropolized multiscale forest recombination for redistricting. *Multiscale Modeling & Simulation*, 19(4):1885–1914, 2021.
- [6] Michel L Balinski and H Peyton Young. *Fair representation: meeting the ideal of one man, one vote*. Brookings Institution Press, second edition, 2010.
- [7] Assaf Bar-Natan, Lorenzo Najt, and Zachary Schutzman. The gerrymandering jumble: map projections permute districts’ compactness scores. *Cartography and Geographic Information Science*, 47(4):321–335, 2020.
- [8] Richard Barnes and Justin Solomon. Gerrymandering and compactness: Implementation flexibility and abuse. *Political Analysis*, 29(4):448–466, 2021.
- [9] Cynthia Barnhart, Ellis L Johnson, George L Nemhauser, Martin WP Savelsbergh, and Pamela H Vance. Branch-and-price: Column generation for solving huge integer programs. *Operations Research*, 46(3):316–329, 1998.
- [10] John E Beasley. Lagrangean heuristics for location problems. *European Journal of Operational Research*, 65(3):383–399, 1993.
- [11] Amariah Becker, Moon Duchin, Dara Gold, and Sam Hirsch. Computational redistricting and the Voting Rights Act. *Election Law Journal: Rules, Politics, and Policy*, 20(4):407–441, 2021.

- [12] Amariah Becker and Dara Gold. The gameability of redistricting criteria. *Journal of Computational Social Science*, 5:1735–1777, 2022.
- [13] Amariah Becker and Justin Solomon. Redistricting algorithms. In Moon Duchin and Olivia Walch, editors, *Political Geometry: Rethinking Redistricting in the US with Math, Law, and Everything In Between*. Birkhauser, 2022.
- [14] Pietro Belotti, Austin Buchanan, and Soraya Ezazipour. Political districting to optimize the Polsby-Popper compactness score, 2023. Available on Optimization-Online.
- [15] Gerdus Benade, Nam Ho-Nguyen, and JN Hooker. Political districting without geography. *Operations Research Perspectives*, 9:100227, 2022.
- [16] Mira Bernstein and Olivia Walch. Measuring partisan fairness. In Moon Duchin and Olivia Walch, editors, *Political Geometry*, pages 39–75. Birkhauser, 2022.
- [17] Ralf Borndörfer, Carlos E Ferreira, and Alexander Martin. Decomposing matrices into blocks. *SIAM Journal on Optimization*, 9(1):236–269, 1998.
- [18] Ralf Borndörfer, Stephan Schwartz, and William Surau. Vertex covering with capacitated trees. *Networks*, 81(2):253–277, 2023.
- [19] Austin Buchanan, Je Sang Sung, Sergiy Butenko, and Eduardo L Pasiliao. An integer programming approach for fault-tolerant connected dominating sets. *INFORMS Journal on Computing*, 27(1):178–188, 2015.
- [20] Charles S Bullock III. *Redistricting: The most political activity in America*. Rowman & Littlefield Publishers, 2010.
- [21] Sarah Cannon, Moon Duchin, Dana Randall, and Parker Rule. Spanning tree methods for sampling graph partitions. *arXiv preprint arXiv:2210.01401*, 2022.
- [22] Sarah Cannon, Ari Goldbloom-Helzner, Varun Gupta, JN Matthews, and Bhushan Suwal. Voting rights, Markov chains, and optimization by short bursts. *Methodology and Computing in Applied Probability*, 25(1):36, 2023.
- [23] Daniel Carter, Zach Hunter, Dan Teague, Gregory Herschlag, and Jonathan Mattingly. Optimal legislative county clustering in North Carolina. *Statistics and Public Policy*, 7(1):19–29, 2020.
- [24] Rodolfo Carvajal, Miguel Constantino, Marcos Goycoolea, Juan Pablo Vielma, and Andrés Weintraub. Imposing connectivity constraints in forest planning models. *Operations Research*, 61(4):824–836, 2013.
- [25] Moses Charikar, Paul Liu, Tianyu Liu, and Thuy-Duong Vuong. On the complexity of sampling redistricting plans. *arXiv preprint arXiv:2206.04883*, 2022.

