ID: ____________________ Name: ______________________________

1. Counting. Please don't miss any of this. You do not have to get numerical value for your results.

   (a) (20)How many binary strings of length 10 are there? Ex: 0011001100.

   2^{10} ← 10 places

   ↑ no. can be 1, 0

   +20

   (b) (20)If we draw 6 cards from a standard deck of card(13 rank, 4 suits). How many 3 pair hands are there? Ex:2♠, 2♣, 3♡, 3♣, 4♡, 4♢.

   $$\binom{13}{3}\underbrace{\binom{4}{2}\binom{4}{2}\binom{4}{2}}_{\text{choose suits for 3 ranks.}} = \binom{13}{3}\binom{4}{2}^3$$

   ↑ choose rank

   +20

   (c) (20)An America lottery is played by drawing 6 balls with number from 00-99. No numbers can appear twice in the list of 6 numbers and the order does *not* matter. Ex: 00,63,43,12,90,05. How many lottery outcome are there?

   $\binom{100}{6}$

   ↑ choose 6 no. from 100 balls

   +20

   (d) (20)With the american lottery described above, we pick 3 different numbers: 12, 34, 56. If at least one of the numbers we pick is in the outcome, then we win the lottery. Guessing two numbers correctly still counts as one winning outcome. How many lottery outcomes are there that we win?

   1 of 3 that we pick must be in outcome → $\dfrac{\binom{3}{1}\binom{99}{5}}{6}$ ← any others is fine (100 balls − 1 that we pick)

   6 ← number we pick can be in any places.

   or

   $\dfrac{3 \cdot 99 \cdot 98 \cdot 97 \cdot 96 \cdot 95}{6!}$

   ↑ swap in any order

   $= \dfrac{3 \cdot 99 \cdot 98 \cdot 97 \cdot 96 \cdot 95}{6!}$

   2.

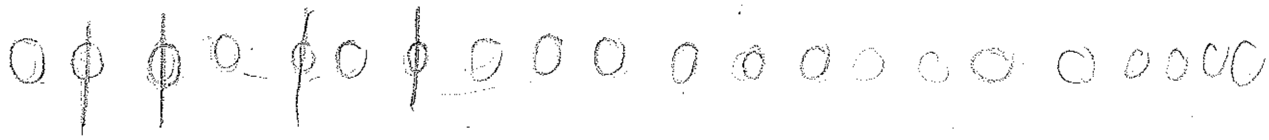
ID: ________________ Name: ______________________

(e) (20)Everyone in discrete math class(17 people) goes to a Somtum shop. The shop only has 5 items on the menu.

- Roasted Porkneck.
- Somtum.
- BBQ Chicken.
- Spicy Chicken Soup.
- Bamboo Shoot Salad.

Zaza

If everyone orders exactly 1 item from the list, then we share the meal afterward. How many ways can we make our order? For example, we can order 5 Somtum, 5 BBQ Chicken, 7 Bamboo Shoot Salad, 0 Spicy Chicken Soup and 0 Roasted pork. This counts as one no matter who order which item.



stick = menu

stone = no. of students

$$\binom{17+4}{4} = \binom{21}{4}$$

20

(f) (Bonus)(20) If you spend more than 5 minutes on this, just skip and come back later. Since Zaza is a pescetarian. She only eats vegetable and fish. So, she only has 2 choices in the menu: Somtum or Bamboo Shoot Salad, which she has to order one. With this constraint, how many ways we can make our order?

Hint: 5 Spicy Chicken Soup, 5 BBQ Chicken, 7 Bamboo Shoot Salad will never happen since Zaza will order either Somtum or Bamboo Shoot.

① No body pick Somtum + Bamboo

$$= \binom{17+2}{2} = \binom{19}{2}$$

20

$$\therefore \text{Total} = \binom{21}{4} - \binom{19}{2}$$

ID: ____________________ Name: ______________________________

2. Easy Prob and Graph

(a) (20)Suppose we throw two fair dice. What is the probability that the *product* of the two dice is even? Ex: If we get 2 and 3 then the product is 2×3 which is even.

