ID: ____________________ Name: ____________________________________

Discrete Mathematics

Midterm Exam T3 2016

Instruction

- Write your name
- Read the questions carefully.
- You have 4 hours to finish the exam.
- There are 6 problems. Each problem worths 100 points. 600 points in total. You only need to get 540 points to get full score.
- Attempt all problems, state your reasons *clearly* and *legibly*, because partial credits will be given.

Question	Full Score	Your Score
1	100	
2	100	
3	100	
4	100	
5	100	
6	100	
Bonus	30	

Total: ______ /540

ID: ___________________ Name: ___________________

Useful Formula and Definitions

Asymptotics

Definiton	Definition			Intuition
Asym. Equal	$f \sim g$	iff	$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$	$f \underbrace{\equiv}_{x\to\infty} g$
Big Oh	$f \in O(g)$	iff	$\lim_{x\to\infty} \frac{f(x)}{g(x)} < \infty$	$f \underbrace{\leq}_{x\to\infty} g$
Little Oh	$f \in o(g)$	iff	$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$	$f \underbrace{<}_{x\to\infty} g$
Little Omega	$f \in \omega(g)$	iff	$\lim_{x\to\infty} \frac{f(x)}{g(x)} \to \infty$	$f \underbrace{>}_{x\to\infty} g$
Big Omega	$f \in \Omega(g)$	iff	$\lim_{x\to\infty} \frac{f(x)}{g(x)} > 0$	$f \underbrace{\geq}_{x\to\infty} g$
Theta	$f \in \Theta(g)$	iff	$\lim_{x\to\infty} \frac{f(x)}{g(x)} = c, c \neq 0$	$f \underbrace{=}_{x\to\infty} g$

Sum

$$
\begin{aligned}
1 + 2 + 3 + \ldots + n &= \frac{n(n+1)}{2} \\
1^2 + 2^2 + 3^2 + \ldots + n^2 &= \frac{n(n+1)(2n+1)}{6} \\
1^3 + 2^3 + 3^3 + \ldots + n^3 &= \left(\frac{n(n+1)}{2}\right)^2
\end{aligned}
$$

Integral

$$\int x^n \; dx = \frac{1}{n+1} x^{n+1} \qquad \text{if } n \neq -1$$

$$\int \frac{1}{x} \; dx = \ln(x)$$

Quadratic

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

ID: ____________________ Name: ____________________________________

1. Easy stuff(100 points. 20 each.)

   (a) Draw the truth table for $(P \to Q) \vee (Q \to P)$

   (b) Which of these asymptotic symbols $(\{\sim, o, O, \omega, \Omega, \Theta\})$ are applicable for

   $$f(x) = \_\_\big(g(x)\big)$$

      i. $f(x) = (x)^2, g(x) = 2x^2$

      ii. $f(x) = x^2, g(x) = x + 10^{999}$

   (c) Find the closed form formula for the following sum/product:

      i. $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{m}(ij + 1)$

      ii. $\displaystyle\prod_{i=1}^{n}\prod_{j=1}^{m}2^{i+j}$

(d) Prove or Disprove the following proposition.(Read the question very carefully)

$$\text{Let } a, b \in I. \text{ If } ab = 0 \text{ then } a = 0.$$

(e) Use integral bound to find the *upperbound* for the following sum. You may leave your answer as integral. But make sure you specify the limits.

$$\sum_{x=1}^{n} \sqrt[2]{x}$$

ID: ____________________ Name: ___________________________________

2. Easy Proof.

   (a) (50 points) Recall that

$$|a| = \begin{cases} -a & \text{if } a \leq 0 \\ a & \text{if } a > 0 \end{cases}$$

   Given $x, y \in \mathbb{R}$ show that $|xy| = |x||y|$.

   (b) (50 points) Read this carefully. Let n be an interger.
   Show that if n^2 is odd then n is odd.

3. (100 points)Pick one. Indicate the one you pick. If you don't, I'll pick the one you got less score. Doing two won't give you more score. Be sure to indicate where you use inductive hypothesis.

