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A widespread belief about county splits in political districting plans is wrong

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Abstract

Consider the task of dividing a state into d contiguous political districts whose populations must not differ by more than one person. A widely held belief among districting experts is that this task requires at least $d-1$ county splits. In this paper, we seek to dispel this belief. To illustrate, we find contiguous congressional plans for Idaho, Iowa, Mississippi, Montana, Nebraska, and West Virginia that use *zero* county splits, i.e., all counties are kept whole, despite satisfying a 1-person deviation. This is not a rare phenomenon; for example, Montana admits more than 10,000 such plans. In practice, district drawers may need to satisfy additional criteria, like compactness and minority representation, which may lead them to believe $d-1$ splits to be minimum. Again, this need not be true. Inspired by the recent Supreme Court case *Allen v. Milligan* (2023), we draw a districting plan for Alabama that uses fewer than $d-1$ county splits. The districts are also reasonably configured (e.g., contiguous, compact, 1-person deviation, five precinct splits), and two of them are majority-Black, in line with *Gingles*. These examples show that $d-1$ county splits should not be assumed minimum.

Keywords: political districting, county splits, political subdivisions, integer programming

1. Introduction

The vast majority of US states require the preservation of political subdivisions (e.g., counties, cities, towns) in their political districts; this is true for both congressional and legislative districts (NCSL 2021). Arguably, the most popular ways to quantify splitting is the number of splits (Carter et al. 2020; Cervas and Grofman 2020; Autry et al. 2021; Nagle 2022; Dave's Redistricting App 2023; McCartan and Imai 2023; Shahmizad and Buchanan 2023), which is nearly equivalent to the number of "parts" or "pieces" (Gladkova et al. 2019; Becker and Gold 2022) or "intersections" (Wachspress and Adler 2021). For example, if a county is wholly assigned to one district, then it contributes zero county splits. If it is divided across two districts, then it contributes one split. Generally, if a county is divided across k districts, then it contributes $k-1$ county splits. Usually, the total number of county splits is reported, summed over all counties.

Another popular splitting score, the number of *split counties*, only captures *how many* counties are split, not how many *times* they are split. In this way, the number of county splits provides a more detailed look than the number of split counties and may be preferable (Cervas and Grofman 2020). The numbers of precinct splits and split precincts can be defined analogously.

The academic literature on redistricting makes several claims about the number of county splits s and how this quantity relates to the number of districts d. Usually, the claim is that, in any districting plan, the number of splits is at least the number of districts minus one, i.e., $s \geq d-1$, especially if districts must not differ in population by more than one person (Autry et al. 2021; Nagle 2022).

Sometimes, it is further asserted that the *minimum* number of splits s^* precisely achieves this quantity for (almost) all instances, i.e., $s^* = d - 1$, see Nagle 2022.

These claims are believed by a wide variety of districting experts, including expert witnesses, attorneys, courts, redistricting commissioners, and special masters tasked with drawing remedial districts. As we will see, such claims have even made their way to the US Supreme Court, e.g., in *Allen v. Milligan* (2023).

In this paper, our aim is to dispel these beliefs. We make three main points:

1. For most states, fewer than $d - 1$ county splits suffice to satisfy the most basic districting criteria (i.e., 1-person deviation and contiguity); for example, we show that several states can do so using *zero* county splits.
2. These examples are not rare flukes; for example, Montana admits more than 10,000 contiguous, whole-county plans with 1-person deviation.
3. Even when constrained by other criteria (e.g., compactness, minority representation, limited precinct splits), $d-1$ need not be the minimum number of county splits; for example, we provide a reasonably configured plan for Alabama with two majority-Black districts and 1-person deviation that nevertheless exhibits fewer than $d - 1$ county splits.

We conclude that $d - 1$ county splits should not be assumed minimum. Going forward, districting experts should either remain agnostic to such statements, or rigorously prove or disprove them using exact methods like ours.

