Convexity

Definition 1 (convex set)**.** *A set $S \subseteq \mathbb{R}^n$ is convex if, for any two points in S, the line segment joining them is also in S, i.e., if $x, y \in S$ and $\lambda \in [0, 1]$, then $\lambda x + (1 - \lambda)y \in S$.*

Definition 2 (convex function)**.** *A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if, for any two points $x, y \in \mathbb{R}^n$ and any $\lambda \in [0, 1]$, we have $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$.*

What is the connection between convex sets and convex functions?

Proposition 1. *A function f is convex if and only if its epigraph* epi(f) *is a convex set, where*

$$\mathrm{epi}(f) := \{(x, z) \mid x \in \mathbb{R}^n,\ z \in \mathbb{R},\ z \geq f(x)\}.$$

$(\implies)$

($\Longleftarrow$)

Convex Hulls

Definition 3 (convex hull)**.** *The convex hull of a set S, denoted* $\text{conv.hull}(S)$ *or simply* $\text{conv}(S)$*, is defined as the intersection of all convex sets that contain S.*

Definition 4 (convex combination)**.** *A point $x \in \mathbb{R}^n$ is said to be a convex combination of points in S if there exists a finite set of points $x^1, \dots, x^p \in S$ and scalars $\lambda_1, \dots, \lambda_p$ such that*

$$x = \sum_{i=1}^{p} \lambda_i x^i, \qquad \sum_{i=1}^{p} \lambda_i = 1, \qquad \lambda_1, \dots, \lambda_p \geq 0.$$

Proposition 2. *The convex hull of S can alternatively be defined as:*

1. $\text{conv}(S)$ *is the inclusion-wise minimal convex set that contains S;*
2. $\text{conv}(S) = \{x \in \mathbb{R}^n \mid x \text{ is a convex combination of points in } S\}$.

Proof. Exercise. □

Proposition 3. *Let $S \subset \mathbb{R}^n$ and $c \in \mathbb{R}^n$. Then,* $\sup\{c^T x \mid x \in S\} = \sup\{c^T x \mid x \in \text{conv}(S)\}$*. Further, the supremum of $c^T x$ is attained over S if and only if it is attained over* $\text{conv}(S)$*.*

Why is this proposition important for us?

Meyer's Theorem

In this class, we are interested in feasible sets of the form $S = \{(x, y) \in \mathbb{Z}_+^n \times \mathbb{R}_+^p \mid Ax + Gy \leq b\}$.

Theorem 1 (Meyer, 1974)**.** *If A, G, and b contain only rational entries, then there exist rational A', G', and b' such that* $\text{conv}(S) = \{(x, y) \mid A'x + G'y \leq b'\}$*, i.e., it is a polyhedron.*

Why is this theorem important for us?

Remark 1 (The rationality assumption is needed)**.** *The following IP has supremum 0, but it is not attained. The convex hull of its feasible points is not a polyhedron.*

$$
\begin{aligned}
&\max -\sqrt{2}x_1 + x_2 \\
&-\sqrt{2}x_1 + x_2 \leq 0 \\
&x_1 \geq 1 \\
&x_2 \geq 0 \\
&x_1,\ x_2 \textit{ integer.}
\end{aligned}
$$

Why is this not too discouraging?