DISCRETE MATHEMATICS

Final Exam(600+60)

Instruction:

- DO NOT LEAVE any problem BLANK including the bonus question.
- SHOW YOUR WORK so I can give partial credits.
- READ THE QUESTIONS.
- There are tons of part for each problem. But don't be scared most of them require at most 3 lines answer.
- You have 4 hours to finish it.
- If you use the back side, indicate that you have done so.
- There are 6 questions with the bonus question.
- This exam comprises of a lot of mini problems. Make sure you answer them all. Check twice before you hand it in.
- You are allowed 2 A4 Cheat Sheet + Calculator.
- Have fun.

Problem	Full Score	Your Score
1	100	
2	100	
3	100	
4	100	
5	100+30	
6	100	
Bonus	30	
Total	600+60	

Useful Formula:

Euler formula: $f + v = e + 2$

Max Edge for planar graph $e \leq 3v - 6$

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

$$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$$

$$E[X] = \sum_{\omega \in S} Pr(\omega)X(\omega)$$

$$Var[X] = E[(X - E[X])^2] = E[X^2] - E[X]^2$$

Name:_____ ID: _____ 3/16

1) (100 points)Easy Counting stuff. For each question, you **DO NOT** need multiply it out. But be sure to explain each term so when you get it wrong I can give you partial credits.

A) (20 points)How 5-digit numbers are there that use 1, 2, 3, 4 and 5 exactly once each?

B) (20 points)How many binary strings(string of 0 and 1) of length 20 are there that has exactly 5 ones?

C) (20 points)If you draw 6(six) cards hand from a normal deck of card, how many hands are there that have 2 three-of-a-kind? (Do not double count)

D) (20 points) How many numbers are there from 1 to 1,000 that is **NOT** divisible by 2, 3 nor 10? (Be careful)

E) (20 points) How many ways are there to split 20 coins among 6 students? (Some students may get no coin.)

Name:_____ ID: _____ 4/16

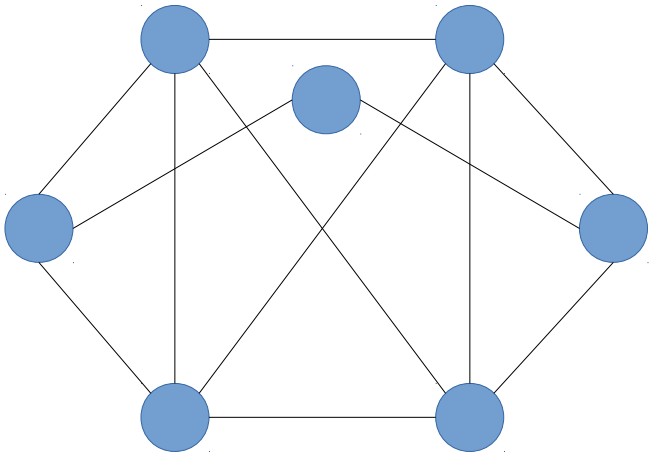
2) (100 points) Easy Stuff 2

A) (20 points)Given a Jar of 3 Red Marbles and 2 Blue Marbles. If I draw two marbles at the same time, what is the probability that the two marbles will be of different colors?

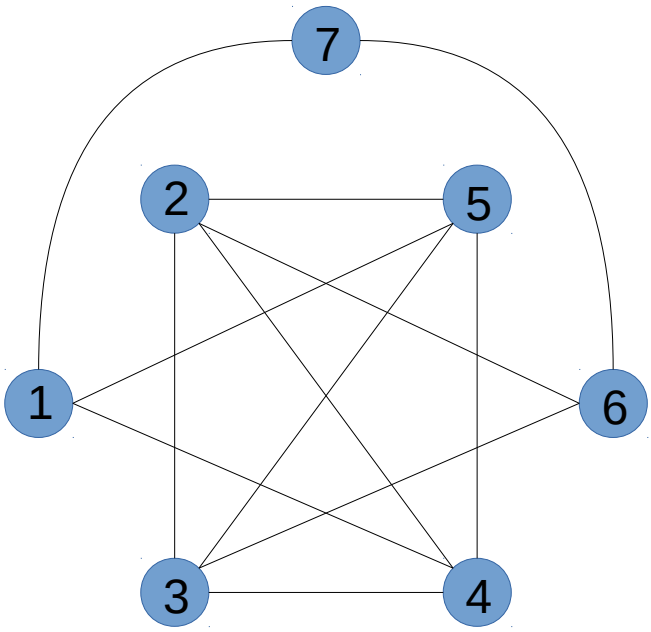
B) (20 points) Given the same Jar of 3 Red Marbles and 2 Blue Marbles. Now I draw one marble, look at the color then place it back in the jar. Then I draw another marble. What is the probability that the two marbles will be of different colors? (Hint: Not the same as A.)