P_1	P_2	$P_1 \times P_2$	
$\frac{1}{2}$ odd	$\frac{1}{2}$ odd	odd	$\frac{1}{4}$
	$\frac{1}{2}$ Even	Even	$\frac{1}{4}$
$\frac{1}{2}$ even	$\frac{1}{2}$ Odd	Even	$\frac{1}{4}$
	$\frac{1}{2}$ Even	Even	$\frac{1}{4}$

$Pr[\text{product is even}] = \frac{3}{4}$

+20

(b) (20)If you drop a smartphone, there is a probability of 0.2 that the screen will crack. On average how many times a smartphone can be dropped before the screen cracks? (Assume that if the screen doesn't crack then it also does *not* make the screen easier to crack next time.)

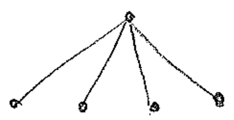
p = probability it fails ; average times it drop $= \frac{1}{p}$

$= \frac{1}{0.2}$

$= 5$ times.

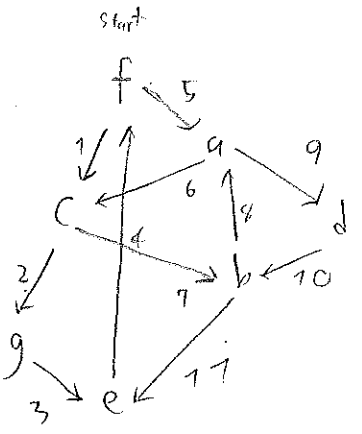
+20

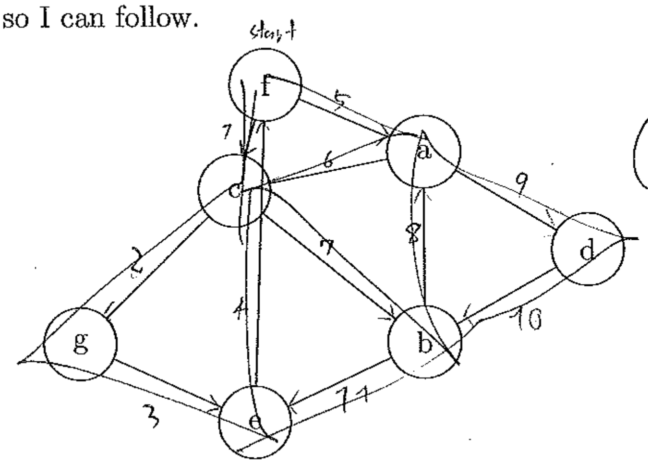
(c) (20)Draw a simple graph of 5 vertex with exactly one node of degree 4. Just draw one.



+20

(d) (20)Euler walk. Find an Euler walk for the following graph. Label each edge with number so I can follow.

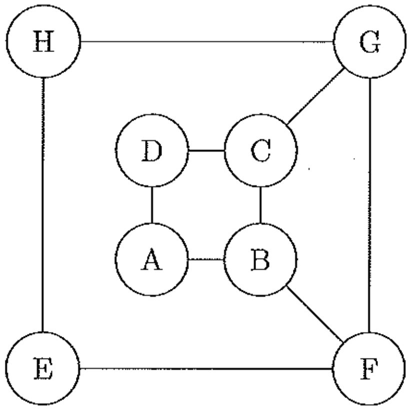


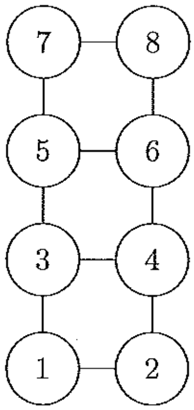


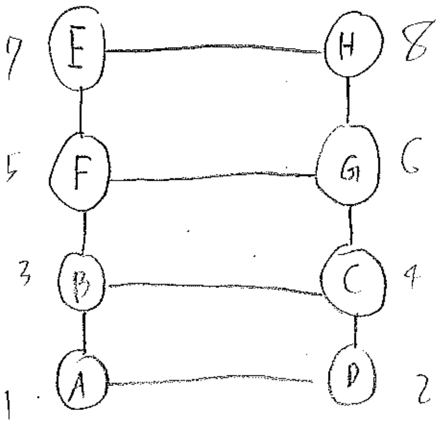
+20

ID: ____________________ Name: ________________________________

(e) (20)Find an isomorphism between these two graphs.







