DISCRETE MATHEMATICS
# Final Exam T3 2016

**Instruction**

- Write your name

- Read the questions carefully.

- You have 4 hours to finish the exam.

- There are 5 problems. 620 points in total. You only need to get 560 points to get full score.

- Attempt all problems, state your reasons *clearly* and *legibly*, because partial credits will be given.

| Question | Full Score | Your Score |
|:--------:|:----------:|:----------:|
| 1 | 100 | |
| 2 | 160 | |
| 3 | 140 | |
| 4 | 100 | |
| 5 | 120 | |

Total: [        ] /560

# Useful Formulas

## Sum

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$
$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$
$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$
$$1 + 3 + 5 + 7 + \ldots + (2n-1) = n^2$$

## Euler's Formula
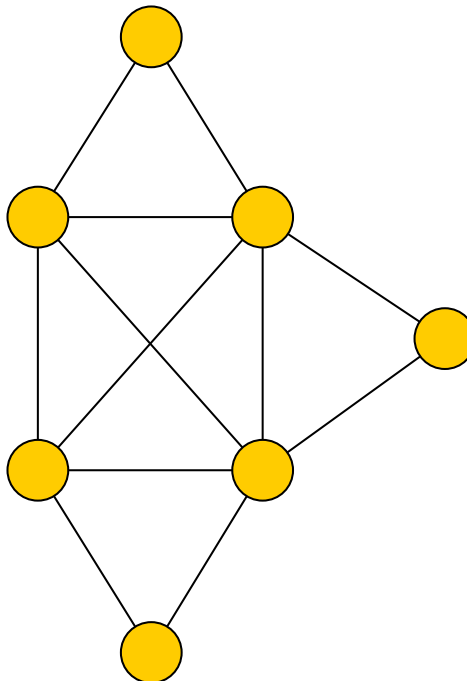
$$e + 2 = v + f$$

## Tree

$$e = v - 1$$

1. Graph Theory(100 points. 20 each)

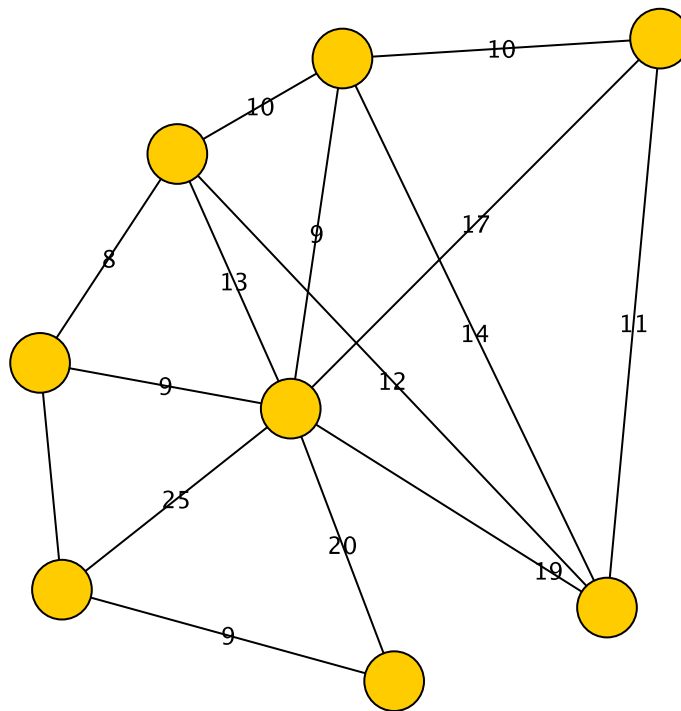   (a) Draw a **tree** that has exactly 2 vertex of degree 3. It can have any number of vertex with other degree.

   ❈❈❈

   (b) Find an Euler walk for the following graph. Label the edges with *numbers* so I can follow.

(c) Find Minimum Spanning Tree from the following graph.

(d) Suppose AJ wants to give a gift to students. However, it would look bad if he give the same present to students who know each other. He came up with the following scheme. He first list out the list of students and find out who is friend with whom. AJ could give out different gift to every one but that would require too much effort. Use something you learn from graph theory to figure out the **minimum** of different gift he needs. (1 means friend 0 means not friend)

|   | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| A | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 |
| B | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| C | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 |
| D | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 |
| E | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 |
| F | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 |
| G | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 1 |
| H | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 0 |

(e) Find an isomorphism between these two graphs.
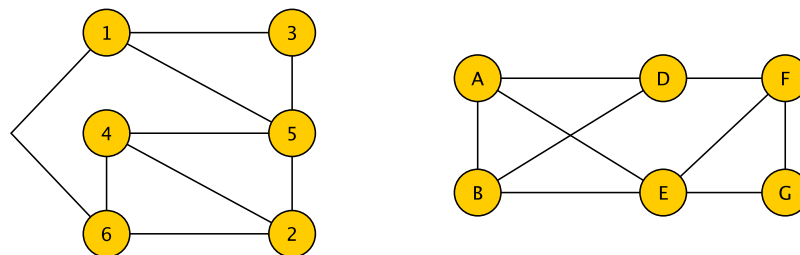
2. In a fictitious first person shooting game there are two types of guns: a sniper rifle and a shot gun.

- A sniper rifle does a very heavy damage but it can miss the target. If the sniper rifle hits the target it does 300 damages. If it miss ther target it does 0 damage. The probability of hitting the target is 1/3.

- A shotgun shoots out 400 pellets. Each pellet does exactly 1 damage. The probability of each bullet hitting the target is 0.25 and are independent of each other.

(a) Find the expected value of damage done by 1 shot of sniper rifle.

