IEM 4013: Two MIPs for redistricting

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Abstract

In the Spring 2022 semester of IEM 4013 (Undergraduate Operations Research), the course project is to create congressional redistricting plans. This document provides two mixed integer programming (MIP) models for redistricting that students may want to use *as starting points* for their project. Parts of this document are reproduced from two papers:

- H. Validi, A. Buchanan, E. Lykhovyd. Imposing contiguity constraints in political districting models. *Operations Research* 70(2):867–892, 2022.
- H. Validi, A. Buchanan. Political districting to minimize cut edges. To appear at *Mathematical Programming Computation*, 2022.

1 Input Data

A map of Oklahoma's counties is given in Figure 1.

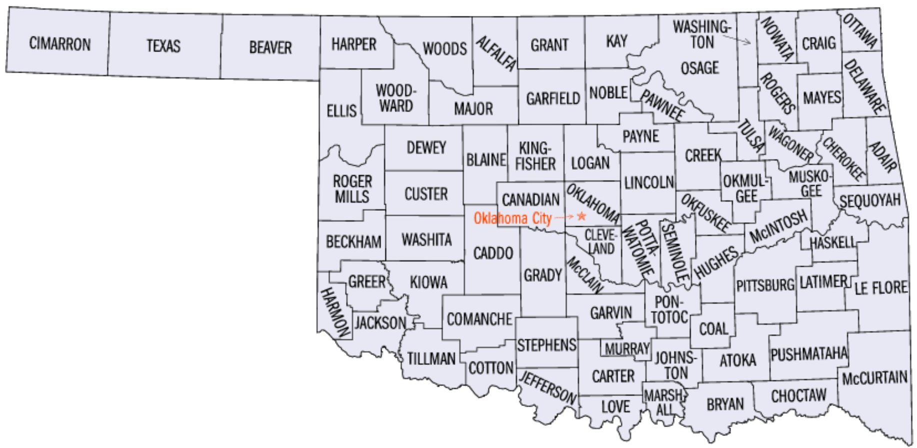

Figure 1: A map of Oklahoma's 77 counties, from Wikipedia.

When trying to impose the contiguity constraints involved in political districting, it is helpful to use the so-called *contiguity graph* [12], also known as the adjacency graph or dual graph. In this graph $G = (V, E)$, each vertex $v \in V$ represents a contiguous geographic unit (e.g., a county or census tract), and there is an edge $\{u, v\} \in E$ connecting vertices u and v when the corresponding geographic units share a border of nonzero length; it is not enough to meet at a point.

The contiguity graph for Oklahoma's 77 counties is depicted in Figure 2.

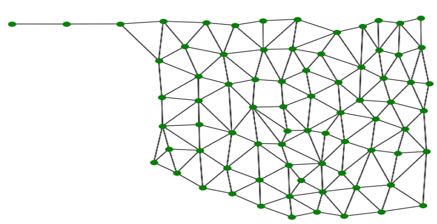

Figure 2: The contiguity graph for Oklahoma's counties, via Daryl DeFord [3].

The set of neighbors of vertex $i \in V$ in the graph G is denoted by

$$N(i) = \{j \in V \mid \{i, j\} \in E\}.$$

For example, Payne County (which is where OSU is located) neighbors the counties: Pawnee, Creek, Lincoln, Logan, and Noble. This is shown in Figure 3.

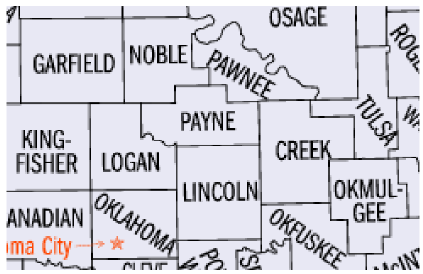

Figure 3: Payne County and vicinity.

We might denote this by

$$N(\text{Payne}) = \{\text{Pawnee}, \text{Creek}, \text{Lincoln}, \text{Logan}, \text{Noble}\}.$$

Other input data includes:

- the number k of districts to be created;
- the population p_v of each land parcel $v \in V$;
- the minimum and maximum population (L and U) allowed in a district.