2. Background and Literature Review

Below we review some common claims regarding the minimum number of county splits s^* and how this quantity relates to the number of districts d and to the maximum number of county clusters c^* (defined later). In short, redistricting folklore states that $s^* \geq d - 1$ or $s^* \leq d - 1$ or possibly both (i.e., $s^* = d - 1$), none of which are generally correct. Meanwhile, researchers like Carter et al. 2020 state that $s^* = d - c^*$, which has been confirmed to hold in practice by Shahmizad and Buchanan 2023, but is also generally incorrect. However, it is indeed always true that $s^* \geq d - c^*$, see Carter et al. 2020; Shahmizad and Buchanan 2023.

2.1 $d - 1$ *is neither an upper nor lower bound*

Redistricting folklore states that, when dividing a state into d contiguous and population-balanced districts, $d - 1$ county splits suffice. This $d - 1$ number appears throughout the academic literature and is also reported on popular districting software like Dave's Redistricting App 2023. For some intuition, consider four counties arranged in a line, each with a population of 75, as shown in Figure 1. Suppose we seek $d = 3$ equipopulous districts. We may create our first district with the leftmost county (population 75) and add to it 25 people from the second county, introducing one split. Then, create our second district with the remaining 50 people from the second county and add to it 50 people from the third county, introducing a second split. Then, create the third district from the remaining 25 people and the entire rightmost county. Thus, we have created three districts using two county splits, as folklore suggests should be possible. Of course, this idea applies to more complicated instances; the important assumption is that we should be able to carve $d - 1$ districts from the state one-by-one, each time introducing one county split, and then take what remains as the final district.

This is not always possible to do, as observed by Carter et al. 2020. Here, we give a modified example from Shahmizad and Buchanan 2023. Consider a hub county with 35 people that is adjacent to three spoke counties, each with a population of 55, as shown in Figure 2. Suppose we are to divide this state into two districts, each with a population between 95 and 105. See that we must split at least one of the spoke counties (otherwise, all spoke counties will be kept whole and some district will contain at least two of them, causing its population to reach 110, which is too much). Now, each

Figure 1. A hypothetical districting instance with four counties in a line

district can take at most 55 people from this split spoke county, which is too little, meaning that each district must extend into the hub county, splitting it as well. Thus, we require at least two county splits, which is more than $d - 1$. Shahmizad and Buchanan extend this example to show that any number of splits $d + q$ might be required, where q is any nonnegative integer. So, there is generally no way to upper bound the minimum number of splits s^* by a function of the number of districts d.

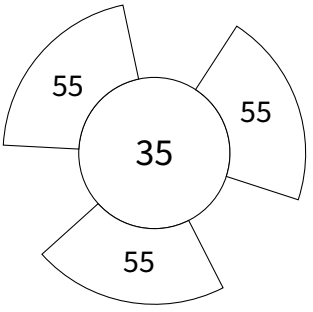

Figure 2. A hypothetical districting instance with a hub county and three spoke counties

Further, the academic literature often claims $d - 1$ to be a lower bound. Nagle 2022 states that forcing districts to satisfy a 1-person deviation makes it "highly probable that the minimum number of county splits is uniquely given as the number of districts minus one". Likewise, Autry et al. 2021 consider this a "reasonable" assumption. As we will see, this belief has been repeated in high-profile court cases, including the Supreme Court case *Allen v. Milligan* (2023), even though it is not true.

2.2 $d - c^*$ is usually minimum in practice (but not always)

Carter et al. 2020 propose a more nuanced claim. They recognize that the number of county splits can sometimes be less than $d - 1$. In an example, consider the task of dividing Alabama's total population of 5,024,279 across seven districts, so each has an ideal population of 5,024,279/7≈717,754.14. Thus, to achieve a 1-person deviation, there must be six districts with a population of L=717,754 and one district with population U=717,755.