- [26] Jowei Chen and Jonathan Rodden. Unintentional gerrymandering: Political geography and electoral bias in legislatures. *Quarterly Journal of Political Science*, 8(3):239–269, 2013.
- [27] Jowei Chen and Nicholas O Stephanopoulos. The race-blind future of voting rights. *Yale Law Journal*, 130:862–947, 2021.
- [28] Maria Chikina, Alan Frieze, and Wesley Pegden. Assessing significance in a markov chain without mixing. *Proceedings of the National Academy of Sciences*, 114(11):2860–2864, 2017.
- [29] Wendy K Tam Cho and Yan Y Liu. Sampling from complicated and unknown distributions: Monte Carlo and Markov chain Monte Carlo methods for redistricting. *Physica A: Statistical Mechanics and its Applications*, 506:170–178, 2018.
- [30] Jeanne Clelland, Haley Colgate, Daryl DeFord, Beth Malmskog, and Flavia Sancier-Barbosa. Colorado in context: Congressional redistricting and competing fairness criteria in Colorado. *Journal of Computational Social Science*, 5:189–226, 2022.
- [31] Thomas J Cova and Richard L Church. Contiguity constraints for single-region site search problems. *Geographical Analysis*, 32(4):306–329, 2000.
- [32] Michelle Davis, Frank Strigari, Wendy Underhill, Jeffrey M Wice, and Christi Zamarripa. *Redistricting Law 2020*. National Conference of State Legislatures, 2019.
- [33] Daryl DeFord, Amariah Becker, and Dara Gold. Brief of Computational Redistricting Experts as Amici Curiae in support of Appellees and Respondents in *Merrill v. Milligan* and *Merrill v. Caster*, 2022.
- [34] Daryl DeFord and Moon Duchin. Redistricting reform in Virginia: Districting criteria in context. *Virginia Policy Review*, 12(2):120–146, 2019.
- [35] Daryl DeFord and Moon Duchin. Random walks and the universe of districting plans. In Moon Duchin and Olivia Walch, editors, *Political Geometry*, pages 341–381. Birkhauser, 2022.
- [36] Daryl DeFord, Moon Duchin, and Justin Solomon. Recombination: A family of Markov chains for redistricting. *Harvard Data Science Review*, 3(1), 2021.
- [37] Moon Duchin. Outlier analysis for Pennsylvania congressional redistricting. *LWV vs. Commonwealth of Pennsylvania Docket No. 159 MM 2017*, 2018.
- [38] Moon Duchin. Presentation of Alternative Congressional Districting Plans for Alabama, 2021. Available in the Supplemental Joint Appendix for *Merrill v. Milligan* at <https://www.supremecourt.gov/DocketPDF/>

21/21-1086/221826/20220425150837756_42140\%20pdf\%20Bowdre\%20IV\%20Supplemental\%20JA.pdf.

- [39] Moon Duchin, Taissa Gladkova, Eugene Henninger-Voss, Ben Klingensmith, Heather Newman, and Hannah Wheelen. Locating the representational baseline: Republicans in Massachusetts. *Election Law Journal: Rules, Politics, and Policy*, 18(4):388–401, 2019.
- [40] Moon Duchin and Gabe Schoenbach. Redistricting for proportionality. *The Forum*, 20(3-4):371–393, 2022.
- [41] Moon Duchin and Doug Spencer. Models, race, and the law. *Yale Law Journal Forum*, 130:744–797, 2021.
- [42] Moon Duchin and Douglas Spencer. Blind justice: Algorithms and neutrality in the case of redistricting. In *Proceedings of the 2022 Symposium on Computer Science and Law*, pages 101–108, 2022.
- [43] Moon Duchin and Olivia Walch, editors. *Political Geometry: Rethinking Redistricting in the US with Math, Law, and Everything In Between*. Birkhauser, 2022.
- [44] Martin E Dyer and Alan M Frieze. On the complexity of partitioning graphs into connected subgraphs. *Discrete Applied Mathematics*, 10(2):139–153, 1985.
- [45] Yuri Faenza and Volker Kaibel. Extended formulations for packing and partitioning orbitopes. *Mathematics of Operations Research*, 34(3):686–697, 2009.
- [46] Carlos E Ferreira, Alexander Martin, C Carvalho de Souza, Robert Weismantel, and Laurence A Wolsey. Formulations and valid inequalities for the node capacitated graph partitioning problem. *Mathematical Programming*, 74(3):247–266, 1996.
- [47] Benjamin Fifield, Michael Higgins, Kosuke Imai, and Alexander Tarr. Automated redistricting simulation using Markov chain Monte Carlo. *Journal of Computational and Graphical Statistics*, 29(4):715–728, 2020.
- [48] Matteo Fischetti, Markus Leitner, Ivana Ljubić, Martin Luipersbeck, Michele Monaci, Max Resch, Domenico Salvagnin, and Markus Sinnl. Thinning out Steiner trees: a node-based model for uniform edge costs. *Mathematical Programming Computation*, 9(2):203–229, 2017.
- [49] Bernhard Fleischmann and Jannis N Paraschis. Solving a large scale districting problem: a case report. *Computers & Operations Research*, 15(6):521–533, 1988.
- [50] Jamie Fravel, Robert Hildebrand, Nicholas Goedert, Laurel Travis, and Matthew Pierson. Dual bounds for redistricting problems with non-convex objectives. *arXiv preprint arXiv:2305.17298*, 2023.

