Homework 2 – Due 6 June 2022

1. Formulate the following as MIPs:

   (a) $u = \min\{x_1, x_2\}$, assuming that $0 \le x_j \le C,\ j = 1, 2$.

   (b) $v = |x_1 - x_2|$ with $0 \le x_j \le C,\ j = 1, 2$.

   (c) the set $X \setminus \{x^*\}$ where $X = \{x \in \mathbb{Z}^n \mid Ax \le b\}$ and $x^* \in X$ is a given integer vector.

2. Draw the convex hull of the following 2-variable integer sets. Note that the first set is pure integer, while the second is mixed. (Distinguish the feasible set S from its convex hull by coloring the points in S.)

$$
\begin{aligned}
S &:= \{x \in \mathbb{Z}^2_+ \mid x_1 + x_2 \le 2, x_1 - x_2 \le 1, x_2 - x_1 \le 1\} \\
S &:= \{x \in \mathbb{Z}_+ \times \mathbb{R}_+ \mid x + y \ge 1.6, x \le 2, y \le 2\}.
\end{aligned}
$$

3. List three branching strategies (e.g., most infeasible branching) and briefly describe the main idea behind each one. For this question, you can use internet resources (e.g., research papers, Wikipedia) to find information, but your answer should be self-contained and in your own words. Note that branching strategies (which variable to branch on) are distinct from node selection strategies (which node in the branch-and-bound tree should be explored next).

4. Show that the following three sets are equal, i.e., that $S_1 = S_2 = S_3$.

$$
\begin{aligned}
S_1 &:= \{x \in \{0,1\}^4 \mid 90x_1 + 35x_2 + 26x_3 + 25x_4 \le 138\} \\
S_2 &:= \{x \in \{0,1\}^4 \mid 2x_1 + x_2 + x_3 + x_4 \le 3\} \\
S_3 &:= \{x \in \{0,1\}^4 \mid 2x_1 + x_2 + x_3 + x_4 \le 3, \\
&\qquad\qquad\qquad\quad x_1 + x_2 + x_3 \phantom{{}+x_4} \le 2, \\
&\qquad\qquad\qquad\quad x_1 + x_2 \phantom{{}+x_3} + x_4 \le 2, \\
&\qquad\qquad\qquad\quad x_1 \phantom{{}+x_2} + x_3 + x_4 \le 2\}.
\end{aligned}
$$

   Next, can you rank these three formulations in terms of the tightness of their linear relaxations? Formally, denote by P_1, P_2, P_3 the associated linear relaxations in which the binary restriction $x \in \{0,1\}^4$ is replaced by $x \in [0,1]^4$. You do **not** need to use the words from our definitions (e.g., "at least as strong as"); instead just write the inclusions (e.g., $P \subseteq Q$). Prove each inclusion that you claim, also showing any strict inclusions (e.g., $P \subsetneq Q$).

5. Recall the two different TSP formulations from Homework 1: cut-based and cycle-elimination. Denote by P_{cut} the feasible region of the cut-based LP relaxation. Similarly, denote by P_{cycle} the feasible region of the cycle-elimination LP relaxation.

   (a) Rigorously define the polytopes P_{cut} and P_{cycle} using set-builder notation;

   (b) Prove that these polytopes are actually the same, i.e., that $P_{cut} = P_{cycle}$. Suggestion: prove $P_{cut} = P_{cycle}$ by showing that $P_{cut} \subseteq P_{cycle}$ and $P_{cut} \supseteq P_{cycle}$.