Integral/Perfect formulations

We would like to know when a formulation is "ideal" or "perfect" (or "integral" in the pure integer case). In other words, when does solving the LP relaxation solve the underlying MIP too?

Definition 1. *A perfect formulation of a mixed integer set $S \subseteq \mathbb{Z}^n \times \mathbb{R}^d$ is a system $Ax + Gy \le b$ of linear inequalities such that* $\operatorname{conv}(S) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^d \mid Ax + Gy \le b\}$.

Definition 2. *A polyhedron $P \subseteq \mathbb{R}^n$ is said to be integral if $P = \operatorname{conv}(P \cap \mathbb{Z}^n)$.*

How can we show that $Ax + Gy \le b$ is a perfect formulation?

Theorem 1. *Let $P := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^d \mid Ax + Gy \le b\}$ be a rational polyhedron and let $S := P \cap (\mathbb{Z}^n \times \mathbb{R}^d)$. The following are equivalent.*

(i) P is a perfect formulation for S, i.e., $P = \operatorname{conv}(S)$;

(ii) Every nonempty face of P contains a point of S;

(iii) Every extreme point (i.e., vertex) of P belongs to S;

(iv) For every $(c, h) \in \mathbb{R}^n \times \mathbb{R}^d$ such that $z := \max\{cx + hy \mid (x, y) \in P\}$ is finite, there is a point $(x^*, y^*) \in S$ for which $z = cx^* + hy^*$.

Definition 3 (extreme point)**.** *A point x from a polyhedron P is an extreme point (a.k.a. vertex) of P if $\{x\}$ is a face of P. More generally, a point x belonging to a convex set P is said to be an extreme point if it cannot be expressed as a strict convex combination of distinct points from P.*

Equivalently, you should not be able to write $x = \lambda x^1 + (1 - \lambda) x^2$ for $\lambda \in (0, 1)$; $x^1 \neq x^2$; $x^1, x^2 \in P$.

Example. Use Theorem 1(iii) to show that P, defined as the set of $(x, y) \in \mathbb{R}^m \times \mathbb{R}$ satisfying:

$$\begin{aligned} &0 \le y \le 1 \\ &0 \le x_i \le y, \;\; i = 1, \ldots, m. \end{aligned}$$

is a perfect formulation for

$$S := \left\{ (x, y) \in \mathbb{R}^m \times \{0, 1\} \;\middle|\; \mathbf{0} \le x \le \mathbf{1}, \;\; \sum_{i=1}^{m} x_i \le my \right\}.$$

Example. Use Theorem 1(iv) to show that P, defined as the set of $(x, y) \in \mathbb{R}^m \times \mathbb{R}$ satisfying:

$$\begin{aligned} 0 \le y \le 1 \\ 0 \le x_i \le y, \; i = 1, \ldots, m. \end{aligned}$$

is a perfect formulation for

$$S := \left\{ (x, y) \in \mathbb{R}^m \times \{0, 1\} \;\middle|\; \mathbf{0} \le x \le \mathbf{1}, \; \sum_{i=1}^{m} x_i \le my \right\}.$$

Total Unimodularity

Definition 4. *A matrix A is said to be totally unimodular (TUM) if the determinant of every square submatrix belongs to $\{0, +1, -1\}$.*

Example. Determine which of the following matrices are totally unimodular and why.

$$\begin{bmatrix} -1 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Why do we care about TUM matrices?

Theorem 2 (Hoffman and Kruskal, 1956)**.** *Let A be an $m \times n$ integral matrix. The polyhedron $P(b) := \{x \mid Ax \leq b,\ x \geq 0\}$ is integral for every $b \in \mathbb{Z}^m$ if and only if A is totally unimodular.*

Notice that TUM refers to the constraint matrix, and not to the formulation as a whole.

Proposition 1. *The following are equivalent regarding a matrix A.*

1. *A is totally unimodular.*
2. *A^T is totally unimodular.*
3. *(A, I) is totally unimodular.*

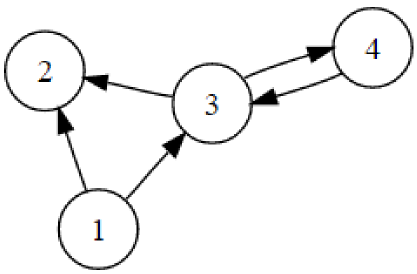
Theorem 3. *Let A be an $m \times n$ integral matrix. The polyhedron $Q := \{x \mid c \leq Ax \leq d,\ l \leq x \leq u\}$ is integral for all integral vectors c, d, l, u if and only if A is totally unimodular.*

Applications of TUM

Consider directed graph $G = (V, A)$. The (node-arc) incidence matrix B_G of G has dimension $n \times m$, where $n = |V|$ and $m = |A|$. There is a row for each vertex v and column for each arc $a = (u, w)$. The associated entry B_{va} takes a value as follows:

$$B_{va} = \begin{cases} -1 & \text{if } v = u \\ 1 & \text{if } v = w \\ 0 & \text{o.w.} \end{cases}$$

Example. Give the incidence matrix for this graph.



Theorem 4. *The incidence matrix of a directed graph is totally unimodular.*

By this theorem, many LP relaxations for network problems are integral. For example, consider the following formulation for the minimum cost flow problem.

Remark 1. *The constraint matrix associated with the flow balance constraints above is the graph's incidence matrix. Thus, the above formulation is integral when b and u are integral.*

Using TUM for non-network problems

Exercise. Use total unimodularity to show that P, defined as the set of $(x, y) \in \mathbb{R}^m \times \mathbb{R}$ satisfying:

$$\begin{aligned} &0 \le y \le 1 \\ &0 \le x_i \le y, \;\; i = 1, \ldots, m. \end{aligned}$$

is integral and is hence a perfect formulation for

$$S := \left\{ (x, y) \in \mathbb{R}^m \times \{0, 1\} \;\middle|\; \mathbf{0} \le x \le \mathbf{1}, \;\; \sum_{i=1}^{m} x_i \le my \right\}.$$