- [51] Robert S Garfinkel and George L Nemhauser. Optimal political districting by implicit enumeration techniques. *Management Science*, 16(8):B-495, 1970.
- [52] Nikhil Garg, Wes Gurnee, David Rothschild, and David Shmoys. Combatting gerrymandering with social choice: the design of multi-member districts, 2022.
- [53] John A George, Bruce W Lamar, and Chris A Wallace. Political district determination using large-scale network optimization. *Socio-Economic Planning Sciences*, 31(1):11–28, 1997.
- [54] Taissa Gladkova, Ari Goldbloom-Helzner, Muniba Khan, Brandon Kolstoe, Jasmine Noory, Zachary Schutzman, Eric Stucky, and Thomas Weighill. Discussion of locality splitting measures, 2019. <https://github.com/vrdi/splitting/blob/master/SplittingReport.pdf>.
- [55] Sebastian Goderbauer and Jeff Winandy. Political districting problem: Literature review and discussion with regard to federal elections in Germany, 2018.
- [56] Nicholas Goedert, Robert Hildebrand, Matt Pierson, Laurel Travis, and Jamie Fravel. Black representation and district compactness in southern congressional districts. *Available at SSRN 4449256*, 2023.
- [57] Arusha Gordon and Douglas M Spencer. Explainer: A brief introduction to the Voting Rights Act. In *Political Geometry: Rethinking Redistricting in the US with Math, Law, and Everything In Between*, pages 131–136. Birkhauser, 2022.
- [58] Bernard Grofman. Criteria for districting: A social science perspective. *UCLA L. Rev.*, 33:77, 1985.
- [59] Bernard Grofman and Jonathan Cervas. The terminology of districting. *Available at SSRN 3540444*, 2020.
- [60] Wes Gurnee and David B Shmoys. Fairmandering: A column generation heuristic for fairness-optimized political districting. In *SIAM Conference on Applied and Computational Discrete Algorithms (ACDA21)*, pages 88–99. SIAM, 2021.
- [61] J Gerald Hebert, Martina E Vandenberg, and Paul Smith. *The Realist’s Guide to Redistricting: Avoiding the Legal Pitfalls*. American Bar Association, 2010.
- [62] Alexandra Henzinger, Alexander Noe, and Christian Schulz. ILP-based local search for graph partitioning. *Journal of Experimental Algorithmics (JEA)*, 25:1–26, 2020.