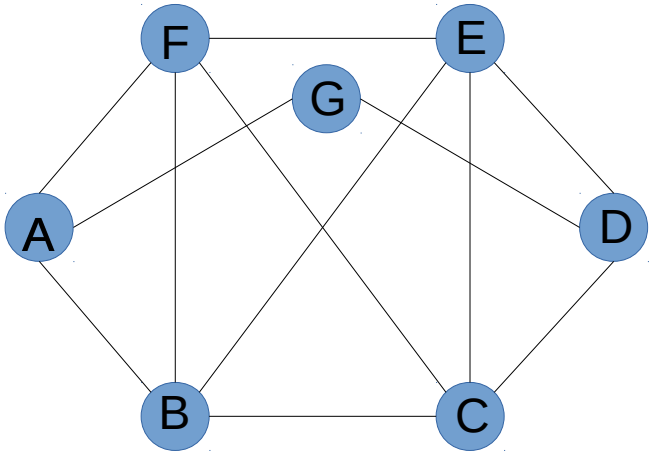
C) (20 points) Draw a graph with 5 vertices, 5 edges with exactly 1 node of degree exactly 3. (Just draw one).

D) (20 points) Find an Eulerian tour(use every edge exactly once) for this graph. Label the edges with numbers so I can follow.



E) (20 points) Find isomorphism(the bijection) between these two graphs. (Hint: start with the node with degree 2)