   (a) Prove the following identity (look at the sign carefully)

   $$-1^2 + 2^2 - 3^2 + 4^2 + \ldots + (-1)^n n^2 = (-1)^n \frac{n(n+1)}{2}$$

   (b) Show that

   $$7 | (8^n - 1) \quad \forall n \in I^+$$

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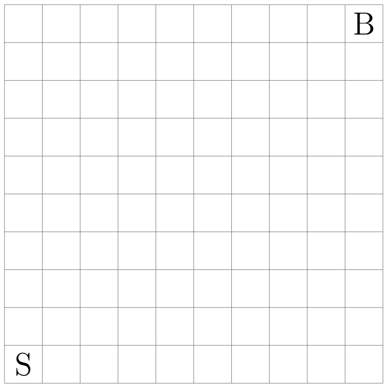
ID: ____________________ Name: ___________________________________

4. Pick one. Indicate the one you pick. If you don't, I'll pick the one you got less score. Doing two won't give you more score.

   (a) New and Nice are trying to go from parking lot to a bookstore. They park their car at S and want to go to the book store at B. The parking lot layout is a 10x10 grid shown below.



   New and Nice devise a game which the goal is the be **the one to reach** the bookstore. The rule is simple. For each turn, the player can pick to do **only one** of the two actions.

   - Go up $n \geq 1$ steps.
   - **Or**, go right $n \geq 1$ steps.

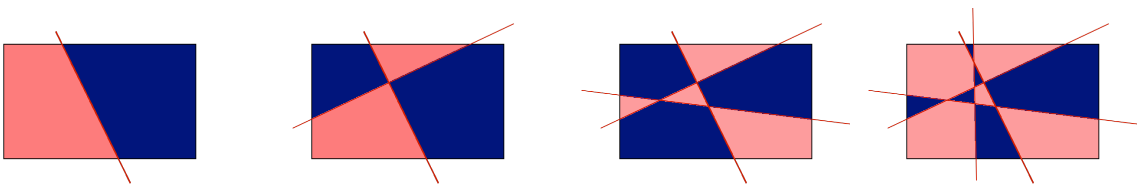
   The restriction is that they cannot go out of bound.

   For example, this is an example of a possible outcome from the game. Let the lower left corner be (1,1).

   i. New go right 3 steps. (They are now at (4, 1))
   ii. Nice go right 6 steps. (They are now at (10, 1))
   iii. Then New go up 9 steps and reach (10,10) which is where the bookstore is. So New wins.

   If **New starts first** who has the winning strategy? Prove also that it is a winning strategy.

   (b) Nop found a new game to play. Given a piece of A4 paper, he notice that if he draw a line **through** the paper. He can paint the paper with exactly 2 colors with no two adjacent areas share the same color. As he keep going, he found that he can do that for any number of straight lines drawn through the paper. Example for some of his plays can be seen below



   Show that Nop can paint the paper such that no two adjacent area share the same color for any number of straight lines drawn.

   Hint: notice from the pictures above how the color change from one to the next one.

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5. Let's consider a bags of 1000 yoyos: 700 grapes and 300 cola. Each students takes turns picking out 3 yoyo at a time. Then eat the flavor that is the majority.

   For example, if the yoyo that got picked are 2 grapes and 1 cola. The student gets to eat the grape yoyo.

   AJ Piti will get to eat the last 2 yoyos. Show that AJ Piti will always get to eat 1 cola and 1 grap yoyo.

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6. (100 points)Solve the following recurrence. Just find the solution. No need to verify it using induction.:

   (a) (30 points)$T(n) = T(n-1) + 2n^3; T(1) = 1$ You may find a formula on the front page useful.

   (b) (30 points)$T(n) = 2T\left(\frac{n}{2}\right) + n \quad \text{where } T(1) = 1$

(c) (40 points)$T(n) = 6T(n-1) + 7T(n-2); T(0) = 9, T(1) = 31$

ID: ____________________ Name: ___________________________________

7. Bonus.(30 points) No partial credit for this one. This is just to keep you sit there for a while. Prove the following:

   Given a set of integer pairs $(a_1, b_1), (a_2, b_2), (a_3, b_3) \dots (a_n, b_n)$ we can always find integer c can d such that

$$\prod_{i=1}^{n}(a_i^2 + b_i^2) = c^2 + d^2$$

   Hint: Find an identity that looks like (This is called Fibbonacci's identity.)

$$(p^2 + q^2)(r^2 + s^2) = (\dots - \dots)^2 + (\dots - \dots)^2$$