It turns out that Alabama's counties can be partitioned into two contiguous sets, one with a population of $L + U$ and another with a population of $5L$, see the left of Figure 3. We can consider them as two separate, miniature districting instances, the first with two districts and the second with five districts. We will see that we can divide up the first using one county split and the second using four county splits, for a total of five county splits. So, by first dividing the state's counties into two miniature districting instances, we save one county split (beyond the folklore $d - 1$ bound).

On the right of Figure 3, we give a different county clustering, this time with $c = 3$ clusters; the northernmost cluster has a population of $L + U$, the central cluster has a population of $2L$, and the southern cluster (which winds around the state) has a population of $3L$. Later, we will see that we can divide up the first using one county split, the second using one county split, and the third using two county splits, for a total of four county splits. So, by first dividing up the counties into three miniature districting instances, we save two county splits (but at the cost of less compact districts).

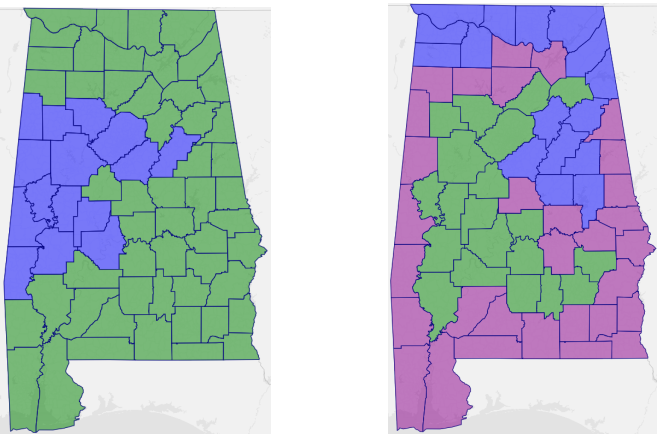

Figure 3. County clusterings for Alabama with two and three clusters

More generally, the idea behind Carter et al.'s claim is that, by first dividing a state's counties into a maximum number c^* of miniature districting instances, we can save $c^* - 1$ county splits, thus giving $(d-1)-(c^*-1) = d-c^*$ county splits. Indeed, in their "basic" theorem, they propose the bold claim that the minimum number of county splits s^* equals $d - c^*$. In a subsequent "enlarged" theorem, they add that the caveat that this equality holds "except in rare circumstances". Most mathematicians would agree that phrasing like this is unusual in a theorem, as theorems are mathematical statements that are proven to hold in all cases. Their theorem statement does not specify *what* these rare circumstances are, nor do they establish *how rare* they are in practice.

However, Shahmizad and Buchanan 2023 point out that half of Carter et al.'s theorem always holds, that is $s^* \geq d - c^*$, a result that they name *weak split duality*. Using mathematical optimization techniques (specifically, integer programming), Shahmizad and Buchanan compute a maximum number of county clusters for each congressional and legislative districting instance across the USA, thus establishing their c^* values. Then, using the inequality $s^* \geq d - c^*$, they establish a lower bound on s^*. With other integer programming techniques, they find districting plans that achieve this lower bound, thus proving optimality in terms of minimum county splits. So, we may empirically conclude that Carter et al. are right; their $s^* = d - c^*$ "theorem" does hold in practice. (Note that the hub-and-spoke instance from earlier provides a synthetic counterexample, as it has a maximum of one county cluster $c^* = 1$ but requires at least $s^* \geq 2$ splits, thus giving an example where $s^* > d - c^*$.)

However, Shahmizad and Buchanan primarily used a 1% deviation ($\pm 0.5\%$) for congressional instances and a 10% deviation ($\pm 5\%$) for legislative instances. Districting experts will complain that 1-person deviation is the norm for congressional districting. In response, Shahmizad and Buchanan found that 79% of these districting instances admit a nontrivial county clustering (i.e., with $c^* \geq 2$) even when subjected to 1-person deviation. So, contrary to speculations by Autry et al. 2021 and Nagle 2022, it is the norm rather than a rare exception for a state to admit a nontrivial county clustering. If these county clusterings can be extended into districting plans (as redistricting folklore would suggest), then this would yield districting plans that satisfy 1-person deviation and have fewer than $d - 1$ county splits.