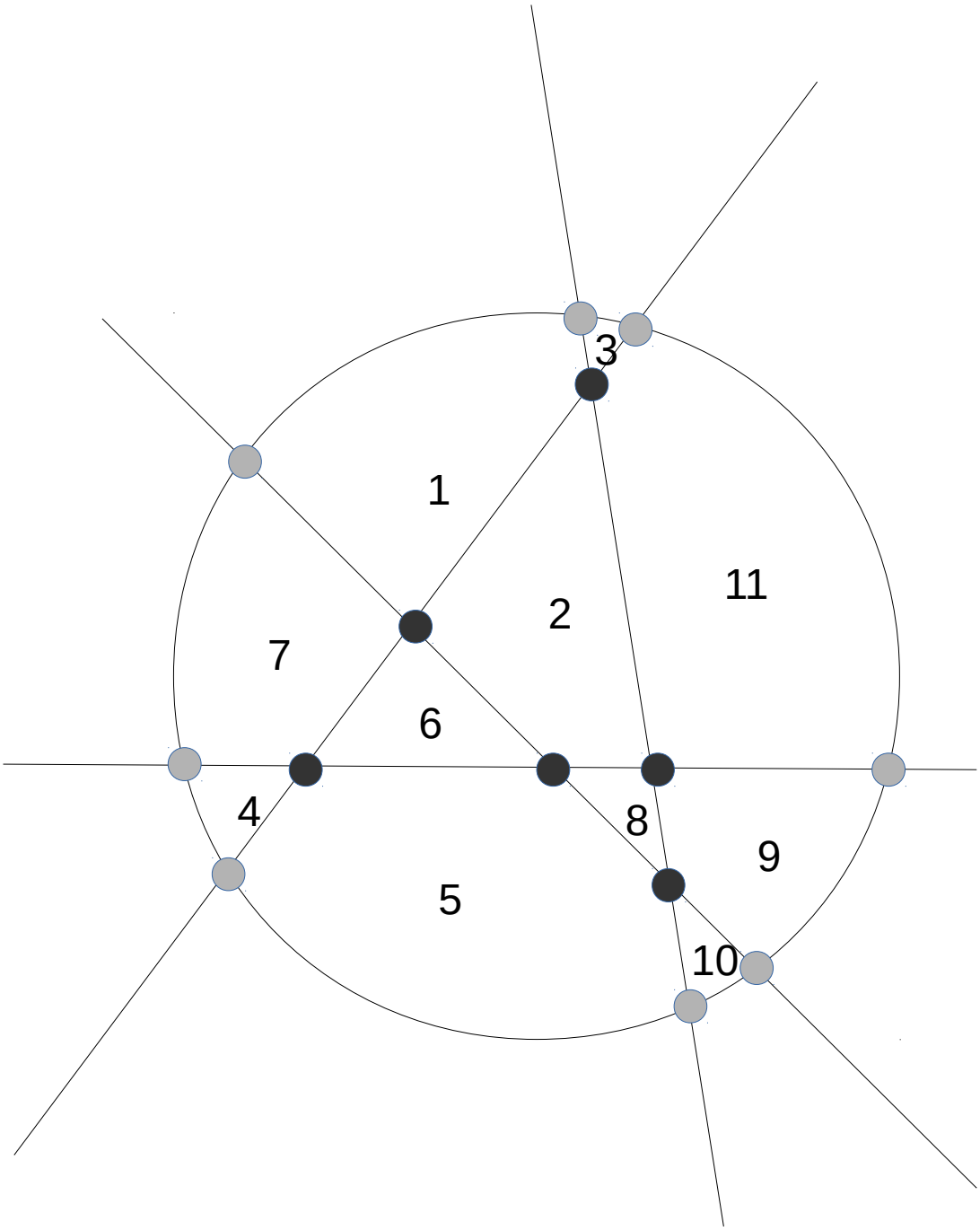




3) Given a circle. Let us draw n straight lines such that every line intersect every other straight lines **inside** the circle.

Moreover, every intersection inside the circle is two line intersection. (No more than 2 line intersect at one point).

For example here is a picture when $n = 4$ lines intersect within a circle.



Name:_____ ID: _____ 7/16

A)(15 points) How many intersection are there for **circle** and the **straight lines**? (Only circle and straight line intersection like ones shown with grey dots in the example)

B) (15 points)How many straight line intersections are there **inside** the circle?(ones shown with black dots in the example)

C) (15 points) How many straight line sections are there **within** the circle? (There are 16 in the example)

D) (15 points) How many circular segments are there? (The number of line segments the circle got divided into. Ex: there are 8 in the example.)

E)(25 points) Use Euler's Formula to find out the number of region **inside** the circle? (I only need the **inside**. There are 11 in the example)

F)(15 points) Now forget the circle. Suppose we have an infinite plane, then we draw n infinite straight lines on the plane such that i) every intersection is a two-line intersection and ii) every line intersect every other lines. How many regions does the infinite plane get divided into? This is actually maximum number of region you can get with n straight lines. (If you find this easy, you are on probably right.)

4) Consider a very simple game of two players.

The dice use in this game has 3 numbers on it $\{1, 2, 3\}$. All are equally likely.

Player A get to roll the dice twice and take the sum as his/her value. Player B get to roll the dice once and then we multiply it by two to obtain as his/her value. The player with the higher value wins.

For example: Player A rolls 1 and 2 : His value is $1 + 2 = 3$
Player B rolls 1: B's value is $1 \times 2 = 2$
Therefore A wins here.

A) (20 points)Is this a fair game(both players are equally likely to win)? If not, who win?
(Be careful of draw)

B) (20 points)What is the probability that the first dice for A is 2 **and** A wins?

C) (20 points)**Given** that the first dice of A is 2 what is the probability that A wins?

