Discrete Mathematics 4

Problem 1:

Theorem: $F_n = \frac{1}{\sqrt{5}}(a^n - b^n)$ where $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$

Inductive Predicate: $P(i) \equiv F_i = \frac{1}{\sqrt{5}}(a^i - b^i)$

Base case:

P(1) $\equiv F_1 = \frac{1}{\sqrt{5}}(a - b)$

$$\begin{aligned} F_1 &= \frac{1}{\sqrt{5}}(a - b) \\ &= \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}) \\ &= \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}-1+\sqrt{5}}{2}) \\ &= \frac{\sqrt{5}}{\sqrt{5}} \\ &= 1 \end{aligned}$$

P(1) is true since $F_1 = 1$ by definition.

P(2) $\equiv F_2 = \frac{1}{\sqrt{5}}(a^2 - b^2)$

$$\begin{aligned} F_2 &= \frac{1}{\sqrt{5}}(a^2 - b^2) \\ &= \frac{1}{\sqrt{5}}((\frac{1+\sqrt{5}}{2})^2 - (\frac{1-\sqrt{5}}{2})^2) \\ &= \frac{1}{\sqrt{5}}\frac{1+2\sqrt{5}+5-1+2\sqrt{5}-5}{2} \\ &= \frac{\sqrt{5}}{\sqrt{5}} \\ &= 1 \end{aligned}$$

P(2) is true since $F_2 = 1$ by definition.

Inductive step:

Assume that $\exists k \in \mathbb{Z}^+ \quad \forall i \in \mathbb{Z}^+ \quad 2 < i < k \quad F_i = \frac{1}{\sqrt{5}}(a^i - b^i)$. We want to show that

$F_k = \frac{1}{\sqrt{5}}(a^k - b^k)$

$$\begin{aligned}
F_k &= F_{k-1} + F_{k-2} \quad \text{by definition} \\
&= \underbrace{\frac{1}{\sqrt{5}}(a^{k-1} - b^{k-1})}_{IH} + \underbrace{\frac{1}{\sqrt{5}}(a^{k-2} - b^{k-2})}_{IH} \\
&= \frac{1}{\sqrt{5}}(a^{k-1} - b^{k-1} + a^{k-2} + b^{k-2}) \\
&= \frac{1}{\sqrt{5}}(a^{k-1} + a^{k-1} \cdot a^{-1} - b^{k-1} - b^{k-1} \cdot b^{-1}) \\
&= \frac{1}{\sqrt{5}}(a^{k-1}(1 + a^{-1}) - b^{k-1}(1 + b^{-1})) \\
&= \frac{1}{\sqrt{5}}(a^{k-1} \cdot a - b^{k-1} \cdot b) \\
&= \frac{1}{\sqrt{5}}(a^k - b^k)
\end{aligned}$$

Therefore, by mathematical induction, $F_n = \frac{1}{\sqrt{5}}(a^n - b^n)$ $\square$

Problem 2:

Prove that ajarn always get the flavor he wants.

State: all permutations of the yoyo

Transitions: the yoyo can be picked in 3 ways. T(g,g), T(c,c) and T(g,c) where g is a grape yoyo and c is a cola yoyo.

Lemma: $I(s) \equiv$ the number of cola yoyo is ≥ 1 after picking from permutation s is a preserved invaraint.

Proof:

- If two grape yoyo are picked, the number of cola yoyo remains the same.
- If one grape and one cola yoyo are picked, the number of cola yoyo remains the same.
- If two cola yoyo are picked, one cola yoyo needs to go back to the bag. There will always be at least one cola yoyo in the bag.

Since we have shown for all the transition there will be at least one cola yoyo left, I is a preserved invariant.

Corollary: After all students pick their yoyo, ajarn will be left with the flavor he wants.

Proof: By the preserved invariant I, after students pick their yoyo from any permutation of 5 cola yoyo and 6 grape yoyo there will be at least one cola yoyo left. The flavor ajarn wants is cola.

$\square$