Homework 1 – Due 30 May 2022

1. Solve the following pure integer program by branch-and-bound. Draw the initial LP relaxation and solve LPs by inspection. Tell why each branch-and-bound node is pruned.

$$\begin{aligned} \max\ & 2x_1 + x_2 \\ & -x_1 + x_2 \leq 0 \\ & 6x_1 + 2x_2 \leq 21 \\ & x_1, x_2 \geq 0 \text{ integer.} \end{aligned}$$

2. A firm is considering projects $A, B, \ldots, H$. Using binary variables $x_a, x_b, \ldots, x_h$ and linear constraints, model the following conditions on the projects to be undertaken.

   (a) At most one of $A, B, \ldots, H$.
   (b) Exactly two of $A, B, \ldots, H$.
   (c) If A then B.
   (d) If A then not B.
   (e) If not A then B.
   (f) If A then B, and if B then A.
   (g) If A then B and C.
   (h) If A then B or C.
   (i) If B or C then A.
   (j) If B and C then A.
   (k) If two or more of $B, C, \ldots, H$ then A.

3. Given jobs $j = 1, 2, \ldots, n$ with release date r_j, due date d_j, and processing time p_j, sequence the jobs to minimize the makespan (on a single machine).

4. Below is a cut-based formulation for the symmetric TSP. Here, the input graph $G = (V, E)$ is a complete undirected graph (i.e., $E = \binom{V}{2}$), and c_e is the length of edge $e \in E$. For each edge $e \in E$, there is a binary variable x_e. When $S \subseteq V$ is a subset of vertices, denote by $\delta(S)$ the subset of edges that have one endpoint in S and one endpoint in $V \setminus S$. In a slight abuse of notation, we write $\delta(v)$ instead of $\delta(\{v\})$ when $v \in V$ is a single vertex.

$$\min \sum_{e \in E} c_e x_e \tag{1}$$

$$\sum_{e \in \delta(v)} x_e = 2 \qquad \forall v \in V \tag{2}$$

$$\sum_{e \in \delta(S)} x_e \geq 2 \qquad \forall S \subseteq V,\ 2 \leq |S| \leq |V| - 2 \tag{3}$$

$$x_e \in \{0, 1\} \qquad \forall e \in E. \tag{4}$$

(a) Rigorously prove that this formulation is *correct*. In other words, show that an edge subset $C \subseteq E$ is an undirected Hamiltonian cycle if and only if its characteristic vector x^C satisfies the formulation's constraints. Here, the characteristic vector of C has:

$$x_e^C = \begin{cases} 1 & \text{if } e \in C \\ 0 & \text{otherwise.} \end{cases}$$

You can take as given that an undirected Hamiltonian cycle is defined as a subset E' of edges such that: (i) every vertex of the subgraph (V, E') has degree two, and (ii) the subgraph (V, E') is connected.

(b) Rigorously prove that a similar formulation that uses *cycle elimination constraints* instead of cut constraints (3) is also correct. Cycle elimination constraints have the form[1]

$$\sum_{e \in E(S)} x_e \leq |S| - 1,$$

where $S \subset V$ and $2 \leq |S| \leq |V| - 2$. Feel free to use the equivalence shown in part (a).

[1] Here, $E(S) := \{\{i, j\} \in E \mid i, j \in S\}$ is the subset of edges that have both endpoints in $S \subseteq V$.