ID: ____________________ Name: ___________________________________

Discrete Mathematics

Midterm Exam T3 2019

Instruction

- Write your name
- Read the questions carefully.
- There are 6 problems. Each problem worths 100 points. 600 points in total. You only need to get 540 points to get full score.
- Attempt all problems, state your reasons *clearly* and *legibly*, because partial credits will be given.

Question	Full Score	Your Score
1	100	
2	100	
3	100	
4	100	
5	100	
6	100	
Bonus	30	

Total: ______ /540

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Useful Formula and Definitions

Asymptotics

Definiton	Definition			Intuition
Asym. Equal	$f \sim g$	iff	$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 1$	$f \underbrace{\equiv}_{x\to\infty} g$
Big Oh	$f \in O(g)$	iff	$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} < \infty$	$f \underbrace{\leq}_{x\to\infty} g$
Little Oh	$f \in o(g)$	iff	$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 0$	$f \underbrace{<}_{x\to\infty} g$
Little Omega	$f \in \omega(g)$	iff	$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} \to \infty$	$f \underbrace{>}_{x\to\infty} g$
Big Omega	$f \in \Omega(g)$	iff	$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} > 0$	$f \underbrace{\geq}_{x\to\infty} g$
Theta	$f \in \Theta(g)$	iff	$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = c, c \neq 0$	$f \underbrace{=}_{x\to\infty} g$

Sum

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$
$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$
$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Integral

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} \qquad \text{if } n \neq -1$$
$$\int \frac{1}{x} \, dx = \ln(x)$$

Quadratic

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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1. Easy stuff(100 points. 20 each.)

   (a) Draw the truth table for $Q \wedge (P \vee \neg Q)$

   (b) Which of these asymptotic symbols ($\{\sim, o, O, \omega, \Omega, \Theta\}$) are applicable for

   $$f(x) = \_\_\big(g(x)\big)$$

      i. $f(x) = n^2 + 3, g(x) = n + 5$

      ii. $f(x) = 2 \log x, g(x) = x \log x$

   (c) Find the closed form formula for the following sum/product:

      i. $\displaystyle\sum_{i=1}^{n} \sum_{j=1}^{m} (ij + 1)$

(d) Disprove the following proposition. (note the strictly greater than sign)

$$\forall p \in I \;\; p^2 > p$$

(e) Use integral bound to find the *upperbound* for the following sum.

$$\sum_{x=1}^{n} \frac{1}{x^2}$$

2. (a) (50 points) Consider $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. If you pick 5 numbers from A, then there is at least a pair that add up to 9. Hint: Pair up.

   (b) (50 points) Show that if we take two positive integers(a, b) such that $a - b = 2$ (ex: (3,5) or (10, 12)). Then,

$$ab + 1 = n^2 \ \ \exists n \in I$$

3. After graducation you becomes a candy reseller. You buy candies from factories and sell it to consumer. You show has two suppliers

   - Hi-Five Candies Factory which only sell candies to you in a multiple of 5. Ex: 5, 10, 15, 16, ...
   - Super-Six Candies Factory which only sell candies to you in a multiple of 6. Ex: 6, 12, 18, ...

   You want to maximize your profit and hold no stock candies. If your customer order, 13 candies, you it's not possible for you to order the exact amount from the two candies factories; forcing you to stock the left over.

   To solve this problem you want to establish the minimum order, m, such that for any number of order $\geq m$ we can fill the order with no stock.

   Find m and prove that it works.

4. (100 points) Show that for $n \geq 3$, we can find n **distinct** numbers $\{a_1, a_2, \ldots a_n\}$ such that

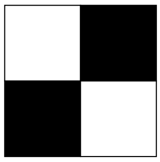
$$1 = \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}$$

For example,

- For $n = 3$, $1 = 1/2 + 1/3 + 1/6$
- For $n = 4$, $1 = 1/2 + 1/4 + 1/6 + 1/12$

Hint: $1_{new} = 1/2 + 1_{old}/2$

5. (100 points) Consider an 2×2 black and white board show below.



Each turn you can pick a square and flip the color the two the adjacent squares(but not itself). For example if you pick the lower left square, then you can flip the color of the top left and bottom right square.

Show that you can't reach grid with exactly 1 black square left.

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6. (100 points)Solve the following recurrence. Just find the solution. No need to prove it using induction.:

   (a) (30 points)$T(n) = T(n-1) + n + 1; T(1) = 1$

   (b) (30 points)$T(n) = 2T\left(\frac{n}{2}\right) + n; T(1) = 1$

(c) (40 points)$T(n) = 9 \times T(n-1) - 20T(n-2); T(0) = 5, T(1) = 23$