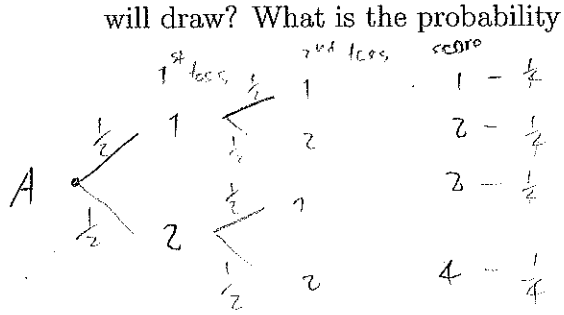
+20

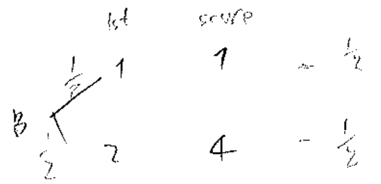
ID: ______________ Name: ______________________

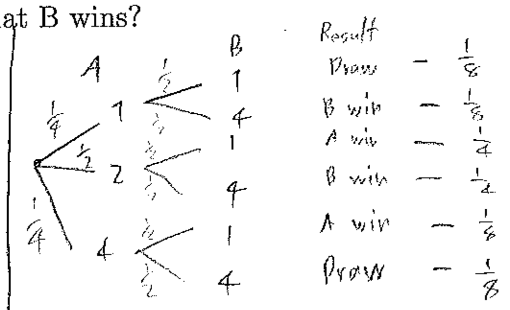
3. (100) Consider a fair coin with number 1 on one side and 2 on the other side. Let us consider a 2-player game with the following rule:

   - The first player(A) throws the coin twice. The product of the two coins toss is the score for the first player. For example, if the first player gets 2 and 1 then his score would be 2×1.
   - The second player(B) throws the coin once. The score would then be the value of the coin square. For example, if the second player gets a 2, then his score is $2^2 = 4$. If the second player gets a 1, then his score is $1^2 = 1$.
   - Whoever has more score wins.

   (a) (20)What is the probability that A wins? What is the probability that they will draw? What is the probability that B wins?







$Pr[A \text{ win}] = \frac{3}{8}$

$Pr[B \text{ win}] = \frac{3}{8}$

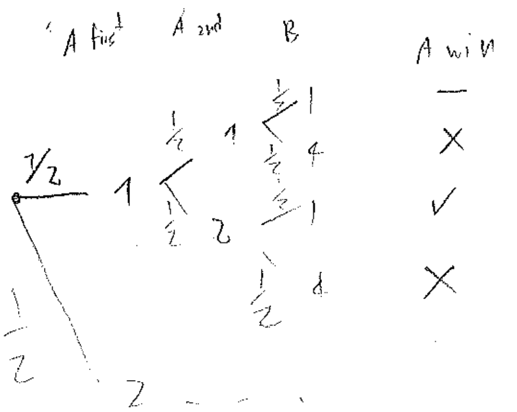
$Pr[\text{Draw}] = \frac{2}{8}$

+20

   (b) (20)What is the probability that the A get 1 on the first coin toss *and* A wins?

$Pr[A \text{ get 1 first} \cap A \text{ wins}] = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

+20



(c) (20)What is the probability that A wins given that A gets 1 on the first coin toss?

$$Pr[A \text{ wins} \mid A \text{ get } 1 \text{ first}] = \frac{Pr[A \text{ wins} \cap A \text{ get } 1 \text{ first}]}{Pr[A \text{ get } 1 \text{ first}]}$$

$$= \frac{\frac{1}{8}}{\frac{1}{2}} = \frac{1}{4}$$

+20

(d) (20)What is the probability that A gets 1 on the first coin toss given that A win?

$$Pr[A \text{ get } 1 \text{ first} \mid A \text{ wins}] = \frac{Pr[A \text{ wins} \cap A \text{ get } 1 \text{ first}]}{Pr[A \text{ wins}]}$$

$$= \frac{\frac{1}{8}}{\frac{3}{8}} = \frac{1}{3}$$

+20

(e) (20)Are the event that A gets 1 on the first coin toss and the event that A win independent?

$$Pr[A \text{ get } 1 \text{ first} \mid A \text{ wins}] = \frac{1}{3}$$

$$Pr[A \text{ get } 1 \text{ first}] = \frac{1}{2}$$

$\therefore$ not independent

+ 20

4. (100)AJ Piti usually buy bags of candies for his class. He noticed that for a bag of yoyo.