- [63] Gregory Herschlag, Han Sung Kang, Justin Luo, Christy Vaughn Graves, Sachet Bangia, Robert Ravier, and Jonathan C Mattingly. Quantifying gerrymandering in North Carolina. *Statistics and Public Policy*, 7(1):30–38, 2020.
- [64] SW Hess, JB Weaver, HJ Siegfeldt, JN Whelan, and PA Zitlau. Nonpartisan political redistricting by computer. *Operations Research*, 13(6):998–1006, 1965.
- [65] Cyrus Hettle, Shixiang Zhu, Swati Gupta, and Yao Xie. Balanced districting on grid graphs with provable compactness and contiguity. *arXiv preprint arXiv:2102.05028*, 2021.
- [66] Mehran Hojati. Optimal political districting. *Computers & Operations Research*, 23(12):1147–1161, 1996.
- [67] Christopher Hojny, Imke Joormann, Hendrik Lüthen, and Martin Schmidt. Mixed-integer programming techniques for the connected max- k -cut problem. *Mathematical Programming Computation*, 13(1):75–132, 2021.
- [68] Ellis L Johnson, Anuj Mehrotra, and George L Nemhauser. Min-cut clustering. *Mathematical Programming*, 62(1-3):133–151, 1993.
- [69] Jörg Kalcsics and Roger Z Ríos-Mercado. Districting problems. In Gilbert Laporte, Stefan Nickel, and Francisco Saldanha da Gama, editors, *Location Science*, pages 705–743. Springer, 2019.
- [70] Christopher T Kenny, Cory McCartan, Tyler Simko, Shiro Kuriwaki, and Kosuke Imai. Widespread partisan gerrymandering mostly cancels nationally, but reduces electoral competition. *Proceedings of the National Academy of Sciences*, 2023. To appear.
- [71] Douglas M King, Sheldon H Jacobson, and Edward C Sewell. Efficient graph contiguity and hole algorithms for geographic zoning and dynamic plane graph partitioning. *Mathematical Programming*, 149(1-2):425–457, 2015.
- [72] Douglas M King, Sheldon H Jacobson, and Edward C Sewell. The graph in practice: creating United States congressional districts from census blocks. *Computational Optimization and Applications*, 69(1):25–49, 2018.
- [73] Douglas M King, Sheldon H Jacobson, Edward C Sewell, and Wendy K Tam Cho. Geo-graphs: an efficient model for enforcing contiguity and hole constraints in planar graph partitioning. *Operations Research*, 60(5):1213–1228, 2012.
- [74] Connor Lawless and Oktay Günlük. Fair minimum representation clustering. *arXiv preprint arXiv:2302.03151*, 2023.

- [75] Justin Levitt. *A citizen's guide to redistricting*. Brennan Center for Justice at New York University School of Law, 2010.
- [76] Cory McCartan. Finding Pareto efficient redistricting plans with short bursts. *arXiv preprint arXiv:2304.00427*, 2023.
- [77] Cory McCartan and Kosuke Imai. Sequential Monte Carlo for sampling balanced and compact redistricting plans. *Annals of Applied Statistics*, 2023. To appear.
- [78] Cory McCartan, Christopher T Kenny, Tyler Simko, George Garcia III, Kevin Wang, Melissa Wu, Shiro Kuriwaki, and Kosuke Imai. Simulated redistricting plans for the analysis and evaluation of redistricting in the United States. *Scientific Data*, 9(1):689, 2022.
- [79] Anuj Mehrotra, Ellis L Johnson, and George L Nemhauser. An optimization based heuristic for political districting. *Management Science*, 44(8):1100–1114, 1998.
- [80] MGGG. GerryChain 0.2.22., 2023. <https://gerrychain.readthedocs.io/en/latest/>.
- [81] Elle Najt, Daryl DeFord, and Justin Solomon. Complexity and geometry of sampling connected graph partitions. *arXiv preprint arXiv:1908.08881*, 2019.
- [82] NCSL. Redistricting criteria. <http://www.ncsl.org/research/redistricting/redistricting-criteria.aspx>, 2021. Accessed: 2023-02-17.
- [83] Richard G Niemi, Bernard Grofman, Carl Carlucci, and Thomas Hofeller. Measuring compactness and the role of a compactness standard in a test for partisan and racial gerrymandering. *The Journal of Politics*, 52(4):1155–1181, 1990.
- [84] Johannes Oehrlein and Jan-Henrik Haunert. A cutting-plane method for contiguity-constrained spatial aggregation. *Journal of Spatial Information Science*, 2017(15):89–120, 2017.
- [85] Hayri Önal and Kevin T Patrick. A mathematical programming approach to political redistricting with compactness and community integrity considerations. Technical report, University of Illinois at Urbana-Champaign, 2016.
- [86] Kanstantsin Pashkovich. *Extended formulations for combinatorial polytopes*. PhD thesis, Otto-von-Guericke-Universität Magdeburg, 2012.
- [87] Daniel D Polsby and Robert D Popper. The third criterion: Compactness as a procedural safeguard against partisan gerrymandering. *Yale Law & Policy Review*, 9:301, 1991.