For example, after the 2010 Census, Oklahoma had a total population of 3,751,351 which must be split into $k = 5$ congressional districts of (roughly) equal population. Thus, the ideal district population is $\bar{p} = 750,270.2$. This fractional population cannot be achieved, and so the bounds L and U must allow some flexibility. A common rule-of-thumb for congressional districting is to allow a 1% population deviation ($+/-0.5\%$), see [5], meaning that

$$L = 0.995(750,270.2) \approx 746,519$$
$$U = 1.005(750,270.2) \approx 754,021.$$

Note, however, that a majority of states created their congressional districts with $L = \lfloor \bar{p} \rfloor$ and $U = \lceil \bar{p} \rceil$, where $\bar{p}$ is the ideal district population [10]. For Oklahoma, this means setting $L = 750,270$ and $U = 750,271$ (which it did!).

For a districting plan to be feasible, it should satisfy the following constraints *at bare minimum*. Geographic units V should be partitioned into *districts* such that:

1. each geographic unit belongs to exactly one district;
2. there are k districts;
3. each district D satisfies the population bounds, i.e., $L \leq \sum_{i \in D} p_i \leq U$;
4. each district D is contiguous on the map, i.e., $G[D]$ is connected.

The last constraint says that the *subgraph induced by* D, which is denoted by $G[D]$ should be connected. This induced subgraph $G[D]$ has the vertices of D and all edges from E that have both endpoints in D.

For example, the subgraph induced by the "district" $D = \{\text{Payne}, \text{Creek}, \text{Tulsa}\}$ is connected and has two edges: $\{\text{Payne}, \text{Creek}\}$ and $\{\text{Creek}, \text{Tulsa}\}$. This is shown in Figure 4.

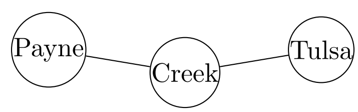

Figure 4: A connected subgraph, induced by $D = \{\text{Payne}, \text{Creek}, \text{Tulsa}\}$.

Meanwhile, the subgraph induced by the "district" $D = \{\text{Payne}, \text{Lincoln}, \text{Tulsa}\}$ is disconnected and has one edge: $\{\text{Payne}, \text{Lincoln}\}$. This is shown in Figure 5.

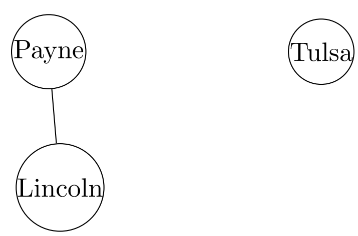

Figure 5: A disconnected subgraph, induced by $D = \{\text{Payne}, \text{Lincoln}, \text{Tulsa}\}$.

2 Hess model (moment-of-inertia compactness)

Hess et al. [6] introduced an IP model for political districting. It uses the following n^2 binary variables, where $n = |V|$ is the number of land parcels.

$$x_{ij} = \begin{cases} 1 & \text{if vertex } i \text{ is assigned to (the district centered at) vertex } j \\ 0 & \text{otherwise.} \end{cases}$$

The IP model is as follows, where objective seeks compact districting plans (defined in terms of moment-of-inertia, as described below).

$$\min \sum_{i \in V} \sum_{j \in V} w_{ij} x_{ij} \tag{1a}$$

$$\sum_{j \in V} x_{ij} = 1 \qquad \forall i \in V \tag{1b}$$

$$\sum_{j \in V} x_{jj} = k \tag{1c}$$

$$L x_{jj} \leq \sum_{i \in V} p_i x_{ij} \leq U x_{jj} \qquad \forall j \in V \tag{1d}$$

$$x_{ij} \leq x_{jj} \qquad \forall i, j \in V \tag{1e}$$

$$x_{ij} \in \{0, 1\} \qquad \forall i, j \in V. \tag{1f}$$

Constraints (1b) ensure that each vertex is assigned to one district. Constraint (1c) ensures that k districts are chosen. Constraints (1d) ensure that the population of each district lies between L and U. Constraints (1e) were not originally included by [6] but are usually added for LP strength [12].

The moment-of-inertia objective function (1a) treats the districts as physical bodies, each with a centroid. Land parcels from a district are located either at the centroid j, or elsewhere at location i at some positive (Euclidean) distance d_{ij} from the centroid j. The mass of parcel i is its population. Then, to capture the moment-of-inertia, Hess et al. define penalties w_{ij} as follows.

$$\text{(penalty for moment-of-inertia objective)} \qquad w_{ij} := p_i d_{ij}^2.$$

Figure 6 gives solutions for the county-level and tract-level.