- Most of the time there are 15 Yoyo in a bag.
- 20% of the time there are 14 Yoyo in a bag
- 20% of the time there are 16 Yoyo in a bag.

$\frac{1}{5}$ 14
$\frac{3}{5}$ 15
$\frac{1}{5}$ 16

For a Sugus bag:

- Most of the time there are 30 Sugus in a bag.
- 20% of the time there are 29 Sugus in a bag.
- 20% of the time there are 31 Sugus in a bag.

$\frac{1}{5}$ 29
$\frac{3}{5}$ 30
$\frac{1}{5}$ 31

If we quantify quality conrol(Q) of packaging by the *ratio* of the standard deviation of number of candies devided in a bag by the expected number of candies in a bag. For example, for Sugus factory the quality control factor Q_S is given by the ratio of the standard deviation of the number Sugus in a bag σ_S and expected number of Sugus in a bag $\mathbb{E}[S]$.

$$Q_S = \frac{\sigma_S}{\mathbb{E}[S]}$$

(a) (15)Find the quality control factor Q_S for Sugus factory.

$E[S] = (\frac{1}{5})(29) + \frac{3}{5}(30) + \frac{1}{5}(31)$

$= \frac{29 + 90 + 31}{5}$

$= \frac{150}{5} = 30$

$Var[S] = \frac{1}{5}(-1)^2 + \frac{3}{5}(0)^2 + \frac{1}{5}(1)^2$

$= \frac{2}{5}$

$\sigma_S = \sqrt{Var[S]} = \frac{\sqrt{10}}{5}$

$Q_S = \frac{\frac{\sqrt{10}}{5}}{30} = \frac{\sqrt{10}}{150} = 0.02$

+15

(b) (15)Find the quality control factor Q_Y for Yoyo factory.

$E[Y] = (\frac{1}{5})(14) + \frac{3}{5}(15) + \frac{1}{5}(16)$

$= \frac{14 + 45 + 16}{5}$

$= \frac{75}{5} = 15$

$Var[Y] = \frac{1}{5}(-1)^2 + \frac{3}{5}(0)^2 + \frac{1}{5}(1)^2$

$= \frac{2}{5}$

$\sigma_Y = \sqrt{Var[Y]}$

$= \frac{\sqrt{10}}{5}$

$Q_Y = \frac{\frac{\sqrt{10}}{5}}{15} = \frac{\sqrt{10}}{75} = 0.04$

+15

(c) (10)Which factory has better quality control?

Sugus

+10

(d) (10)A Box of Sugus has 30 bags. What is expected number of Sugus candies in a box?

$E[30S] = 30(E[S])$

$= 900$ Ans

+10

ID: ____________________ Name: ________________________________

(e) (10)A Box of Yoyo has 30 bags. What is expected number of Yoyo candies in a box?

$E[30Y] = 30E[Y]$

$= 30(15)$

$= 450$ +10

(f) (10)What is the *standard deviation*, $\sigma_{S,Box}$, for the number of Sugus candies in a box assuming that the numbers of candies in each bag are independent.

$Var[30S] = 30^2 Var[S]$

$30^2(\frac{2}{5})$

$\sigma_{S,Box} = \sqrt{30^2(\frac{2}{5})} = \frac{30\sqrt{10}}{5} = 6\sqrt{10} \approx 18.97$

(g) (10)Yoyo factory however pack the box such that all bags in a box have exactly the same amount of candies. That is either all 14, all 15 or all 16. What is the *standard deviation* $\sigma_{Y,Box}$ for number of yoyo in a box?

