Name:______________________________ ID:________________

Discrete Mathematics

Midterm (600 + 40 + 50)

Instruction:

- Do not leave any problem blank! Partial credits will be given.(Except No. 7)
- You have 4 hours to finish this. You should not need that much though.
- You are allowed to use 2 **Handwritten** A4 Cheat Sheet in the exam.
- Ask if the question is unclear. No hint will be given though.
- The full score for Problem 1 to 6 is 640 but you only need to get 600.
- Problem 7 is there just to keep you in the exam room long enough if you are bored. 50 points doesn't worth the headache.
- Make me proud.

Warning:

- Proposition is **not** a number!
- If your work get messy, write down a road map where I should start reading.
- Do not bother with problem 7 unless you are done with all others. No partial credit will be given for problem 7. Partial credit will be given to all others though.

Problem	Full Score	Score
1	120	
2	100	
3	100	
4	100	
5	120	
6	100	
7	50	
Total	600	

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Useful Definition/Formula

- $f \in O(g)$ iff $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ is finite
- $f \in o(g)$ iff $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$
- $f \in \Omega(g)$ iff $\lim_{n\to\infty} \frac{f(n)}{g(n)} > 0$
- $f \in \omega(g)$ iff $\lim_{n\to\infty} \frac{f(n)}{g(n)} \to \infty$
- $f \in \Theta(g)$ iff $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c; c \neq 0$

Definition: A is divisible by B iff $A = kB \quad \exists k \in I$

Useful Sum:

- $\sum_{x=1}^{n} x = \frac{n(n+1)}{2}$
- $\sum_{x=1}^{n} x^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{x=1}^{n} x^3 = \left(\frac{n(n+1)}{2}\right)^2$

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Problem 1) (120 points) Easy Stuff. 30 points each.

a) Use truth tables to show that $\sim (P \to Q)$ is the same thing as $P \wedge \sim Q$

b) Fill in O, o, Ω, ω or Θ in the blank spot. Fill all that apply.

1. $x^2 + 2x + 1 \in \boxed{\ ?\ }(x^2)$

2. $x + 2\log x + 1 \in \boxed{\ ?\ }(\log x)$

c) Find the close form expression of $\sum_{y=1}^{n}\sum_{x=1}^{n} 3^x + y$

d) Prove or Disprove the following statement.
If the sum of two numbers is even then one of the number must be even.

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Problem 2) (100 points) Show that

The sum of n consecutive numbers is divisible by n for every positive odd integer n.

Note that the sum does not have to start with 1. For example,

$123 + 124 + 125 = 124 \times 3$ which is divisible by 3.

$111 + 112 + 113 + 114 + 115 = 113 \times 5$ which is divisible by 5.

Hint: You don't need to use induction for this. You could if you want though.

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Problem 3) (100 points) Do one of these problems. Doing two will not give you extra credit. Be sure to state your proposition clearly and make sure your propositions are not numbers. State which one you pick or I'll pick the one you got the lowest score.

a) Let F_n be Fibonacci number such that $F_1 = 1$, $F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$.
Show that $F_1^2 + F_2^2 + \ldots + F_n^2 = F_n \times F_{n+1}$

b) Use induction to show that $\sum_{i=1}^{n} i \times 2^i = (n-1) \times 2^{n+1} + 2$. You must use induction.

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Problem 4) (100 points) Do one of these problems. Doing two won't give you extra credit. State your proposition clearly. State which one you pick or I'll pick the one you got the lowest score.

a) Brew&Bev issue another series of coupons. This time they are 5 Baht and 3 Baht coupon. Find the **minimum** price w such that everything in Brew&Bev with price $\geq w$ can be bought with coupons with no change required. Prove it as well.

b) Show that every positive integer is a product of a **power of two** and an **odd integer**.
Ex: $7 = 2^0 \times 7$, $36 = 2^2 \times 9$, $100 = 2^2 \times 25$

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Problem 5) (120 points) Recurrence. No need to prove it. Just give the answer with justification.

5.1) (40 points) Solve this recurrence: $T(n) = T(n-1) + 2n; T(1) = 1$

5.2) (40 points) Solve this recurrence: $T(n) = 2T(n/2) + 3n$; $T(1) = 1$

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5.3) (40 points) Solve this recurrence $T(n) = 6T(n-1) - 8T(n-2); T(0) = 3, T(1) = 8$

Hint: The answer should be all integer.

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Problem 6) (100 points) Consider a board game similar to HW3 Chessboard problem:

1. The game is played on a 4x4 grid with +1 or -1 on it.
2. The player can pick a row or a column. Every number on that row/column will be flipped. Specifically, all the 1's in that row/column will become -1 and all the -1's in that row/column will become 1.
3. After, the row/column is flipped, the player can repeat the process of picking a row/column and flip the entire row/column.

The question is if you start the game in the following state (notice the -1 in the last row):

1	1	1	1
1	1	1	1
1	1	1	1
1	1	-1	1

Show that you **cannot** make all the cells 1.

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Problem 7) (50 points)

Do not even attempt on this if you have not finish the other 6. This is a lot harder than its look.

Consider the same game in problem 6. But this time, in addition to just being able to pick any row or column. I'll allow you to pick any line parallel to the diagonal lines as well. Specifically you are allowed to change the corners freely. Show that you still can't make all the cells 1 if you start with the state given in problem 6.

Hint: You may need a new invariant.

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