D) (20 points) **Given** that A wins what is the probability that the first dice is 2?

E) (20 points) Are the event that A wins and that the first dice is 2 **independent**?

Name:_____ ID: _____ 11/16

5) (100+30 points)Consider a market for Yoyo(The candies I often give you in class).

Pink Yoyo is now valued at 20 Baht a piece. But, at the end of month, it could go up to 25 Baht with probability 3/5 and down to 15 Baht with probability 2/5.

Red Yoyo is now also valued at 20 Baht a piece. But, at the end of month, it could go up to 200 Baht with probability 1/10 and down to 10 Baht with probability 9/10.

For A, B, C, D, E, assume that the price at the end of the month for Pink Yoyo and Red Yoyo are independent.

A) (10 points)Find the **expected gain** and **variance on gain** at the end of the month for buying 1 **Pink Yoyo.**

B) (10 points)Find the **expected gain** and **variance on gain** at the end of the month for buying 1 **Red Yoyo.**

D) (10 points) If I buy **50 Pink Yoyo and 50 Red Yoyo** today, what is my **expected gain/loss** at the end of the month?

E) (10 points)What would be the **variance** of my return if I buy **50 Pink Yoyo and 50 Red Yoyo** today?

F) (30 points)Now we do not assume that the price of Red and Pink Yoyo are independent. In fact, whenever the price of Pink Yoyo go ***up*** the price of Red Yoyo always ***down***.

If the probability of Red Yoyo going ***down*** **given that** the price of Pink Yoyo going ***up*** is 1:

What is the probability of Red Yoyo going ***up*** **given that** the price of Pink Yoyo going ***up***?
What is the probability of Red Yoyo going ***down*** **given that** the price of Pink Yoyo going ***down***?
What is the probability of Red Yoyo going ***up*** **given that** the price of Pink Yoyo going ***down***?
(Hint: Draw the tree.)

G) (10 points) What would be the **expected gain/loss** for the condition given in F) if I buy 50 Pink Yoyo and 50 Red Yoyo?

H) (10 points) What would be the **variance of the return** for the condition given in F) if I buy 50 Pink Yoyo and 50 Red Yoyo? (Be careful!)

I) Bonus.(30 points). If I have exactly 2,000 baht now, I can buy any amount of pink yoyo and red yoyo I want within given budget. Find the relation between the variance and expected return. This is the basis for an important practical concept called Efficient Frontier in Finance.

Name:_____ ID: _____ 14/16

6) (100 points)Graph Theory Proof.

Pick One from these two. Doing two won't result in extra credit. Indicate clearly which one you picked. This is much easier than it looks.

6-1) Six Color Theorem. This question has two parts

A) Show that for any planar graph there is at least one vertex with degree 5 or less. Note that we do not assume connectedness for planar graph here. (Hint: use the inequality we got from Euler's Formula in class).

B) Show that any planar graph is 6-colorable. (Hint: Use A. This is similar to d+1-colorable we did in class. They aren't the same though. You may also use the fact that a planar graph with 1 vertex taken out is still a planar graph.)

6-2) Use induction to show that every tree has at least 2 leaves.

Blank Page for Problem 6

Name:_____ ID: _____ 16/16

Bonus:(Partial Credit will be given(30)):

Despite the wordiness, this is quite simple though.

Thailand national lottery is announced twice a month: on the first, and on the sixteenth.

The most common reward can be simplified as follows. The player pick a number from 00-99 (100 numbers). Then if the announced number matches the number the player picked, then the player wins. The player loses otherwise.

Since superstition is still running strong in Thailand, people are gullible to all sort of random tricks for predicting the next lottery number. Every once in a while there will be news about random people which psychic power to predict the next lottery number. I hope that after you have been fooled so many times in class you will think twice before believing in such thing.

This question is about how often such news will show up.

Suppose these are 2,000 people guessing random numbers independently. The news reporter will write a news about that person if they guess the number correctly 2 times within that same month.

How long would I expect to hear such news? (Or, how often should I expect such news to come up?)