$E[30Y]: 450$

$Var[Box] = Var[Y]$

$= \left(\frac{1}{5}(420-450)^2 + \frac{3}{5}(450-450)^2 + \frac{1}{5}(480-450)^2\right)$

$= \left(\frac{1800}{5}\right) = 360$

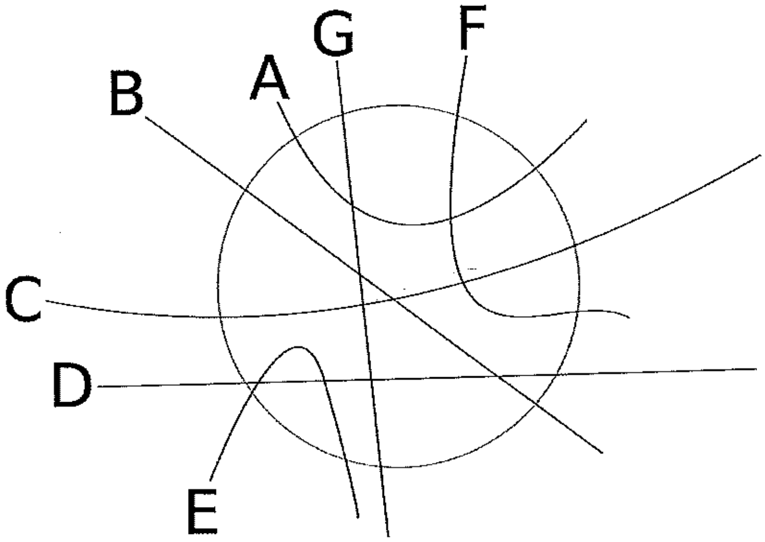
9

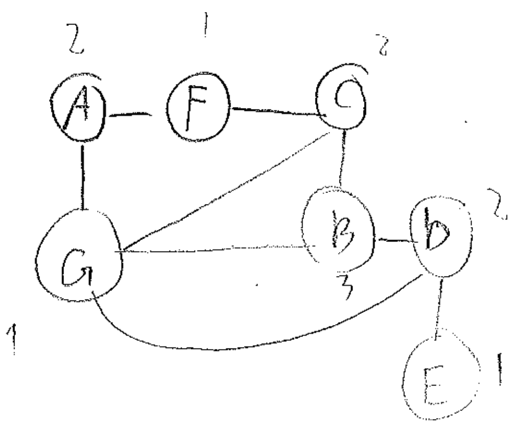
(h) (10)Which factory has better strategy for putting packages in a box? Explain.

Sugus, because the amount of Yoyo in Yoyo box is much varies compared to sugus.

+5

5. (50) Consider the following messy road intersection. You want to let multiple lanes go at the same time. However, you cannot let two lanes run at the same time if they intersect inside the intersection since they will crash. Use graph coloring to solve this problem. What do nodes, edges and colors represent? How would you schedule the traffic light(which lanes go together)?