In this paper, we go further. We show that *zero* county splits suffice for states like Idaho, Iowa, Mississippi, Montana, Nebraska, and West Virginia. In other words, we have found county clusterings for them with $c^* = d$ clusters, with each cluster serving as a district. Further, we establish that states like Montana admit literally *thousands* of contiguous, whole-county plans that satisfy 1-person deviation, contrary to the belief of a Montana redistricting commissioner who thought none existed. Even when plans must satisfy other criteria, such as compactness and minority representation, it is

still possible for states like Alabama to draw plans with fewer than $d-1$ county splits, contrary to claims from *Allen v. Milligan* (2023), as we will see.

3. Fewer county splits usually suffice

As shown by Shahmizad and Buchanan 2023, 79% of congressional and legislative districting instances across the USA admit nontrivial county clusterings (i.e., $c^* \geq 2$), even when subjected to 1-person deviation. This includes all instances with more than 30 counties. Presumably, this would mean that it is the norm, rather than a rare exception, to admit plans with fewer than $d-1$ county splits. Here, we go even further; we give plans that have *zero* county splits, despite satisfying a 1-person deviation.

We begin with two states that have two congressional districts: Idaho and West Virginia. Both admit whole-county plans with *zero*-person deviation, as shown in Figure 4 and available on DRA[1,2]. Later, we will see that Montana admits *thousands* of whole-county plans with 1-person deviation.

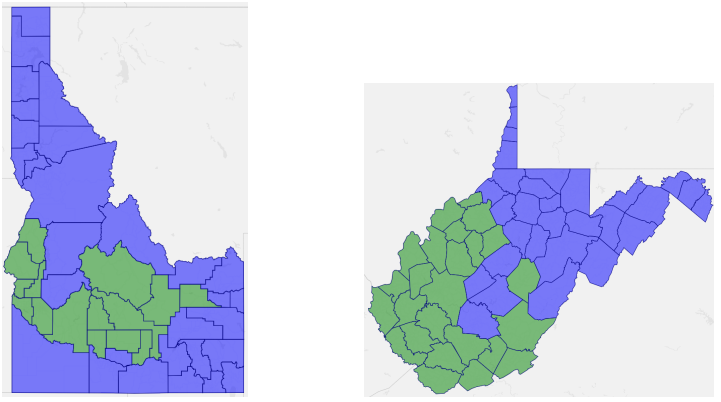

Figure 4. Plans for Idaho and West Virginia with zero county splits and zero-person deviation

West Virginia is particularly interesting for us, as it draws whole-county plans in practice. Its 2010 districts were upheld by the Supreme Court in *Tennant v. Jefferson County* (2011) in a *per curiam* opinion, despite exhibiting a 4871-person deviation. The current map has a 1582-person deviation.

Next, we consider Nebraska and Mississippi, which have three and four congressional districts, respectively. As Figure 5 shows, each admits a whole-county plan with 1-person deviation[3,4]. So, the typical $d-1$ assumption overestimates the true minimum number of county splits by one and two units in these cases.

Last, we consider Iowa, which has four congressional districts and draws whole-county maps in practice. After both the 2010 and 2020 censuses, it admitted whole-county plans with 1-person deviation[5,6], as shown in Figure 6. We admit that they are unappealing, e.g., for compactness reasons, but they *do* exist. It is conceivable that more compact plans exist.

4. These examples are not rare flukes

In this section, we remark that plans with 1-person deviation and fewer than $d-1$ county splits are not rare flukes. In particular, consider Montana, which has two congressional districts and 56 counties.

1. https://davesredistricting.org/join/beb669cd-6a4d-4714-807f-f11193df3069
2. https://davesredistricting.org/join/b94ffa26-22cc-41d9-b213-b99883aeabe8
3. https://davesredistricting.org/join/e017c76e-9c91-462b-b604-d0964e1d2d9d
4. https://davesredistricting.org/join/9c2d051f-0a57-44df-9652-805e5c1e1007
5. https://davesredistricting.org/join/e2280c9a-cfeb-4f6d-9cba-06bf03567d06
6. https://davesredistricting.org/join/93229cb1-c198-4b69-8907-e7263c0bd9e6

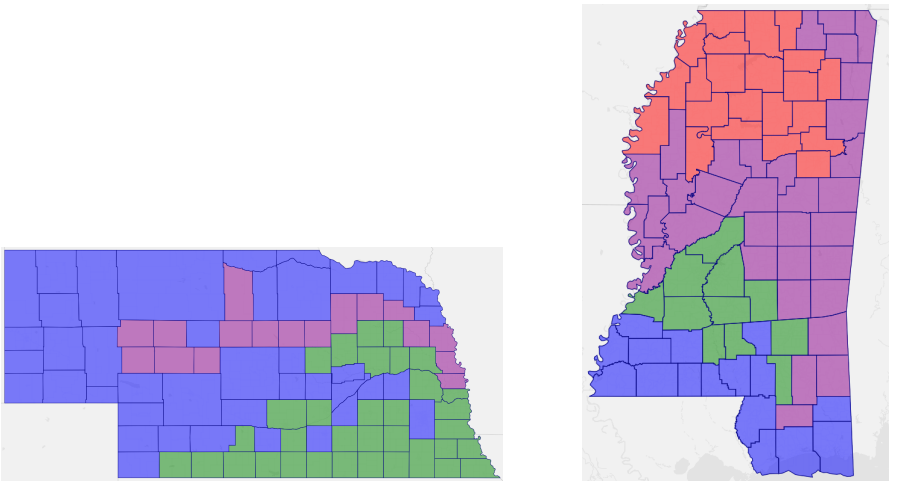

Figure 5. Plans for Nebraska and Mississippi with zero county splits and one-person deviation

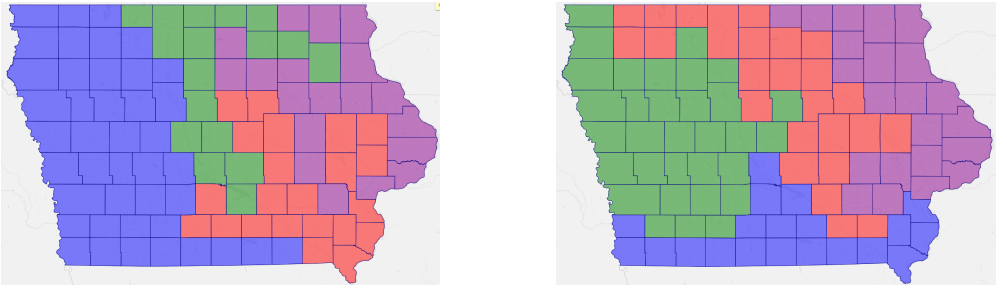

Figure 6. Plans for Iowa (2010 and 2020) with zero county splits and one-person deviation

Although a state redistricting commissioner believed that $d - 1$ county splits was minimum[7], we have found more than 10,000 plans with fewer county splits (in fact, no splits). Intuitively, such plans should be more and more plentiful as the number of districts and counties increase, see Shahmizad and Buchanan 2023. Pertinently, most states have more counties than Montana, and the vast majority of congressional and legislative instances have more than two districts. For computational niceties, however, we focus our efforts on Montana.

To enumerate whole-county plans with 1-person deviation, we prepared an implementation in the Python programming language that calls Gurobi, a popular commercial mixed-integer programming solver. Using a standard desktop PC, it enumerates 75 plans in the first minute, 2,290 plans in the first hour, and 10,001 plans within 40 hours (at which point we stopped the code). The supporting code and data files are publicly available on GitHub[8]. We refer the reader to Buchanan 2023 for more information about integer programming approaches to districting problems.

Due to the large file size of images, the code only draws the first 20 plans that are found, although the 10,001 plans are listed in the code printouts (identified by their county node numbers). Figure 7 shows the top 10 most compact plans, according to the average inverse Polsby-Popper score (Belotti, Buchanan, and Ezazipour 2023) also called the L^{-1} average (Duchin 2018), which is equivalent to the sum of inverse Polsby-Popper scores (Chikina, Frieze, and Pegden 2017).

7. https://x.com/dstusek11/status/1632112243696599040?s=20
8. https://github.com/AustinLBuchanan/refuting_a_widespread_belief_about_county_splits

Figure 7. Ten selected plans (among the 10,000+ that exist) for Montana with zero county splits and one-person deviation

5. Additional criteria can still be satisfied

The belief that the minimum number of county splits is $d - 1$ has been repeated in high-profile court cases, including in the Supreme Court case *Allen v. Milligan* (2023). In it, Alabama's seven congressional districts were challenged under Section 2 of the Voting Rights Act (VRA). Below, we provide excerpts from expert testimony, cross examination, Supreme Court oral arguments, and the Special Master's report. From these quotes, it is clear that many people involved in the case (from both sides) believe that $d - 1$ county splits are necessary.

- From expert testimony (*Allen v. Milligan* 2021):

  > *In order to make seven finely population-tuned districts, it is necessary to split at least six of Alabama's 67 counties into two pieces, or to split some counties into more than two pieces.*

- From cross examination before a three-judge district court (*Allen v. Milligan* 2022a):

  > Q: *At least six times, a county must be split to get the one person one vote minimal deviation that we're looking for, right?*
  >
  > A: *I think a precise way to phrase it would be that there have to be at least six additional county pieces as a way of phrasing.*
  >
  > Q: *And that's simple math that counties rarely line up where– you're unlikely to have a county that's exactly 717,000 whatever people in it to form that one perfect district, so you are going to probably have to split it at least a little to equalize it, right?*
  >
  > A: *That's the idea, yes.*
  >
  > …
  >
  > Q: *So question here: So we have got seven county splits instead of the bare minimum six that would be needed if all we were trying to do was equalize population; is that correct?*
  >
  > A: *No. As we heard, you can get under six if you're willing to tolerate more pieces.*
  >
  > Q: *Splits of like – we will stipulate when we say county splits, we're referring to splits of counties. Would that be the more accurate –*
  >
  > A: *No. So the number of split counties can – I'm sorry. I don't mean to be – I want to be accurate. The number of split counties can get below six, but if you are going to split them only two ways, then six is the minimum.*
  >
  > Q: *Okay. So number of two-way county splits is one greater here than would be necessary if all you were trying to do was equalize population; is that fair?*
  >
  > A: *I'm sorry. Yes, that's fair.*

- During Supreme Court oral arguments (*Allen v. Milligan* 2022b):

  > JUSTICE KAVANAUGH: *…you look at respecting county lines, for example, right? That's an important one. And this did. This new district did just as well, if not better, in respecting county lines. At least that's the argument. So I want to hear your response to that…*
  >
  > MR. LACOUR: *Well, three of the Duchin plans split more counties than necessary. The Cooper plans keep them together but the same number of splits. Six is the minimum you have to have.*

- From the Special Master's report, whose remedial plans all have at least six county splits (*Allen v. Milligan* 2023):

  > *Second, to minimize county splits, the Special Master proposes placing Elmore County…entirely in District 6…Finally, after avoiding county splits where possible, the Special Master also sought to minimize the number of split precincts…The three proposed remedial plans perform comparably to the 2023 Plan when measured by voting district splits, splitting 14, 13, and 14 voting districts, respectively, compared to 11 in the 2023 Plan.*

Despite these claims, we provide a plan for Alabama that splits just five counties (once each) and has one-person deviation. It also has five precinct splits. The plan is shown on the left of Figure 8 and is available on DRA[9]. It was drawn by hand by subdividing the two county clusters from Figure 3 into districts.

