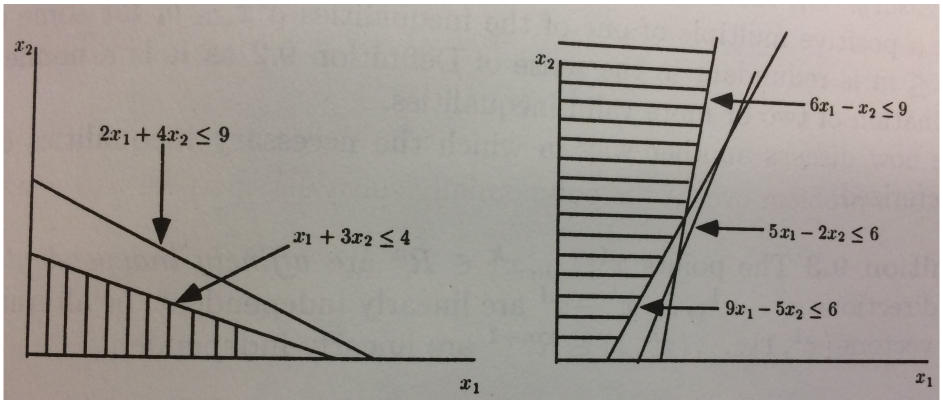
Strong Valid Inequalities

Given an integer set $X \subseteq \mathbb{Z}^n_+$, we would like to characterize which inequalities are *strong* or *useful* in a formulation for conv(X). There are varying levels of strength, but a strong valid inequality $\alpha x \leq \beta$ should improve the formulation's tightness. Dominated/redundant inequalities are undesirable.



Definition 1 (dominated)**.** *If $\alpha x \leq \beta$ and $\pi x \leq \pi_0$ are valid inequalities for $P \subseteq \mathbb{R}^n_+$, $\pi x \leq \pi_0$* dominates *$\alpha x \leq \beta$ if there exists $u > 0$ such that $\pi \geq u\alpha$ and $\pi_0 \leq u\beta$ and $(\pi, \pi_0) \neq (u\alpha, u\beta)$.*

Observe that if $\pi x \leq \pi_0$ dominates $\alpha x \leq \beta$, then

$$\{x \in \mathbb{R}^n_+ \mid \pi x \leq \pi_0\} \subseteq \{x \in \mathbb{R}^n_+ \mid \alpha x \leq \beta\}.$$

Example. If $2x_1 + 4x_2 \leq 9$ and $x_1 + 3x_2 \leq 4$ are valid inequalities, which dominates the other? With what value of u?

Definition 2 (redundant)**.** *A valid inequality $\alpha x \leq \beta$ is* redundant *if there exist $k \geq 2$ valid inequalities $\pi^i x \leq \pi^i_0$, $i = 1, \ldots, k$ and multipliers $u_i > 0$, $i = 1, \ldots, k$ such that the aggregated inequality $\left(\sum_{i=1}^k u_i \pi^i\right) x \leq \sum_{i=1}^k u_i \pi^i_0$ dominates $\alpha x \leq \beta$.*

Example. Suppose inequalities $6x_1 - x_2 \leq 9$ and $9x_1 - 5x_2 \leq 6$ are valid. Argue that the inequality $5x_1 - 2x_2 \leq 6$ (which is assumed to be valid) is redundant. (Hint: use $u_1 = u_2 = \frac{1}{3}$.)

It can be computationally expensive to test for redundancy. The important point is to avoid using an inequality when one that dominates it is readily available.

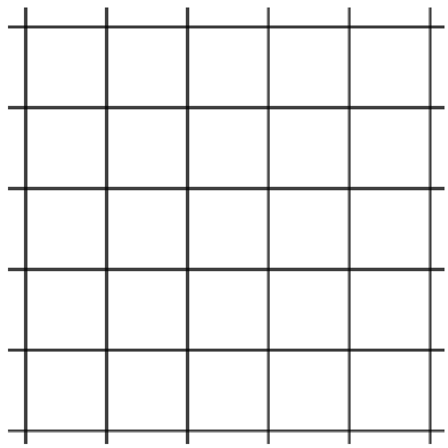
Polyhedra, faces, and facets

Theorem 1. *If P is a full-dimensional polyhedron, then it has a unique minimal description*

$$P = \{x \in \mathbb{R}^n \mid a^i \leq b_i,\ i = 1, \dots, m\}, \tag{1}$$

where each inequality is unique to within a positive multiple.