Group	Lanes
1	G, F, E
2	A, C, D
3	B

x 50

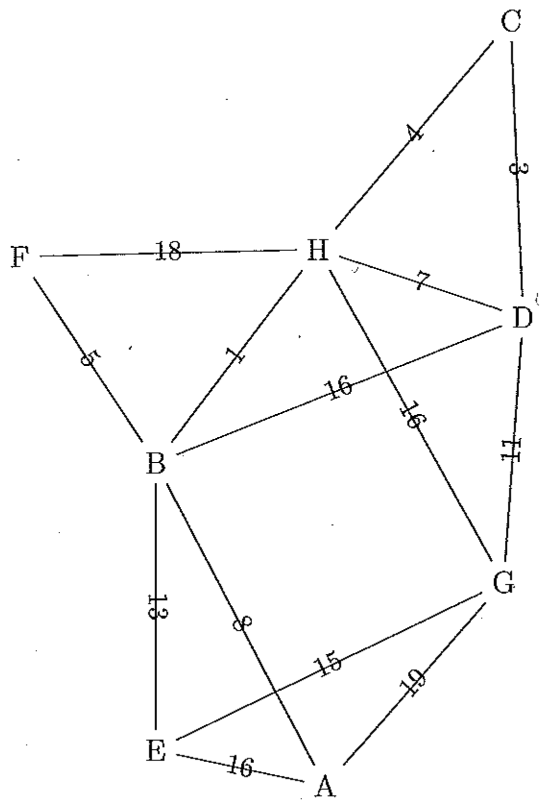
Nodes : lane

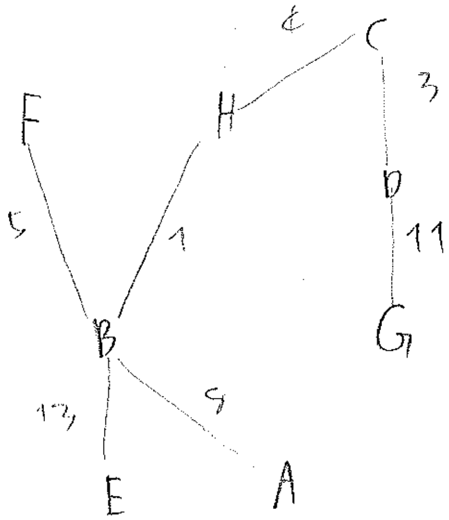
Edge : intersection

number = groups that lanes can go together.

ID: ____________ Name: ____________

6. (50) You are in charge of a cable company. You want to figure out a way to connect everyone to the internet with minimum cost. The cost for connecting 2 regions are shown in the following graph. Find the way to connect everyone with the minimum cost.



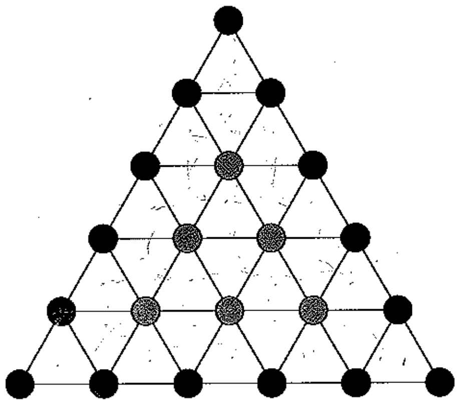


x50

ID: ____________ Name: ____________________

7. (100)For this problem if you need to borrow an answer from previous subquestion which you can't do. Just write it down as variable. You also don't need to simplify your result. Consider a triangular grid with n level. For example, the figure shown below is of $n = 6$ level.

$3 : 3 \quad +6$
$4 - 4$
$5 .. 18 \quad +9$
$6 = 20$
30



$n-2+2(n-2)$

$n-2+2n-4$
$3n-6$

(a) (10)How many vertex are there for n-level triangular grid(v_{total})? (Both black and gray dots)

$1+2+3+4+\dots+n = \frac{n(n+1)}{2}$

+10

(b) (10)How many vertex are there outside for n-level triangular grid(v_{out})? (Just the black dots)

$n+1+2(n-2) = n+1+2n-4$
$= 3n-3$

+10

(c) (10) How many vertex are there inside n-level triangular grid(v_{in})?(Just the gray dots)

$\frac{n(n+1)}{2} - 3n-3 = \frac{n^2+n-6n+6}{2} = \frac{n^2-5n+6}{2}$

$= \frac{(n-3)(n-2)}{2}$

+10

(d) (10)How many edges are there around the boundary of the triangle(e_{out})? We will call this outside edges. Give your answer in terms of n.

$3n-3$ → ~~same~~ as outside vertices

+10

(e) (10)How many edges are there inside the triangle (e_{in})? Give your answer in terms of n.

$= 3(1+2+3+\dots+n-2)$

$= \frac{3(n-1)(n-2)}{2}$

+10

ID: ____________________ Name: ________________________________

$f = e + 1 - V$

(f) (10)How many unit triangle are there? Unit triangle is defined as a triangle that are formed by three vertices that are all adjacent to each other. Give your answer in terms of n. You may find one of the sum on the formula page useful