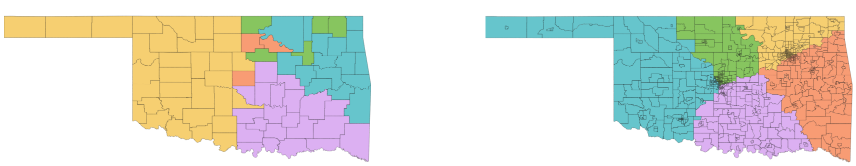

Figure 6: Optimal solutions of the Hess model (moment-of-inertia objective) for Oklahoma at the county-level and tract-level.

2.1 Adding contiguity constraints

The model of Hess et al. does not impose contiguity. For that, we can turn to some "flow" constraints, conceptualized by Shirabe [13, 14] and detailed by Oehrlein and Haunert [11]. See also Validi et al. [16]. The idea is to create flow at each district center, send this flow along the district's edges, and then consume one unit of flow at the district's other nodes. It uses flow variables:

$$f_{ij}^v = \text{the amount of flow, originating at district center } v,$$
$$\text{that is sent across edge } \{i,j\} \text{ (from } i \text{ to } j).$$

Then, to impose that the districts of the Hess model are contiguous, we can add the following constraints.

$$\sum_{u \in N(i)} (f_{ui}^j - f_{iu}^j) = x_{ij} \qquad \forall i \in V \setminus \{j\},\ \forall j \in V \tag{2a}$$

$$\sum_{u \in N(i)} f_{ui}^j \le (n-1)x_{ij} \qquad \forall i \in V \setminus \{j\},\ \forall j \in V \tag{2b}$$

$$\sum_{u \in N(i)} f_{uj}^j = 0 \qquad \forall j \in V \tag{2c}$$

$$f_{ij}^v,\ f_{ji}^v \ge 0 \qquad \forall \{i,j\} \in E,\ \forall v \in V. \tag{2d}$$

Constraints (2a) ensure that if vertex i is assigned to center j, then i consumes one unit of flow of type j; otherwise, it consumes none. Constraints (2b) ensure that vertex i can receive flow of type j only if i is assigned to center j. Constraints (2c) prevent flow circulations.

Figure 7 gives solutions for the county-level and tract-level with contiguity.

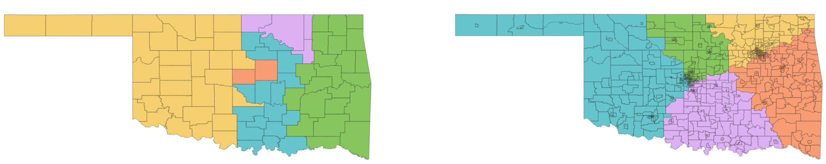

Figure 7: Optimal solutions of the Hess model (moment-of-inertia objective) *with Shirabe flow constraints* for Oklahoma at the county-level and tract-level.

3 Labeling model (cut edges compactness)

Figure 8 provides two ways to partition the 4x4 grid graph into 4 contiguous districts having equal numbers of nodes. If compactness were measured via cut edges, then the *columns* plan, which cuts 12 edges, is the least compact plan possible. Meanwhile, the *squares* plan, which cuts 8 edges, is the most compact.

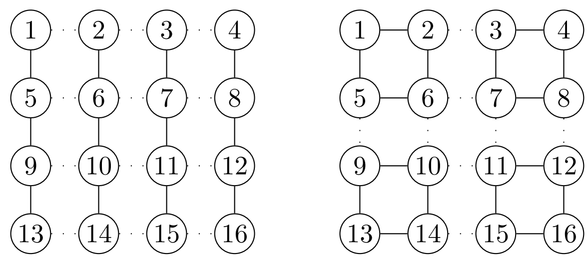

Figure 8: The *columns* plan cuts 12 edges, while the *squares* plan cuts 8 edges.

The edges $\{i, j\} \in E$ that are "cut" are those whose endpoints i and j belong to different districts. Intuitively, the cut edges are those edges that would need to be snipped with a pair of scissors to break the graph into its districts.

