Generating valid inequalities through aggregation and rounding

Recall the *stable set* polytope STAB(G) of a graph $G = (V, E)$:

$$\begin{aligned} \mathrm{STAB}(G) :&= \mathrm{conv}\left\{x^S \;\middle|\; S \subseteq V \text{ is a stable set}\right\} && (1)\\ &= \mathrm{conv}\left\{x \in \{0,1\}^n \mid x_i + x_j \leq 1,\ \forall \{i,j\} \in E\right\}. && (2) \end{aligned}$$

Here, x^S denotes the characteristic vector of S, i.e.,

$$x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S. \end{cases}$$

Consider the 3-node complete graph K_3 with vertices $\{1, 2, 3\}$. Prove that the *clique* inequality $x_1 + x_2 + x_3 \leq 1$ is valid for STAB(G) *only using information from equation* (2).

Consider the 5-node cycle graph C_5 with vertices $\{1, 2, 3, 4, 5\}$. Prove that the *odd-hole* inequality $x_1 + x_2 + x_3 + x_4 + x_5 \leq 2$ is valid for STAB(G) *only using information from equation* (2).

Proposition 1. *For any clique $C \subseteq V$, the inequality $\sum_{i \in C} x_i \leq 1$ is valid for* $\mathrm{STAB}(G)$.

Consider the IP feasible region:

$$\begin{aligned} 7x_1 - 2x_2 &\leq 14 \\ x_2 &\leq 3 \\ 2x_1 - 2x_2 &\leq 3 \\ x \geq 0&,\ \text{integer.} \end{aligned}$$

Weight the constraints with multipliers $(\frac{2}{7}, \frac{37}{63}, 0)$ to obtain a valid inequality *for the LP relaxation*.

Round the coefficients on the left-hand-side (LHS) down to be integers to get:

Why can we safely round down?

By integrality of LHS, generate an inequality that is valid for the IP (but perhaps not for the LP):

Can you generate a stronger inequality?

The Chvátal procedure for generating valid inequalities

Consider an IP feasible region of the form:

$$X := \left\{ x \in \mathbb{Z}_+^n \;\middle|\; \sum_{j=1}^{n} a_{ij} x_j \leq b_i,\ i = 1, \ldots, m \right\}.$$

For any $u \in \mathbb{R}_+^m$ we can generate the following inequality which is valid for the LP relaxation:

In other words,

Since x is nonnegative, we can write the (weaker) inequality:

The LHS is integer (since x is integer), so we can write:

We have just argued for the validity of the Chvátal inequality $\lfloor uA \rfloor x \leq \lfloor ub \rfloor$.

Remark 1. *Chvátal inequalities are sometimes called Chvátal-Gomory (CG) inequalities.*

The Chvátal closure of a polyhedron

Consider a pure integer set $S := P \cap \mathbb{Z}^n$ where $P := \{x \in \mathbb{R}^n_+ \mid Ax \leq b\}$ is a rational polyhedron.

Definition 1. *The Chvátal closure of P, denoted $P^{(1)}$, is the set of points that satisfy all Chvátal inequalities constructed by applying the Chvátal procedure to the original inequalities, i.e.,*

$$P^{(1)} := \left\{x \in \mathbb{R}^n_+ \;\middle|\; \lfloor uA \rfloor x \leq \lfloor ub \rfloor, \; \forall u \in \mathbb{R}^m_+ \right\}.$$

When P is rational, $P^{(1)}$ will be rational. This needs proof; see Chvátal (1973). And, in some cases, $P^{(1)}$ is integral. But not always! This motivates the 2nd Chvátal closure $P^{(2)}$, which is:

In general, the rank t Chvátal closure $P^{(t)}$ is:

Similarly, we can define the Chvátal rank of a valid inequality to be the smallest t such that it belongs to definition of $P^{(t)}$.

What can we say about the Chvátal rank of the k-clique inequalities for the stable set polytope?

Does every valid inequality for the integer hull $P_I := \text{conv}(S)$ have finite rank? In other words, is there always some t for which $P^{(t)} = P_I$? YES!

Theorem 1 (Chvátal, 1973)**.** *Every valid inequality for S can be obtained by applying the Chvátal procedure a finite number of times. Thus, there exists a finite t such that $P^{(t)} = P_I$.*

When $P \subseteq [0,1]^n$, $t = O(n^2 \log n)$ is sufficient (Eisenbrand and Schulz, 2003) and there are examples with $t = \Omega(n^2)$, see Rothvoss and Sanità (2013).