Cuts for MIPs

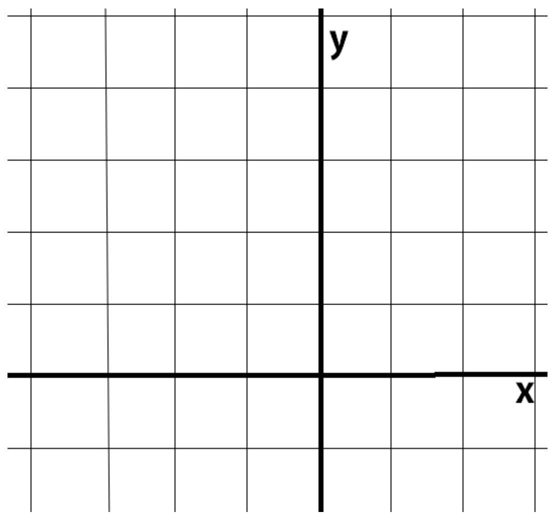
So far, we have seen many valid inequalities for 0-1 programs, and the class of CG inequalities for pure integer programs. Now we discuss cuts for more general MIPs.

The simple rounding inequality for mixed integer sets

Consider the mixed integer set X. (Notice that x has no sign restriction.)

$$X := \{(x, y) \in \mathbb{Z} \times \mathbb{R}_+ \mid x - y \leq b\} .$$

Draw X for the case $b = -1.5$.



See that the LP relaxation is missing one inequality of $\operatorname{conv}(X)$. What is it?

The simple rounding inequality

Proposition 1. *The following inequality is valid for* $X := \{(x, y) \in \mathbb{Z} \times \mathbb{R}_+ \mid x - y \leq b\}$*, where* $f := b - \lfloor b \rfloor > 0$.

$$x - \frac{1}{1-f} y \leq \lfloor b \rfloor.$$

Mixed integer rounding (MIR) inequalities

Corollary 1. *Denote by S the set of all $(x, y) \in \mathbb{Z}^n \times \mathbb{R}^d$ that satisfies:*

$$\sum_{i=1}^{n} a_i x_i - \left(t - \sum_{i=1}^{n} c_i x_i - \sum_{i=1}^{d} g_i y_i \right) \le b$$

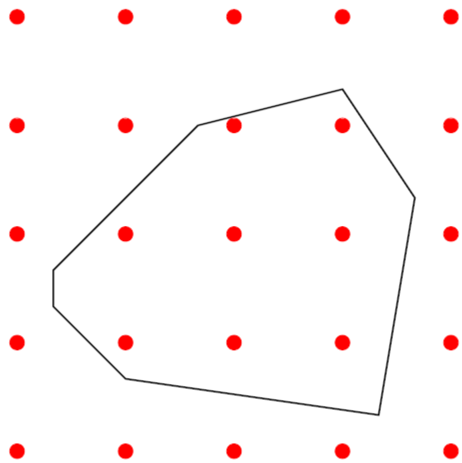
$$\sum_{i=1}^{n} c_i x_i + \sum_{i=1}^{d} g_i y_i \le t,$$

where $a \in \mathbb{Z}^n$. Then, the following mixed integer rounding (MIR) inequality is valid for S:

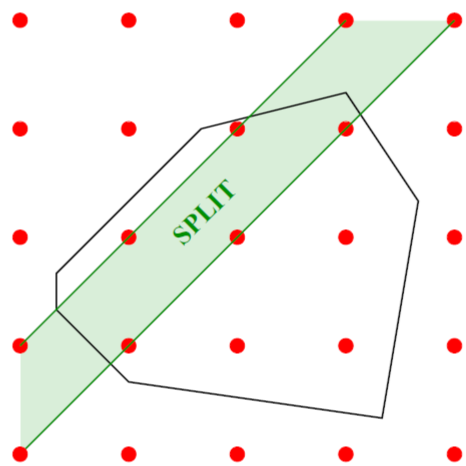
$$\sum_{i=1}^{n} a_i x_i - \frac{1}{1-f} \left(t - \sum_{i=1}^{n} c_i x_i - \sum_{i=1}^{d} g_i y_i \right) \le \lfloor b \rfloor. \tag{1}$$

Split/disjunctive inequalities

These illustrations for split cuts are for the pure integer case, but they apply to MIPs too.



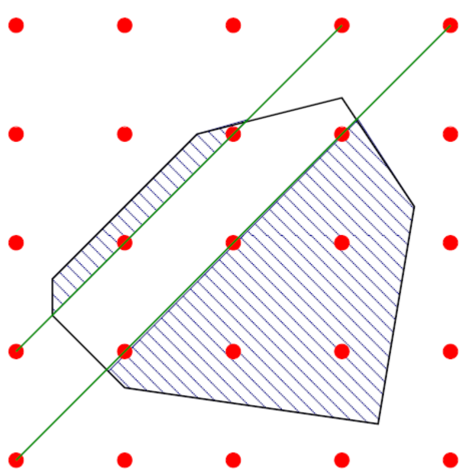
$P = \{x \mid Ax \leq b\}$



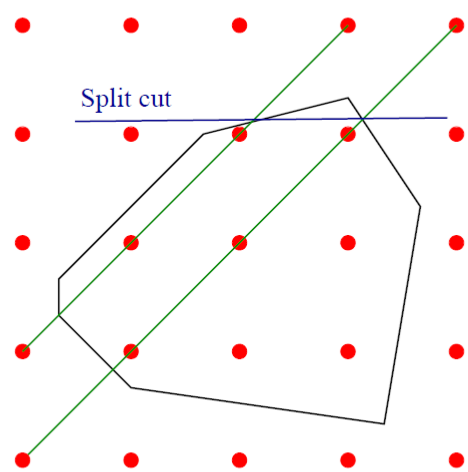
$(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$ defines a split disjunction