Example. Draw the set $X = \{(2,0), (1,1), (2,1), (0,2), (1,2), (2,2), (0,3)\}$. Identify *two* minimal inequality descriptions of the polyhedron $P = \mathrm{conv}(X)$.



Definition 3. *Vectors $x^1, x^2, \dots, x^k \in \mathbb{R}^n$ are* affinely independent *if $\sum_{i=1}^k \lambda_i x^i = \boldsymbol{0}$ and $\sum_{i=1}^k \lambda_i = 0$ implies $\lambda_i = 0$ for all $i \in [k]$. (Each $\lambda_i \in \mathbb{R}$ is a scalar, and $\boldsymbol{0}$ is an n-dimensional vector of 0s.)*

Equivalently, $x^1, x^2, \dots, x^k$ are affinely independent if $x^2 - x^1, \dots, x^k - x^1$ are linearly independent. Or, if $(x^1, 1), \dots, (x^k, 1)$ are linearly independent.

Definition 4. *Vectors $x^1, x^2, \dots, x^k \in \mathbb{R}^n$ are* linearly independent *if $\sum_{i=1}^k \lambda_i x^i = \boldsymbol{0}$ implies $\lambda = \boldsymbol{0}$.*

Example. Argue that the three points $\{(2,0), (1,1), (2,2)\}$ from X are affinely independent.

Definition 5. *The dimension of $P \subseteq \mathbb{R}^n$, denoted* $\dim(P)$*, is the maximum number of affinely independent points in P minus one. Note* $\dim(P) \leq n$*. When* $\dim(P) = n$*, P is full-dimensional.*

By this definition, what is the dimension of $P = \emptyset$?

What about the dimension of $P \subseteq \mathbb{R}^n$, when P contains only one point?

What about the dimension of $\mathrm{conv}(X)$ for the X from above?

Is $P = \mathrm{conv}\{x \in \{0,1\}^n \mid \sum_{i=1}^n x_i \leq 1\}$ full-dimensional? Why?

Definition 6 (face, facet)**.** *Let $\alpha x \leq \beta$ be a valid inequality for $P \subseteq \mathbb{R}^n$. The polyhedron*

$$F := \{x \in P \mid \alpha x = \beta\}$$

where this inequality holds at equality is a called a face of P. The inequality $\alpha x \leq \beta$ is said to induce or define the face F. The face F is a facet of P if $\dim(F) = \dim(P) - 1$.

Give a facet of $P = \mathrm{conv}(X)$. What is its dimension? Give an inequality that induces this facet.

Give a face of $P = \mathrm{conv}(X)$ that is not a facet. What is its dimension? Give an inequality that defines it.

A quick polyhedral study

Recall the *stable set* polytope $\mathrm{STAB}(G)$ of a graph $G = (V, E)$:

$$\begin{aligned}\mathrm{STAB}(G) :&= \operatorname{conv}\left\{x^S \mid S \subseteq V \text{ is a stable set}\right\}\\ &= \operatorname{conv}\left\{x \in \{0,1\}^n \mid x_i + x_j \leq 1,\ \forall \{i,j\} \in E\right\}.\end{aligned}$$

What is the dimension of $\mathrm{STAB}(G)$? Why?

Prove that the nonnegativity bounds $x_i \geq 0$ induce facets of $\mathrm{STAB}(G)$.

Prove that the clique inequality $x_1 + x_2 + x_3 \leq 1$ induces a facet of $\mathrm{STAB}(K_4 - e)$ for edge $e = \{1, 4\}$. (Also notice that the clique inequality $x_1 + x_2 \leq 1$ is not facet-defining here.)

Lemma 1. *If polyhedron P is full-dimensional, then a redundant inequality cannot induce a facet.*

Theorem 2. *The clique inequality $\sum_{i \in C} x_i \leq 1$ induces a facet of $\mathrm{STAB}(G)$ if and only if C is a <u>maximal</u> clique of G. (Here, maximality is by inclusion; there should be no clique C' with $C \subsetneq C'$.)*

Knapsack covers and lifting

Consider the knapsack set $X = \{x \in \{0,1\}^n \mid \sum_{j=1}^n a_j x_j \le b\}$. Without loss, assume:

1. $a_j \ge 0$ (otherwise replace x_j by $\overline{x}_j = 1 - x_j$ and update RHS);
2. $a_j \ne 0$ (otherwise remove variable x_j from problem);
3. $a_j \le b$ (otherwise you might as well fix $x_j = 0$);
4. $b > 0$ (otherwise the first three assumptions force $X = \{\mathbf{0}\}$ or $X = \emptyset$).

