Preprocessing/Presolve

Preprocessing refers to elementary operations that can be used to strengthen/simplify a formulation.

- It is a phase between formulation and solution.
- It can greatly enhance the speed of algorithms.

Tightening bounds and variable fixing

For an example of bound tightening, consider the following IP formulation.

$$\begin{aligned} \max \quad & 2x_1 + x_2 - x_3 \\ & 5x_1 - 2x_2 + 8x_3 \leq 15 \\ & 8x_1 + 3x_2 - x_3 \geq 9 \\ & x_1 + x_2 + x_3 \leq 6 \\ & 0 \leq x_1 \leq 3 \\ & 0 \leq x_2 \leq 1 \\ & 1 \leq x_3 \\ & x_1,\ x_2,\ x_3 \text{ integer.} \end{aligned}$$

Isolate x_1 in the first constraint to improve its upper bound.

Isolate x_3 in the first constraint to improve its upper bound.

Isolate x_2 in the first constraint to improve its lower bound.

Isolate x_1 in the second constraint to improve its lower bound.

Observe that we must have $x_1 = 1$, so we can simplify the formulation:

$$\begin{aligned} \max \quad & 2 + x_2 - x_3 \\ & -2x_2 + 8x_3 \leq 10 \\ & 3x_2 - x_3 \geq 1 \\ & x_2 + x_3 \leq 5 \\ & 0 \leq x_2 \leq 1 \\ & 1 \leq x_3 \\ & x_2,\ x_3 \text{ integer.} \end{aligned}$$

Redundant constraints

Argue that the last constraint is redundant.

Rewrite the further simplified formulation.

Isolate x_3 in the first constraint and improve its bound.

Isolate x_2 in the second constraint and improve its bound.

What is the solution?

Simple observations used for bound tightening

Consider the set $S := \{x \in \mathbb{Z}^n \mid a^T x \leq b,\ l \leq x \leq u\}$, where we assume $l, u \in \mathbb{Z}^n$ and $0 \leq l \leq u$.

Assuming $a_1 > 0$, what is the best conceivable objective for $\max\{x_1 \mid x \in S\}$?

Assuming $a_1 < 0$, what is the best conceivable objective for $\min\{x_1 \mid x \in S\}$.

Proposition 1. *Consider the set* $S = \{x \in \mathbb{Z}^n_+ \mid a^T x \leq b,\ l \leq x \leq u\}$*, where* $0 \leq l \leq u$*.*

1. <u>*Bound Tightening*</u>*. If* $a_1 > 0$*, then* x_1 *is bounded above by:*

   And, if $a_1 < 0$*, then* x_1 *is bounded below by:*

2. <u>*Redundancy*</u>*. The constraint* $a^T x \leq b$ *is redundant if:*

3. <u>*Infeasibility*</u>*. The set* S *is empty if:*

Generating logical inequalities by "probing"

For 0-1 programs, it is often easy to generate (practically useful) "logical" or "boolean" constraints. For example, consider the following 0-1 set.

$$\begin{aligned} 7x_1 + 3x_2 - 4x_3 - 2x_4 &\leq 1 \\ -2x_1 + 7x_2 + 3x_3 + x_4 &\leq 6 \\ -2x_2 - 3x_3 - 6x_4 &\leq -5 \\ 3x_1 - 2x_3 &\geq -1 \\ x_1,\ x_2,\ x_3,\ x_4 &\text{ binary.} \end{aligned}$$

Looking at the first constraint, does setting $x_1 = 1$ imply anything?

Similarly, from the second constraint, can we conclude anything if $x_2 = 1$?

What about setting $x_4 = 0$ in the third constraint?

Anything from the fourth constraint?

Combining logical inequalities

Can we say anything about the relationship between x_1 and x_3?

What about between x_1 and x_2?

Now what can we say about x_4?

What does the constraint set look like after these simplifications?

Conflict graphs: used to store/generate logical inequalities

Introduced by Atamtürk, Nemhauser, and Savelsbergh in a 1998 EJOR paper.

A conflict graph $G = (V, E)$ for an IP with n binary variables has:

- $2n$ vertices: one for $x_i = 0$ and one for $x_i = 1$;
- an edge between two vertices if fixing the variables to associated values yields in a "conflict."

Draw the conflict graph for the previous problem.

Any feasible solution to 0-1 program is ____ in conflict graph.

Corollary:

The following two classes of valid inequalities were studied by Padberg (1973).

Proposition 2 (Clique inequalities)**.** *If $C \subseteq V$ is a clique in a graph $G = (V, E)$, then the inequality $\sum_{i \in C} x_i \leq 1$ is valid for independent sets in G.*

Identify two cliques from the conflict graph that you drew and write the corresponding inequalities.

Proposition 3 (Odd-hole inequalities)**.** *If $C \subseteq V$ induces an odd cycle in graph $G = (V, E)$, then the inequality $\sum_{i \in C} x_i \leq (|C| - 1)/2$ is valid for independent sets in G.*

Identify an odd hole (with $|C| \geq 5$) from the conflict graph and give the associated inequality.