Let $P := \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be the LP relaxation, $I \subseteq [n]$ be the (index set of) integer variables, and $C := [n] \setminus I$. Our mixed integer set is $S := P \cap \{x \mid x_j \in \mathbb{Z},\ \forall j \in I\}$.

Definition 1 (split). *A split is a vector $(\pi, \pi_0) \in \mathbb{Z}^n \times \mathbb{Z}$ such that $\pi_j = 0$ for all $j \in C$.*



$\Pi_1 = P \cap \{x \mid \pi x \leq \pi_0\}$
$\Pi_2 = P \cap \{x \mid \pi x \geq \pi_0 + 1\}$



split cut $\alpha x \leq \beta$ is valid for Π_1 and for Π_2

The simple rounding inequality is an example of a split cut. What is the associated split?

Definition 2. *An inequality $\alpha x \leq \beta$ is said to be a split cut if there exists a split (π, π_0) for which $\alpha x \leq \beta$ is valid for $\Pi_1 \cup \Pi_2$.*

Remark 1. *If (π, π_0) is a split, then $S \subseteq \Pi_1 \cup \Pi_2$. Split cuts $\alpha x \leq \beta$ are thus valid for S.*

We can characterize all "good" split cuts. But first we need some notation. Let $u \in \mathbb{R}^m$. Define u^+ so that $u_i^+ := \max\{0, u_i\},\ i = 1, \ldots, m$ and write $u^- := (-u)^+$. Notice $u^- \geq 0$! Thus $u = u^+ - u^-$.

Theorem 1. *Consider $u \in \mathbb{R}^m$ such that:*

1. *uA is integral;*
2. *$uA_C = \mathbf{0}$ (A_C denotes the submatrix of A having the columns for the continuous variables);*
3. *ub is fractional.*

Then, the following inequality is valid for S, where $f := ub - \lfloor ub \rfloor > 0$.

$$\frac{u^+(b - Ax)}{f} + \frac{u^-(b - Ax)}{1 - f} \geq 1. \tag{2}$$

*Moreover, inequality (**??**) is a split cut with respect to the split where $\pi := uA$ and $\pi_0 := \lfloor ub \rfloor$.*

Gomory mixed integer (GMI) cuts

Recall that Gomory fractional cuts only apply to the pure integer case. This led Gomory to develop inequalities for the mixed integer case. The resulting inequalities (which can be seen as a special case of the split inequalities) are called Gomory mixed integer (GMI) cuts. Surprisingly, they are stronger than the Gomory fractional cuts—even in the pure integer case!

Theorem 2. *Let $I = [n]$ and $C = [d]$ and consider the mixed integer set*

$$X = \left\{ (x, y) \in \mathbb{Z}^n_+ \times \mathbb{R}^d_+ \;\middle|\; \sum_{i \in I} a_i x_i + \sum_{i \in C} g_i y_i = b \right\}.$$

The following GMI cut is valid for X, where $f := b - \lfloor b \rfloor$ and $f_i := a_i - \lfloor a_i \rfloor$ for $i = 1, \dots, n$.

$$\sum_{i \in I : f_i \le f} \frac{f_i}{f} x_i + \sum_{i \in I : f_i > f} \frac{1 - f_i}{1 - f} x_i + \sum_{i \in C : g_i \ge 0} \frac{g_i}{f} y_i + \sum_{i \in C : g_i < 0} \frac{-g_i}{1 - f} y_i \ge 1. \tag{3}$$

Example. Consider the following IP that we used to introduce the Gomory fractional cut.

$$\begin{aligned}
\max \quad & 5.5x_1 + 2.1x_2 \\
& -x_1 + x_2 + x_3 = 2 \\
& 8x_1 + 2x_2 + x_4 = 17 \\
& x_1,\ x_2,\ x_3,\ x_4 \ge 0 \\
& x_1,\ x_2,\ x_3,\ x_4 \text{ integer.}
\end{aligned}$$

The optimal LP solution $x^* = (1.3, 3.3, 0, 0)$ with objective $z = 14.08$ can be read from the tableau:

$$\begin{aligned}
z \qquad\qquad\qquad + 0.58x_3 + 0.76x_4 &= 14.08 \\
x_2 + 0.8x_3 + 0.1x_4 &= 3.3 \\
x_1 \qquad - 0.2x_3 + 0.1x_4 &= 1.3.
\end{aligned}$$

The Gomory fractional cut associated with $x_2 + 0.8x_3 + 0.1x_4 = 3.3$ is:

The GMI cut associated with $x_2 + 0.8x_3 + 0.1x_4 = 3.3$ is: