Non-linear Schrödinger Equation

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Introduction:

"Waves occur in most scientific and engineering disciplines, for example: fluid mechanics, optics, electromagnetism, solid mechanics, structural mechanics, quantum mechanics, etc. The waves for all these applications are described by solutions to either linear or nonlinear PDEs."[4] In non-linear PDEs, dispersing and non-linear properties can change the profile of these waves as they propagate. For particular circumstances, these properties can cancel each other out to create stable waveforms called solitary waves or 'solitons' for short [5].

The nonlinear Schrödinger equation(NLS) is a variation of the well known Schrödinger equation(SE) found in quantum mechanics. In place of the potential is a function of the wave profile ψ, making this PDE non-linear. Furthermore, this and other non-linear PDEs have been shown to admit exact solutions in the form of solitons. This class of solutions models waves that propagate with a constant speed and shape. The fixed wave shape arises from the dispersion generated by the medium canceling with the non-linearity of the governing PDE. Furthermore, when multiple solitons collide, their amplitude, shape, and velocity are conserved [2]. The study of the NLS and solitons appears in various areas of physics and mathematics. Some of these areas include, non-linear optics, plasma physics, superconductivity, and Bose-Einstein Condensates [1].

In this paper, we begin by examining the (1+1)-D Cubic Non-linear Schrödinger equation (NLS^3), and propose a general form for our solutions. Then, in the next section, we limit our search to a single case of the NLS^3 to obtain bright soliton solutions. In section 3 we examine the obtained solution, its parameters, and how the wave properties evolve in time. Finally, ending with some conclusions obtained from this discussion.

1 The Cubic NLS

The (1+1)-D Cubic NLS has the general form shown in (1.1), where α and β are constants. The non-linear term $\beta|\psi^2|$ plays the role of the potential in the SE analog. There are two forms of this PDE, denoted by the $\pm$. These are commonly referred to as the focusing (+) and defocusing (-) cases.

$$i\,\frac{\partial\psi}{\partial t} \;+\; \alpha\frac{\partial^2\psi}{\partial x^2} \;\pm\; \beta|\psi^2|\psi \;=\; 0 \tag{1.1}$$

In this discussion we only consider the focusing case of this PDE, aptly named due to its property of concentrating the wave as it propagates. This constraint requires that the constants α and β be either both positive or negative.

$$\left[i\,\frac{\partial}{\partial t} \;+\; \alpha\frac{\partial^2}{\partial x^2} \;+\; \beta|\psi^2|\right]\psi \;=\; 0 \tag{1.2}$$

Note that all we really require here is that second and third terms in (1.2) be non-zero and have the same sign i.e. $\alpha\beta > 0$. Given both $\alpha < 0$ and $\beta < 0$ we have a representation for the complex conjugate solution ψ^* [3].

Lets assume plane wave solutions of the form:

$$\psi(x,t) \;=\; A(x,t)e^{i\phi(x,t)} \tag{1.3}$$

Where $A(x,t)$ corresponds to the wave amplitude, and $\phi(x,t)$ to the phase. In general, we want these arbitrary functions to take the form:

$$A(x,t) \;=\; A(x - v_e t) \tag{1.4}$$

$$\phi(x,t) \;=\; \phi(x - v_p t) \tag{1.5}$$

Where v_e corresponds to the envelope velocity, determining how quickly the wave propagates. While v_p corresponds to the phase velocity, determining its oscillation frequency in phase space.

For this model, we require that the solutions be localized. This is satisfied if the energy density (wave amplitude) only exists in a concentrated portion of space. If we have a solution with an energy density distributed throughout the spatial dimension then it is not localized. This requirement plays a role in limiting our range of solutions.

2 Bright Soliton Solution

In this section we will derive the NLS^3's bright soliton solution.

Placing the general form (1.3) into (1.2) yields:

$$\Big[iA_t \;-\; \phi_t A + \alpha A_{xx} + i\alpha\phi_{xx}A + 2i\alpha\phi_x A_x - \alpha\phi_x^2 A + \beta A^3\Big]e^{i\phi} \;=\; 0$$

Then, grouping the real and imaginary terms gives the following coupled PDEs:

$$\begin{aligned} -\phi_t A + \alpha A_{xx} - \alpha\phi_x^2 A + \beta A^3 \;&=\; 0 \\ i\Big[A_t + \alpha\phi_{xx}A + 2\alpha\phi_x A_x\Big] \;&=\; 0 \end{aligned}$$

From (1.4&1.5) we note that $A_t = -v_e A_x$ and $\phi_t = -v_p\phi_x$.

Using these relations in the coupled system:

$$-v_p\phi_x A + \alpha A_{xx} - \alpha\phi_x^2 A + \beta A^3 \;=\; 0 \tag{2.1}$$

$$-v_e A_x + \alpha\phi_{xx}A + 2\alpha\phi_x A_x \;=\; 0 \tag{2.2}$$

Multiplying (2.2) by A leads to an integrable form:

$$-v_e A_x A + \alpha\phi_{xx}A^2 + 2\alpha\phi_x A_x A \;=\; 0$$

$$\Downarrow$$

$$\frac{\partial}{\partial x}\Big(\frac{-v_e A^2}{2} + \alpha\phi_x A^2\Big) = 0$$

$$\Downarrow$$

$$\frac{-v_e A^2}{2} + \alpha\phi_x A^2 \;=\; \cancelto{0}{C_1}$$

We want to obtain spatially localized solutions that vanish in the infinite limit. Therefore, we require that as $|x| \to \infty$, $A \to 0$ so $C_1 = 0$ [3].

From this we find:

$$\phi_x \;= \frac{v_e}{2\alpha} \tag{2.3}$$

$$\Downarrow$$

$$\phi(x,t) \;=\; \frac{v_e}{2\alpha}(x - v_p t) + \cancelto{0}{C_2} \tag{2.4}$$

We can impose $C_2 = 0$ by choosing the appropriate reference time [3].

Using $k = \frac{v_e}{2\alpha}$ and $\mu = \frac{v_e v_p}{2\alpha}$ we have:

$$\phi(x,t) \;=\; kx - \mu t$$

Now we want to solve the other coupled PDE.

Using (2.3) in (2.1) and multiplying by αA_x yields:

$$\frac{v_e v_p}{2} A_x A + \alpha^2 A_{xx} A_x - \frac{v_e^2}{4} A_x A + \alpha\beta A_x A^3 \;=\; 0$$

Which leads to the integrable form:

$$\frac{\partial}{\partial x}\left[\frac{\alpha^2 A_x^2}{2} + \frac{\alpha\beta}{4}A^4 + \frac{v_e v_p}{4}A^2 - \frac{v_e^2}{8}A^2\right] \;=\; 0$$

$$\Downarrow$$

$$\frac{\alpha^2 A_x^2}{2} + \frac{\alpha\beta}{4}A^4 + \frac{v_e v_p}{4}A^2 - \frac{v_e^2}{8}A^2 \;=\; \cancelto{0}{C_3}$$

Again, setting $C_3 = 0$ for localized solutions. This result can be written as:

$$\frac{1}{2}\alpha^2 A_x^2 + V_{eff}(A) \;=\; 0 \qquad (2.5)$$

Where $V_{eff}(A)$ functions as the pseudo-potential for the solution [3].

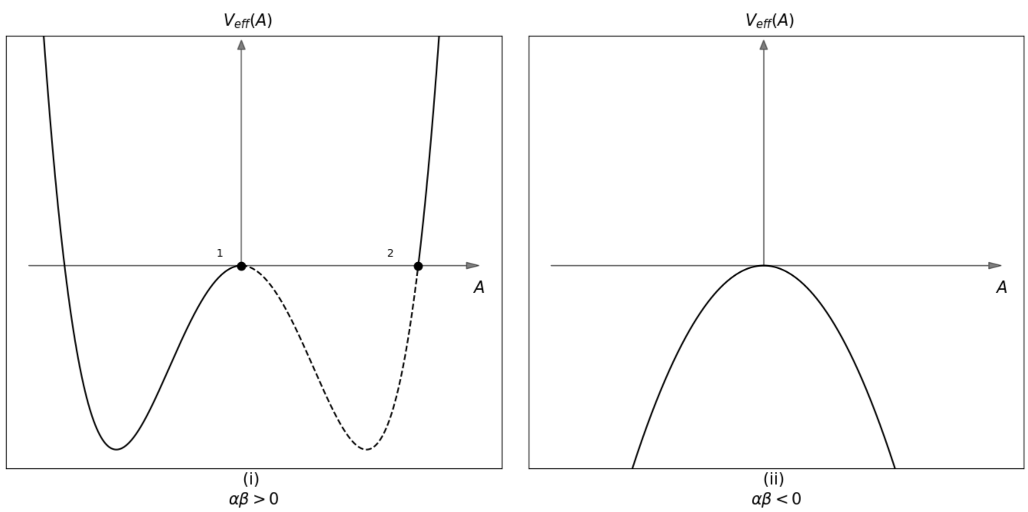

Figure 1: Shape of $V_{eff}(A)$ for varying $\alpha\beta$. In (i) $\alpha\beta > 0$ and in (ii) $\alpha\beta < 0$

Figure 1 depicts V_{eff} vs A as a function of $\alpha\beta$, where (i) is for $\alpha\beta > 0$ and (ii) for $\alpha\beta < 0$. Note that only Fig.1(i) admits localized solutions, where the soliton oscillates between points 1 and 2 on the graph. In Fig.1(ii) we see that particle motion is unstable, due to the pseudo-potential lacking a local minimum. This highlights our requirement of $\alpha\beta > 0$ to obtain stable localized solutions.

Solving for A_x in (2.5) leaves the following PDE:

$$A_x \ = \ \Big[(\frac{v_e^2 - 2v_e v_p}{4\alpha^2})\phi^2 - (\frac{\beta}{2\alpha})\phi^4 \Big]^{1/2}$$

Let $\lambda_1 = \frac{v_e^2 - 2v_e v_p}{4\alpha^2}$ and $\lambda_2 = \frac{\beta}{2\alpha}$ to yield the separable PDE:

$$\frac{\partial A}{(\lambda_1 A^2 - \lambda_2 A^4)^{1/2}} \ = \ \partial x$$

Where integrating[1] gives the following result:

$$\frac{-\tanh^{-1}\left(\frac{\lambda_1 - \lambda_2 A^2}{A}\right)^{1/2}}{\lambda_1^{1/2}} \ = \ x - v_e t$$

Solving for A using $\tanh^2(-x) = \tanh^2(x)$ and $1 - \tanh^2(x) = \text{sech}^2(x)$:

$$A \ = \ \left(\frac{\lambda_1}{\lambda_2}\right)^{1/2} \text{sech}(\lambda_1^{1/2}[x - v_e t])$$

To obtain non-zero real solutions, we require that $\lambda_1, \lambda_2 > 0$. $\lambda_2 > 0$ is already satisfied by the restriction $\alpha\beta > 0$. For $\lambda_1 > 0$ we must make another restriction, requiring that $v_e^2 - 2v_e v_p > 0$.

Using $A_0 = \left(\frac{\lambda_1}{\lambda_2}\right)^{1/2} = \left(\frac{v_e^2 - 2v_e v_p}{2\alpha\beta}\right)^{1/2}$ and $\lambda_0 = \left[\frac{2\alpha}{\beta A_0^2}\right]^{1/2}$ we have:

$$A(x,t) \ = \ A_0 \,\text{sech}\left(\frac{x - v_e t}{\lambda_0}\right) \tag{2.6}$$

Using (2.4) and (2.6) in (1.3) we obtain the soliton solution:

$$\psi(x,t) \ = \ A_0 \,\text{sech}\left(\frac{x - v_e t}{\lambda_0}\right) e^{i(kx - \mu t)} \tag{2.7}$$

This solution to the focusing NLS^3 is called a bright soliton. As previously mentioned, solitons are waveforms that propagate with a constant shape and speed. This soliton is called 'bright' due to its intensity profile. Furthermore, as can be checked, this exact solution satisfies the focusing NLS^3 under the provided restrictions.

[1] Integrated using Wolfram Alpha

3 Solution Analysis

In the previous section we found bright soliton solutions to the NLS^3:

$$\psi(x,t) \;=\; A_0 \,\mathrm{sech}\left(\frac{x - v_e t}{\lambda_0}\right) e^{i(kx-\mu t)}$$

Where the constants are:

- $A_0 \;=\; \left(\frac{v_e^2 - 2v_e v_p}{2\alpha\beta}\right)^{1/2}$
- $\lambda_0 \;=\; \left[\frac{4\alpha^2}{v_e^2 - v_e v_p}\right]^{1/2}$
- $k \;=\; \frac{v_e}{2\alpha}$
- $\mu \;=\; \frac{v_e v_p}{2\alpha}$

In doing this, we defined the pseudo-potential (V_{eff}) which obeyed the relation in (3). This term, named due to its physical significance, is a propagating potential created by the soliton. This potential has a double-well shape which functions to localize the wave.

$$V_{eff} \;=\; \frac{\alpha\beta}{4}A(x,t)^4 + \frac{v_e v_p}{4}A(x,t)^2 - \frac{v_e^2}{8}A(x,t)^2$$

Also physically significant is the soliton intensity[2] which is given by:

$$|\psi(x,t)|^2 \;=\; |A_0|^2\,\mathrm{sech}^2\left(\frac{x - v_e t}{\lambda_0}\right)$$

Solitons, their intensity profile, and the potential propagate as shown below.

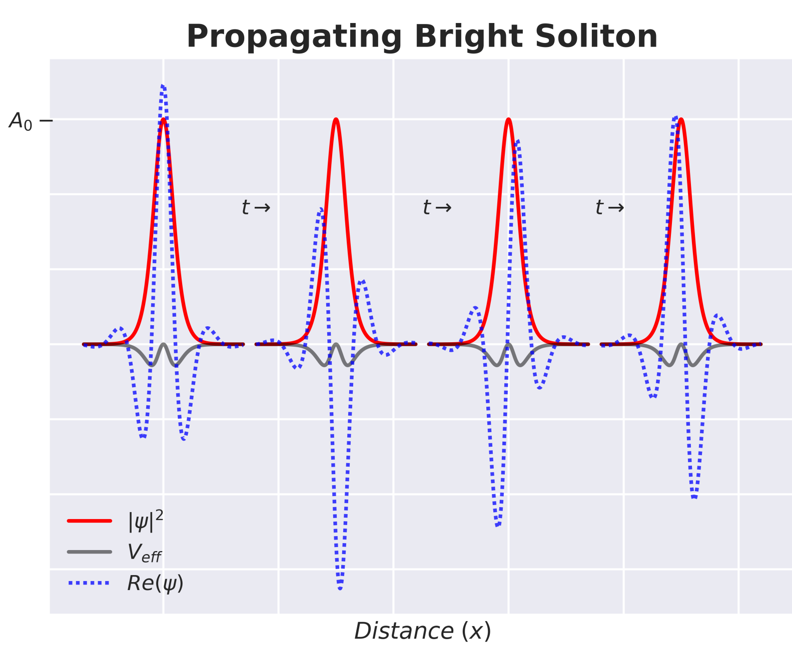

Figure 2: Soliton intensity, wave form, and pseudo-potential propagation

[2]Typically the intensity of electromagnetic or mechanical waves

In Figure 2, the first wave packet on the left represents the soliton and related quantities at time $t = 0$. Each of the following packets is a snapshot of the propagating waveforms as a function of distance for subsequent time steps. Inside each packet, there are three waveforms. The soliton intensity, $|\psi|^2$, is shown by the solid red line. While the dotted blue line shows the real part of the soliton wave. Furthermore, the double-well potential discussed in Figure 1 is depicted by the light grayline. The interesting properties of this system are twofold: 1) The magnitude of the soliton intensity, A_0, remains constant as it propagates and 2) the potential created by the soliton, V_{eff}, propagates with the wave to create stable localized solutions.

In deriving this solution we limited our search to the focusing case of the NLS^3. We also assumed the wave must be concentrated within some finite region, a requirement for localized solutions. These assumptions led to the following restrictions:

- $\alpha\beta > 0$
- $v_e^2 - 2v_e v_p > 0$

Under these restrictions, we obtained solitary wave solutions in the form of bright solitons. Bright solitons are solitary waves with an intensity above that of the background [6]. For our requirement that $\alpha\beta > 0$, the solution ψ generates its own potential well (V_{eff}) creating a stable localized wave. This phenomenon is known as self-trapping. Furthermore, the constant A_0 present in the amplitude function represents the maximum amplitude of the wave intensity. The wave width, determined by the constant λ_0, is inversely proportional to the amplitude. This relation between the amplitude and wave width suggests that the wave's energy density is conserved based on the constants α and β.

4 Conclusion

This paper was motivated by the existence of wave solutions to many physical systems governed by differential equations. In our case, we focused on a PDE called the non-linear schrödinger equation and its soliton solutions which appear in various fields of physics, biology, and mathematics research [3].

We began by defining the cubic non-linear schrödinger equation in one time and spatial dimension. There were two cases of the NLS^3 and we limited our discussion to the focusing case. Then, we assumed that this equation would obey some form of plane wave solutions. Using this assumption, we obtained a system of coupled PDEs. To solve this system and obtain stable solutions, we imposed two restrictions. For our requirement of stable localized solutions, we found that the solution ψ generates its own potential well, a property known as self-trapping.

After solving the coupled PDEs, and obtaining functions for the amplitude and phase of the wave, we then combined them to obtain exact solitary wave solutions in the form of bright solitons. Finally, we looked at how bright solitons and their accompanying quantities such as waveform, intensity, and pseudo-potential evolve in time. In doing this, we also identified some of the unique properties exhibited by propagating solitons. Continuations of this discussion would consider the defocusing case for the NLS^3 and its admittance of another class of solitary wave solutions.

References

[1] Mark Ablowitz and Barbara Prinari. "Nonlinear Schrodinger systems: continuous and discrete". In: Scholarpedia 3.8 (2008), p. 5561.

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[3] Mauro Sergio Dorsa Cattani, José Maria Filardo Bassalo, et al. "Solitons in nonlinear Schrödinger equations." In: Publicação IFUSP 1678 (2013), pp. 1–16.

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[5] Michael Stone and Paul Goldbart. Mathematics for Physics. 2002.

[6] Norman J Zabusky and Mason A Porter. "Soliton". In: Scholarpedia 5.8 (2010), p. 2068.

Figures

- Figure 1: adapted from figure 1 in reference [3].
- Figure 2: Plot generated using *Python* and the equations derived in this document.