Examples of Perfect Formulations

Definition 1. *A matching in a graph $G = (V, E)$ is a subset $E' \subseteq E$ of edges that are pairwise disjoint. In other words, E' is a matching if the graph $G' = (V, E')$ has maximum degree at most 1.*

The maximum (cardinality) matching problem admits the following IP formulation, where $m = |E|$.

$$\max \sum_{e \in E} x_e$$
$$\sum_{e \in \delta(v)} x_e \leq 1, \;\; v \in V$$
$$x \in \{0,1\}^m.$$

Theorem 1. *The constraint matrix for the degree-based matching formulation is TUM if and only if the graph is bipartite.*

In general, the degree-based matching formulation is not integral. For example, consider the case where $G = C_3$ or $G = C_5$.

This observation leads to Edmonds' *Blossom inequalities*, where $E[U] := \{\{u, v\} \in E \mid u, v \in U\}$.

$$\sum_{e \in E[U]} x_e \leq \frac{|U| - 1}{2}, \quad U \subseteq V, |U| \text{ odd.}$$

Argue that blossom inequalities are CG inequalities.

Theorem 2 (Edmonds, 1965)**.** *The following is a perfect formulation for matchings.*

$$\begin{aligned} \sum_{e \in \delta(v)} x_e &\leq 1, && v \in V \\ \sum_{e \in E[U]} x_e &\leq \frac{|U| - 1}{2}, && U \subseteq V,\ |U| \text{ odd} \\ x &\geq 0. \end{aligned}$$

Edmonds proved this algorithmically. Later, the system was proven totally dual integral (TDI).

Matroids and their polyhedra

A classical example of a perfect formulation is for matroids, as shown by Edmonds. Matroids generalize the notion of linear independence in vector spaces.

Definition 2 (Whitney, 1935)**.** *A matroid M is a pair $(I, \mathcal{I})$, where I is a finite "ground" set and $\mathcal{I}$ is a collection of subsets of I that satisifes the following properties:*

1. *(nonemptiness) $\emptyset \in \mathcal{I}$;*
2. *(heredity) if $J \in \mathcal{I}$ and $J' \subseteq J$, then $J' \in \mathcal{I}$;*
3. *(augmentation/exchange) if $S, B \in \mathcal{I}$ and $|S| < |B|$, then there exists $b \in B \setminus S$ such that $S \cup \{b\} \in \mathcal{I}$.*

If $(I, \mathcal{I})$ satisfies the first two properties, it is called an independence system.

Definition 3 (rank)**.** *If $(I, \mathcal{I})$ is an independence system, the* rank *of a subset $J \subseteq I$ is:*

$$r(J) := \max\{|J'| : J' \in \mathcal{I},\ J' \subseteq J\}.$$

Exercise. Give a matroid over the ground set $I = \{1, 2, 3\}$ for which $r(I) = 2$.

Exercise. Give an independence system over $I = \{1, 2, 3\}$ with $r(I) = 2$ that is not a matroid.

Remark 1. *Independence systems are common in combinatorial optimization. Examples include:*

- *feasible solutions to a 0-1 knapsack problem;*
- *the set of cliques in a graph.*

Exercise. Suppose $G = (V, E)$ is a simple graph. Prove that $(E, \mathcal{E})$ is a matroid, where $\mathcal{E}$ contains those edge subsets E' for which (V, E') is a forest. (This is called the graphic matroid.)

Matroids and the greedy algorithm

Definition 4. *If $M = (I, \mathcal{I})$ is a matroid, then a subset $J \subseteq I$ with $J \in \mathcal{I}$ and $r(J) = r(I)$ is called a basis. In other words, it is of maximum cardinality.*

What are the bases for the graphic matroid assuming G is connected?

Theorem 3 (Edmonds, 1971)**.** *The following greedy algorithm finds a minimum cost basis for a matroid $M = (I, \mathcal{I})$ where $i \in I$ has cost c_i:*

1. *$S \leftarrow \emptyset$*
2. *while S is not a basis do:*
   (a) *pick $i \in I \setminus S$ of minimum cost c_i such that $S \cup \{i\} \in \mathcal{I}$;*
   (b) *$S \leftarrow S \cup \{i\}$;*
3. *return S.*

If you apply this greedy algorithm to the graphic matroid, what algorithm do you get?

Theorem 4 (Edmonds, 1971)**.** *If $M = (I, \mathcal{I})$ is a matroid with rank function r, then the matroid polytope*

$$P_M := \operatorname{conv}\left\{x^J \;\middle|\; J \in \mathcal{I}\right\}$$

is fully described by the nonnegativity bounds and rank inequalities, i.e.,

$$P_M = \left\{ x \in \mathbb{R}_+^{|I|} \;\middle|\; \sum_{j \in J} x_j \leq r(J),\ \forall J \subseteq I \right\}.$$

Given this theorem, how would you get a perfect formulation for bases of a matroid?

Corollary 1. *The spanning tree polytope of $G = (V, E)$ is fully described by the inequalities:*

$$\sum_{e \in E} x_e = n - 1 \tag{1}$$

$$\sum_{e \in E[U]} x_e \leq |U| - 1,\ U \subseteq V,\ 2 \leq |U| \leq n - 1 \tag{2}$$

$$x_e \geq 0. \tag{3}$$