Homework 4 – Due 20 June 2022

1. Consider the polyhedron P given below.

$$P := \{(x_1, x_2, y) \in \mathbb{R}^3 \mid x_1 \geq y, x_2 \geq y, x_1 + x_2 + 2y \leq 2, y \geq 0\}.$$

   Let $S := P \cap (\mathbb{Z}^2 \times \mathbb{R})$. Prove that $x_1 \geq 3y$ and $x_2 \geq 3y$ are split inequalities.

2. Consider the following MIP:

$$\begin{array}{rlllllllll}
z = & \max & 7x_1 & +5x_2 & +x_3 & +y_1 & & & & \\
 & & x_1 & +3x_2 & & +4y_1 & +y_2 & & & = 11 \\
 & & 5x_1 & +x_2 & +3x_3 & & & +y_3 & & = 12 \\
 & & & & 2x_3 & +2y_1 & & & -y_4 & = 3 \\
 & & x_1, & x_2, & x_3 & \in \mathbb{Z}_+ & & & & \\
 & & y_1, & y_2, & y_3, & y_4 & \in \mathbb{R}_+. & & &
\end{array}$$

   The optimal tableau of the LP relaxation is:

$$\begin{array}{rrrrrcr}
z & +0.357x_3 & +1.286y_2 & +1.143y_3 & +2.071y_4 & = & 21.643 \\
x_1 & +0.786x_3 & -0.071y_2 & +0.214y_3 & -0.143y_4 & = & 2.214 \\
x_2 & -0.929x_3 & +0.357y_2 & -0.071y_3 & +0.714y_4 & = & 0.929 \\
y_1 & +0.500x_3 & & & -0.500y_4 & = & 1.500
\end{array}$$

   (a) The optimal LP solution is $x_1 = 2.214$, $x_2 = 0.929$, $y_1 = 1.5$ and $x_3 = y_2 = y_3 = y_4 = 0$. Use the equations where x_1 and x_2 are basic to derive two Gomory mixed integer cuts.

   (b) The coefficients in the above optimal simplex tableau are rounded to three decimal places. Discuss how this may affect the validity of the GMI cuts you generated above.

3. Give a polyhedron $P \subseteq \mathbb{R}^2$ that has:

   (a) $\dim(P) = -1$;

   (b) $\dim(P) = 0$;

   (c) $\dim(P) = 1$;

   (d) $\dim(P) = 2$.

4. Construct a polyhedron $P \subseteq \mathbb{R}^2$ (that is *not* of full dimension) for which a redundant inequality is facet-defining. (This is one reason why we prefer to work with full-dimensional polyhedra!)

5. Recall the wheel inequalities for stable set from Homework 3. They are defined for *wheel* graphs W_p consisting of a cycle on an odd $p-1$ number of vertices along with a "hub" vertex (labeled node 1) that is adjacent to all of the other $p-1$ nodes. (Thus p is even.) The inequalities take the form

$$\frac{p-2}{2}x_1 + \sum_{i=2}^{p} x_i \leq \frac{p-2}{2}. \tag{1}$$

- Rigorously prove that the stable set polytope STAB(G) of an *arbitrary* n-vertex graph $G = (V, E)$ is full-dimensional. I want to see your $n + 1$ points and a complete proof that they are affinely independent.
- Prove that the wheel inequality (1) induces a facet of STAB(W_p)—when p is even—directly using the definition of a facet-defining inequality. You can assume that the inequality is valid, since that was covered on Homework 3.
- Prove that the wheel inequality (1) induces a facet of STAB(W_p)—when p is even—using lifting arguments. Be sure to identify the *seed* inequality (and quickly argue why it is valid for the restricted polytope), identify the variable that you are lifting, and *explicitly* state the lifting problem.