If we would like to minimize the number of cut edges in our districting plan, it is convenient to use the labeling model, which uses binary variables:

$$x_{ij} = \begin{cases} 1 & \text{if vertex } i \in V \text{ is assigned to district } j \in \{1, 2, \ldots, k\} \\ 0 & \text{otherwise.} \end{cases}$$

$$y_e = \begin{cases} 1 & \text{if edge } e \in E \text{ is cut} \\ 0 & \text{otherwise.} \end{cases}$$

This leads to the following IP model, see [1, 2, 4, 8, 14, 15].

$$\min \sum_{e \in E} y_e \tag{3a}$$

$$x_{uj} - x_{vj} \leq y_e \qquad \forall e = \{u, v\} \in E,\ \forall j \in \{1, 2, \ldots, k\} \tag{3b}$$

$$\sum_{j=1}^{k} x_{ij} = 1 \qquad \forall i \in V \tag{3c}$$

$$L \leq \sum_{i \in V} p_i x_{ij} \leq U \qquad \forall j \in \{1, 2, \ldots, k\} \tag{3d}$$

$$x_{ij} \in \{0, 1\} \qquad \forall i \in V,\ \forall j \in \{1, 2, \ldots, k\} \tag{3e}$$

$$y_e \in \{0, 1\} \qquad \forall e \in E. \tag{3f}$$

The objective (3a) minimizes the number of cut edges. Constraints (3b) indicate that edge $e = \{u, v\}$ is cut if vertex u but not $v \in V$ is assigned to district j. Constraints (3c) ensure that each vertex $i \in V$ is assigned to one district. Constraints (3d) ensure that each district's population is between L and U.

As written, the labeling model lacks contiguity constraints. If we solve it for Oklahoma, we get the solution depicted in Figure 9.

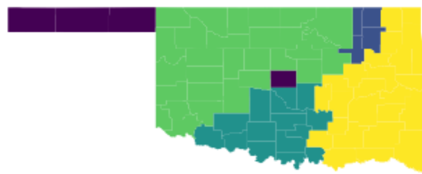

Figure 9: Optimal solution of the Labeling model (cut edge objective).

3.1 Adding contiguity constraints

For imposing contiguity, we will consider a single-commodity flow (SCF) formulation that is written over the labeling variables [7], cf. [9]. It has flow variables f_{ij} and f_{ji} associated with each edge $\{i, j\} \in E$. There are also binary variables r_{ij} indicating whether vertex $i \in V$ is the root of district $j \in [k]$.

$$\sum_{i \in V} r_{ij} = 1 \qquad \forall j \in \{1, 2, \ldots, k\} \tag{4a}$$

$$r_{ij} \leq x_{ij} \qquad \forall i \in V, \ \forall j \in \{1, 2, \ldots, k\} \tag{4b}$$

$$\sum_{u \in N(i)} (f_{ui} - f_{iu}) \geq 1 - M \sum_{j=1}^{k} r_{ij} \qquad \forall i \in V \tag{4c}$$

$$f_{ij} + f_{ji} \leq M(1 - y_e) \qquad \forall e = \{i, j\} \in E \tag{4d}$$

$$f_{ij}, \ f_{ji} \geq 0 \qquad \forall \{i, j\} \in E \tag{4e}$$

$$r_{ij} \in \{0, 1\} \qquad \forall i \in V, \ \forall j \in \{1, 2, \ldots, k\}, \tag{4f}$$

where Hojny et al. [7] mention setting $M = n - k + 1$. Constraints (4a) force each district to have one root. Constraints (4b) state that vertex $i \in V$ cannot root a district j to which it does not belong. Constraints (4c) force vertex i to consume flow if it is not a root. Constraints (4d) disallow flow across cut edges.

By adding these constraints, we get the map in Figure 10.

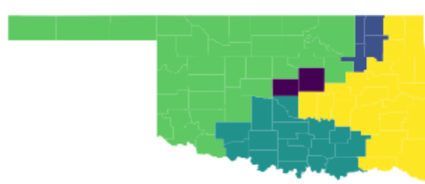

Figure 10: Optimal solution of the Labeling model (cut edge objective) for Oklahoma at the county-level with single-commodity flow constraints for contiguity.

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