(b) Find the variance of damage done by 1 shot of sniper rifle.

(c) Find the expected value of damage done by 1 shot of shotgun.

(d) Find the variance of damage done by 1 shot of shotgun.

(e) Given 1 shotgun shot find the probability that **exactly** 100 pellets will hit the target.

(f) Given a monster of 500 health. Find the probability that 2 sniper rifle shots will kill the monster.

(g) Given a monster of 250 health. Find the expected number of sniper rifle shot needed to kill the monster.

(h) Again, given a monster of 500 health. Find the expected number of sniper rifle shot needed to kill the monster. (Be careful)

3. Dominion Card Counting. You may have seen AJ Piti playing board game called Dominion. This problem is about various aspect of counting in this game. If you need to use answer from previous part that you cannot figure out, just use it as $A_x$ for answer from part $x$. For example $A_b$ refers to answer from part $b$

(a) Dominion game is best played with 4 players sitting in a round table. Suppose that Bossy, AJ Piti, Majeed, and John decided to play. How many arrangement of seating in a round table are there?

(b) Now TaeWon and Poon wants to play too. The game is limited to 4 people but we now have 6 people. So we need to choose who gets to play then have them sit around a round table. How many combination of players and seating arrangement are there?

(c) TaeWon and Bossy are actually eternal rival. This means that if TaeWon gets to play, Bossy will refuse to play the game. Given this condition, how many combination of players and seating arrangement are there?

(d) At the start of them game, each player are given 7 copper cards and 3 estate cards. You then need to shuffle these cards together. How many permutation are there? You may consider all of them distinct as they all have some defects; eg. think of them as copper1, copper2, ....

(e) At the first turn you need to pick up top 5 cards from the pile you shuffled. How many 5 cards hand are there? Remember all cards are distinct.

❈ 9of 12 ❈

(f) What is the probability that you get exactly 3 coppers as the first 5 cards?

(g) What is the expected number of coppers in the first 5 cards?

4. Wit(Junior) has a reputation of sleeping class. He will either sleep through entire class or stay up entire class. His decision to sleep for each class are independent. Here is the details of his in-class sleeping habit.

- In Discrete Math class which runs total of 4 hours a week. For each of the session, he has 70% chance of sleeping in class.
- In Programming I class which runs total of 5 hours a week. He has 50% chance of sleeping in class.
- These are the only two classes he take.

Entire MUIC runs classes at total of 1000 hours a week.

(a) If Ply walks by a random classroom(Wit may or may not be in the class) what is the probability that Ply will find Wit sleeping in class.

(b) If Ply walks by a classroom that Wit attend, what is the probablity that Ply will find Wit sleeping in class.

(c) If Ply found Wit sleeping in a classroom, what is the probability that the class is Discrete Math class?

(d) If Ply found Wit **not** sleeping, the probability that it is Discrete Math class?

(e) What is the expected number of hours Wit would sleep in class in a week?

5. Binary Tree VS Termites. A complete binary tree of depth $n$ shown below is a tree where each node has exactly 2 children nodes unless it is at depth $n$. The figure below is an example of a complete binary tree of depth 4.
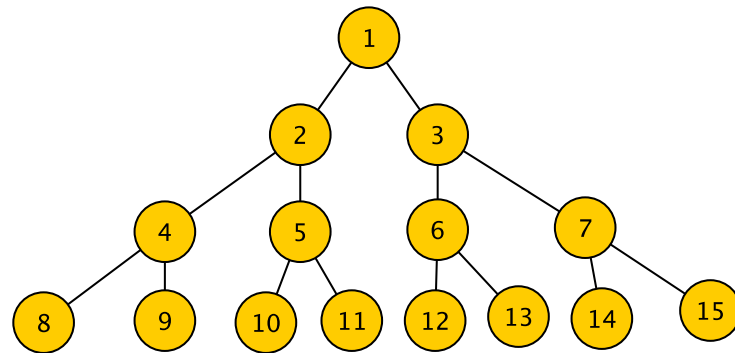


A group of edge-eating termites eat edges of the complete binary tree randomly. Each edge has probability $p$ of surving termites attack independently. After the termite attack the left over graph may have numbers of disconnected tree left over an example is shown below. The below surviving tree has 7 trees. With 3 trees with exactly 1 vertex(3,9,11) and 1 tree with exactly 2 vertices (4-8).
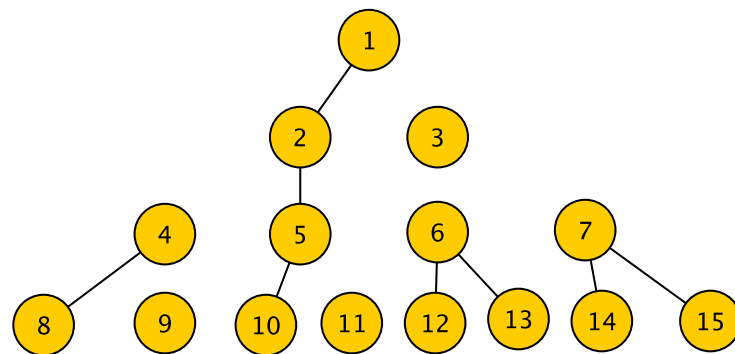


This problem concern various aspects of the surviving tree.

Note: The last 3 problems are actually a lot easier than it looks. Do not panic calm down and think.

(a) Find the probability that every edge survive termite attack.

(b) Find the expected number of edges left over.

(c) Find the variance of the number of edges left over.

(d) Find the expected number of trees left over.

(e) Find the expected number of left over trees with exactly one vertex.

(f) Find the expected number of left over trees with exactly two vertices.