Definition 7. *A set $C \subseteq [n]$ is a knapsack cover (or simply cover) if $\sum_{j\in C} a_j > b$.*

Proposition 1. *If C is a cover, then the cover inequality $\sum_{j\in C} x_j \le |C| - 1$ is valid for X.*

All cover inequalities are valid, but the only interesting ones are *minimal* cover inequalities. Why?

Example. List a couple minimal cover inequalities for

$$X = \{x \in \{0,1\}^7 \mid 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \le 19\}.$$

Not all minimal cover inequalities induce facets! For example, $x_3+x_4+x_5+x_6 \le 3$ for $C = \{3,4,5,6\}$ does not induce a facet of $\text{conv}(X)$. It is dominated by $2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \le 3$.

However, $x_3+x_4+x_5+x_6 \le 3$ does induce a facet F of the lower-dimensional polyhedron $\text{conv}(X_C)$,

$$X_{3,4,5,6} = \{(x_3, x_4, x_5, x_6) \in \{0,1\}^4 \mid 6x_3 + 5x_4 + 5x_5 + 4x_6 \le 19\}.$$

Informally, the inequality is facet-defining when we have fixed the variables x_1, x_2, x_7 to zero.

Suppose we wanted to "free" variable x_1 from being fixed to zero to get a "strongest" valid inequality

$$\alpha_1 x_1 + x_3 + x_4 + x_5 + x_6 \leq 3$$

for the set $X' := \{(x_1, x_3, x_4, x_5, x_6) \in \{0,1\}^5 \mid 11x_1 + 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19\}$. How big can α_1 be? For example, is the inequality valid if we set $\alpha_1 = 0$? What about $\alpha_1 = 3$?

Supposing we found the right α_1, how could we show its validity? (Hint: two cases.)

Using that insight, formulate an optimization problem to find the best value of α_1.

Argue that the inequality $2x_1 + x_3 + x_4 + x_5 + x_6 \leq 3$ induces a facet of $\text{conv}(X') \subseteq \mathbb{R}^5$.

We have just (optimally) *lifted* the variable x_1 into the inequality. We can now lift in x_2.

$$2x_1 + \alpha_2 x_2 + x_3 + x_4 + x_5 + x_6 \leq 3$$

Find the optimal value of α_2 that still yields a valid inequality for

$$\{(x_1, x_2, x_3, x_4, x_5, x_6) \in \{0,1\}^6 \mid 11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 \leq 19\}.$$

Finally, find the best value of α_7 to get a valid inequality for X.

$$2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + \alpha_7 x_7 \leq 3$$

Lifting, more generally

This lifting technique applies much more generally. The feasible set X need not be a knapsack set, and it need not be 0-1 either. However, we will stick with 0-1 problems in this class for simplicity.

Lifting can be applied repeatedly based on some ordering of the variables not currently in the inequality. This is *sequential lifting*. The resulting inequality *can be different* depending on the order.

Proposition 2. *Consider a set $S \subseteq \{0,1\}^n$ such that $S \cap \{x \mid x_n = 1\}$ is nonempty. If $\sum_{i=1}^{n-1} a_i x_i \leq b$ is a valid inequality for $S \cap \{x \mid x_n = 0\}$, then*

$$a_n := b - \max \left\{ \sum_{i=1}^{n-1} a_i x_i \;\middle|\; x \in S,\ x_n = 1 \right\}$$

is the largest coefficient such that $ax \leq b$ is valid for S. Moreover, the dimension of the face of $\operatorname{conv}(S)$ *where $ax = b$ has dimension at least one more than that of the face of* $\operatorname{conv}(S)$ *where $\sum_{i=1}^{n-1} a_i x_i = b$ and $x_n = 0$.*

Exercise. Prove that the maximal clique inequality $\sum_{i \in C} x_i \leq 1$ induces a facet of $\mathrm{STAB}(G)$ using sequential lifting arguments.