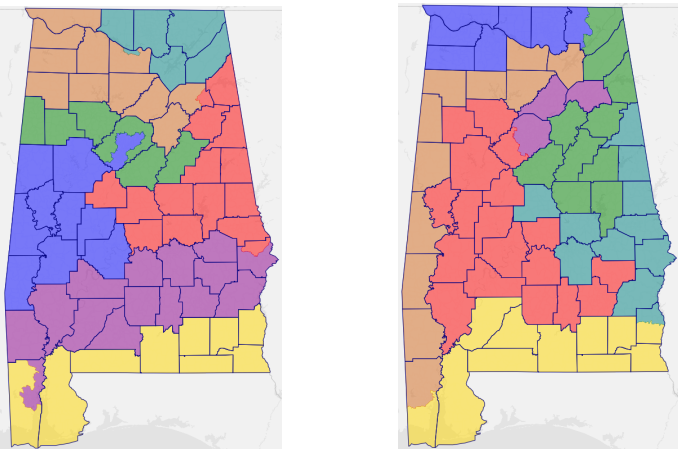

Figure 8. A reasonably configured plan for Alabama with five county splits and one-person deviation; a plan for Alabama with four county splits and 1-person deviation

This plan has two majority-Black districts, 51.33% and 50.58% by voting age population (VAP). Thus, it clears the 50% VAP threshold that the Milligan plaintiffs needed to satisfy to meet the first *Gingles* precondition and bring a §2 VRA lawsuit. We refer to the Supreme Court opinions in *Thornburg v. Gingles* (1986) and *Bartlett v. Strickland* (2009) for more details about *Gingles* 1.

The most popular way to measure district compactness is the Polsby-Popper score (Duchin 2018). According to DRA, the plan shown on the left of Figure 8 scores a 0.2211, which is comparable to that of Alabama's originally enacted plan (0.2203) and the Special Master's remedial plans (which were reported as 0.23, 0.24, 0.24).

So, we conclude that the plan on the left of Figure 8 is reasonably configured and speculate that it might well survive a *Shaw* challenge, see *Shaw v. Reno* (1993). This shows that $d-1$ county splits (or $d-1$ precinct splits) should not be assumed minimum, even if required to satisfy additional nontrivial criteria (e.g., compactness, 1-person deviation, minority representation à la *Gingles* 1).

We remark that *even fewer* splits are possible. The right of Figure 8 shows a plan with four county splits and four precinct splits[10], drawn by hand by subdividing the three clusters from Figure 3. However, it performs worse with respect to compactness and minority representation. The average Polsby-Popper score has dropped to 0.1800, and no district is majority-Black (the highest BVAP is 37.01%). We presume that it would be impermissible under §2 of the VRA.

6. Conclusion

As we have seen, it is not unusual for a state to admit a districting plan with fewer than $d-1$ county splits, even when subjected to 1-person deviation. In fact, this is more common than not. This can remain true even when additional criteria are imposed. This runs contrary to assumptions made by various districting experts, motivating us to write this paper.

To be clear, we make no normative claims about how many county splits is best in actual districting plans. It may well be that $d-1$ county splits can be justified when seeking to satisfy other

9. https://davesredistricting.org/join/7016a5b6-3bfe-46ec-b5e8-6010c26de508
10. https://davesredistricting.org/join/938221bb-cddf-4c07-9076-4665d05f45bc

criteria. Courts have also stated that they would like to avoid "county-split beauty contests" (*Allen v. Milligan* 2023). But, we should not treat a mathematical suspicion about county splits as fact until it has been verified, nor should we confuse a normative belief with a fact about reality.

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