A brief tutorial on Benders decomposition

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To illustrate Benders decomposition, consider the k-median problem in which we are tasked with locating k facilities to satisfy customer demands at minimum cost. We denote the set of candidate sites by J and the set of customers by I. The cost to assign customer $i \in I$ to (a facility at) site $j \in J$ is denoted c_{ij}.

We can express the k-median problem as the following mixed integer program (MIP), where y_j is a binary variable that equals one if we use site j, and x_{ij} is a binary assignment variable indicating if customer i is assigned to site j. Ultimately, however, it is safe to relax the x variables to be continuous.

$$\begin{aligned}
\min \quad & \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} && && \text{(1a)} \\
\text{s.t.} \quad & \sum_{j \in J} y_j = k && && \text{(1b)} \\
& \sum_{j \in J} x_{ij} = 1 && \forall i \in I && \text{(1c)} \\
& 0 \leq x_{ij} \leq y_j && \forall i \in I,\ j \in J && \text{(1d)} \\
& y_j \in \{0, 1\} && \forall j \in J. && \text{(1e)}
\end{aligned}$$

How well does this MIP work in practice? For instances with $1,000$ sites and $1,000$ customers, the associated LP relaxation can often be solved in a few minutes with commercial solvers, while the MIP takes a little longer. As the instances grow larger, we will eventually run out of RAM. How can we deal with this? One possibility is to use Benders decomposition.

Suppose that we have already decided the facility locations, i.e., we have fixed $y = \bar{y}$ for some $\bar{y}$ that satisfies $\sum_{j \in J} \bar{y}_j = k$ and $\bar{y}$ binary (or continuous 0-1). Then, problem (1) (or its LP relaxation) reduces to the subproblem:

$$\begin{aligned}
\Phi(\bar{y}) = \min \quad & \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} && && \text{(2a)} \\
\text{s.t.} \quad & \sum_{j \in J} x_{ij} = 1 && \forall i \in I && \text{(2b)} \\
& 0 \leq x_{ij} \leq \bar{y}_j && \forall i \in I,\ j \in J. && \text{(2c)}
\end{aligned}$$

If $\bar{y}$ is binary, then subproblem (2) asks to assign each customer i to its "closest" opened facility. Even if $\bar{y}$ is fractional, an optimal x can still be found efficiently by solving $|I|$ knapsack problems, one for each customer i:

$$\min \left\{ \sum_{j \in J} c_{ij} x_{ij} \;\middle|\; \sum_{j \in J} x_{ij} = 1,\ 0 \leq x_{ij} \leq \bar{y}_j\ \forall j \in J \right\}.$$

This can be done efficiently, e.g., by using a sorting algorithm.

Next, we write the dual of subproblem (2). Associate a dual variable α_i to each assignment constraint (2b) and a dual variable β_{ij} to each (upper) bound constraint (2c). This gives:

$$\begin{aligned}
\max \quad & \sum_{i \in I} \alpha_i + \sum_{i \in I} \sum_{j \in J} \bar{y}_j \beta_{ij} && && \text{(3a)}\\
\text{s.t.} \quad & \alpha_i + \beta_{ij} \leq c_{ij} && \forall i \in I,\ j \in J && \text{(3b)}\\
& \alpha_i \text{ unrestricted} && \forall i \in I && \text{(3c)}\\
& \beta_{ij} \leq 0 && \forall i \in I,\ j \in J. && \text{(3d)}
\end{aligned}$$

By strong duality, it has the same objective value $\Phi(\bar{y})$ as the primal problem (2).

We are now ready to give the Benders reformulation of the k-median problem (1), where z is a new variable used to represent the objective function (1a).

$$\begin{aligned}
\min \quad & z && && \text{(4a)}\\
\text{s.t.} \quad & z \geq \Phi(y) && && \text{(4b)}\\
& \sum_{j \in J} y_j = k && && \text{(4c)}\\
& y_j \in \{0, 1\} && \forall j \in J. && \text{(4d)}
\end{aligned}$$

Note that constraint (4b) involves the nonlinear (but convex) function $\Phi(y)$.

To solve this problem, we can take a cutting plane approach in which constraint (4b) is initially omitted, and linear inequalities are added as needed to approximate it. That is, our (initial) Benders main problem is:

$$\begin{aligned}
\min \quad & z \\
& \sum_{j \in J} y_j = k \\
& y_j \in \{0, 1\} && \forall j \in J.
\end{aligned}$$

Given a feasible solution $(\bar{y}, \bar{z})$ to the main problem (or to its LP relaxation), we may generate a Benders cut by computing a feasible (preferably optimal) solution $(\bar{\alpha}, \bar{\beta})$ to the subproblem's dual (3) and then imposing the following constraint in the main problem:

$$z \geq \sum_{i \in I} \bar{\alpha}_i + \sum_{i \in I} \sum_{j \in J} y_j \bar{\beta}_{ij}, \tag{6}$$

or equivalently

$$z \geq \sum_{i \in I} \bar{\alpha}_i + \sum_{j \in J} \left(\sum_{i \in I} \bar{\beta}_{ij} \right) y_j. \tag{7}$$

Observe that the right-hand-side of constraint (6) is simply the subproblem's dual objective in which the dual *variables* now take *fixed values*, and the *values* of y variables revert back to being *variables*. We repeat this process (i.e., solve main problem, then solve subproblem's dual[1], then add cut) until convergence.

Further Reading

We have seen a simple Benders decomposition approach for the k-median problem. A (slow) Python implementation is available at: `https://github.com/AustinLBuchanan/kmedian_Benders`. For best results, one needs a careful implementation and/or a different Benders reformulation [3, 4, 5, 8].

When Benders is applied to other problems, *feasibility cuts* may also be needed [1], not just the *optimality cuts* that were discussed here in inequality (6). Variants of Benders decomposition have also been developed for problems in which the subproblem is no longer a linear program [2, 6, 7].

References

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[1] Alternatively, one may solve the primal subproblem and retrieve associated optimal dual multipliers from the MIP solver. Note that $\bar{y}$ will typically be quite sparse, implying that most x_{ij} will be fixed to zero in the subproblem. This can be exploited.