- [88] Ariel D Procaccia and Jamie Tucker-Foltz. Compact redistricting plans have many spanning trees. In *Proceedings of the 2022 Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 3754–3771. SIAM, 2022.
- [89] Daniel Rehfeldt, Henriette Franz, and Thorsten Koch. Optimal connected subgraphs: Integer programming formulations and polyhedra. *Networks*, 80(3):314–332, 2022.
- [90] Federica Ricca, Andrea Scozzari, and Bruno Simeone. Political districting: from classical models to recent approaches. *Annals of Operations Research*, 204(1):271–299, 2013.
- [91] Parker Rule, Matthew Sun, and Bhushan Suwal. mggg/gerrychainjulia, March 2021. <https://doi.org/10.5281/zenodo.4649464>.
- [92] Stephan Schwartz. An overview of graph covering and partitioning. *Discrete Mathematics*, 345(8):112884, 2022.
- [93] Maral Shahmizad and Austin Buchanan. Political districting to minimize county splits, 2023. Available on Optimization-Online.
- [94] Takeshi Shirabe. A model of contiguity for spatial unit allocation. *Geographical Analysis*, 37(1):2–16, 2005.
- [95] Takeshi Shirabe. Districting modeling with exact contiguity constraints. *Environment and Planning B: Planning and Design*, 36(6):1053–1066, 2009.
- [96] Nicholas O Stephanopoulos and Eric M McGhee. Partisan gerrymandering and the efficiency gap. *The University of Chicago Law Review*, pages 831–900, 2015.
- [97] Brendan W Sullivan. *An Introduction to the Math of Voting Methods*. 619 Wreath, 2022.
- [98] Rahul Swamy, Kiera W Dobbs, Douglas M King, Ian G Ludden, and Sheldon H Jacobson. Draft maps for Arizona’s 2021 congressional districts, 2021.
- [99] Rahul Swamy, Douglas M King, and Sheldon H Jacobson. Multiobjective optimization for politically fair districting: A scalable multilevel approach. *Operations Research*, 71(2):536–562, 2023.
- [100] Hamidreza Validi and Austin Buchanan. A note on “A linear-size zero-one programming model for the minimum spanning tree problem in planar graphs”. *Networks*, 73(1):135–142, 2019.
- [101] Hamidreza Validi and Austin Buchanan. Political districting to minimize cut edges. *Mathematical Programming Computation*, 14:623–672, 2022.

- [102] Hamidreza Validi, Austin Buchanan, and Eugene Lykhovyd. Imposing contiguity constraints in political districting models. *Operations Research*, 70(2):867–892, 2022.
- [103] William Vickrey. On the prevention of gerrymandering. *Political Science Quarterly*, 76(1):105–110, 1961.
- [104] Jacob Wachspress and William T Adler. Split decisions: Guidance for measuring locality preservation in district maps. Center for Democracy & Technology, November 2021.
- [105] Jose L Walteros. Graph partitioning. In Panos M. Pardalos and Oleg A. Prokopyev, editors, *Encyclopedia of Optimization*. Springer, 3rd edition, 2022. To appear.
- [106] Yiming Wang, Austin Buchanan, and Sergiy Butenko. On imposing connectivity constraints in integer programs. *Mathematical Programming*, 166(1-2):241–271, 2017.
- [107] James B Weaver and Sidney W Hess. A procedure for nonpartisan districting: Development of computer techniques. *Yale Law Journal*, 73:288, 1963.
- [108] Thomas Weighill and Moon Duchin. Explainer: ranked choice voting. In Moon Duchin and Olivia Walch, editors, *Political Geometry*, pages 415–421. Birkhauser, 2022.
- [109] Justin C Williams. A linear-size zero-one programming model for the minimum spanning tree problem in planar graphs. *Networks*, 39(1):53–60, 2002.
- [110] Justin C Williams. A zero-one programming model for contiguous land acquisition. *Geographical Analysis*, 34(4):330–349, 2002.
- [111] Justin C Williams Jr. Political redistricting: a review. *Papers in Regional Science*, 74(1):13–40, 1995.
- [112] H Peyton Young. Measuring the compactness of legislative districts. *Legislative Studies Quarterly*, 13(1):105–115, 1988.
- [113] Jack Zhang, Hamidreza Validi, Austin Buchanan, and Illya V. Hicks. Linear-size formulations for connected planar graph partitioning and political districting, 2022. Available on Optimization-Online.
- [114] Zhanzhan Zhao, Cyrus Hettle, Swati Gupta, Jonathan Christopher Mattingly, Dana Randall, and Gregory Joseph Herschlag. Mathematically quantifying non-responsiveness of the 2021 Georgia congressional districting plan. In *Equity and Access in Algorithms, Mechanisms, and Optimization*, EAAMO ’22, New York, NY, USA, 2022. ACM.
- [115] Andris A Zoltners and Prabhakant Sinha. Sales territory alignment: A review and model. *Management Science*, 29(11):1237–1256, 1983.