$f = \frac{3(n-1)(n-2)}{2} + 3n - 3 + 1 - \frac{n(n+1)}{2}$

$= \frac{3n^2 - 9n + 6 + 6n - 6 + 2 - n^2 - n}{2}$

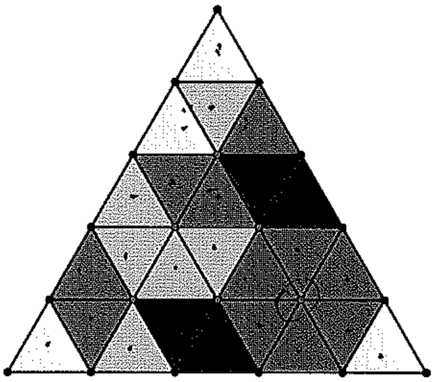
$f = \frac{2n^2 - 4n + 2}{2}$

$= n^2 - 2n + 1$

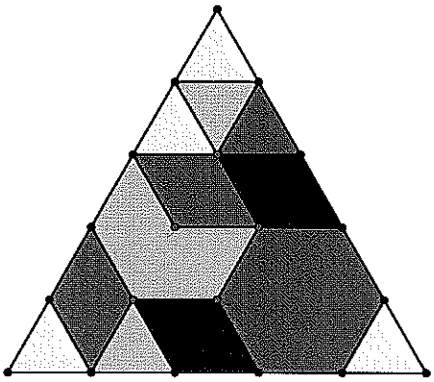
$= (n-1)^2$ Ans

+10

(g) (10)Now we start coloring each unit triangle independently with c color. Each grid has equal probability of being any color. For example, if $c = 4$ the grid may look like the following.



Our goal is to draw a that only has edges and vertex that form the boundary. An example of such graph is shown below.



First, let us consider an inside edge. We want to remove an inside edge if the two colors on both side are the same. For example, if two side of an edge are both red. What is the probability of an inside edge being removed? Give your answer in terms of c.

$\frac{1}{c}$.

+10

ID: ____________ Name: ____________

(h) (10)Find the expected number of edge we will have to remove and what is the expected number of edge we have left? Do not forget to count the outside edges which we never remove. Give your answer in terms of n and c.

$$T_i \begin{cases} 1 & \text{if } i\text{-th edge is removed} \\ 0 & \text{other wise} \end{cases}$$

$$T = T_1 + T_2 + T_3 + \dots + T_{\frac{3(n-1)(n-2)}{2}} \leftarrow \text{last edge.}$$

$$E[\text{removed edges}] = E[T] = E[T_1 + T_2 + T_3 + T_4 + \dots + T_{\frac{3(n-1)(n-2)}{2}}]$$

$$= \frac{1}{c} + \frac{1}{c} + \frac{1}{c} + \dots + \frac{1}{c}$$

$$= \frac{3(n-1)(n-2)}{2c}.$$

+10

$$E[\text{Edge left}] = E[\underset{\substack{\uparrow \\ \text{constant}}}{\text{Total edge}} - \text{removed edge}]$$

$$= 3n-3 + \frac{3(n-1)(n-2)}{2} - E[\text{removed}]$$

$$= 3n-3 + \frac{3(n-1)(n-2)}{2} - \frac{3(n-1)(n-2)}{2c}$$

$$= 3n-3 + \frac{3(n-1)(n-2)}{2}\left(1 - \frac{1}{c}\right)$$

(i) (10)Now we are going to remove inside vertices too if all 6 triangles around a vertex have the same color. What is the probability of an inside vertex being removed? Give your answer in terms of c.

$$\frac{1}{c^5}$$

+10

(j) (10)What is the expected number of vertex we will have to remove? How many vertex will we have left? Give your answer in terms of n and c.

$$T_i \begin{cases} 1 & \text{if } i\text{-th vertex is removed} \\ 0 & \text{other wise} \end{cases}$$

$$T = T_1 + T_2 + T_3 + \dots + T_{\frac{(n-3)(n-2)}{2}} \leftarrow \text{last vertex inside grid.}$$

$$E[\text{removed vertices}] = E[T] = E[T_1] + E[T_2] + \dots + E[T_{\frac{(n-3)(n-2)}{2}}]$$

$$= \frac{1}{c^5} + \frac{1}{c^5} + \frac{1}{c^5} + \frac{1}{c^5} + \dots + \frac{1}{c^5}$$

$$= \frac{(n-3)(n-2)}{2c^5}$$

$$E[\text{Vertices left}] = E[\underset{\substack{\uparrow \\ \text{constant}}}{\text{total vertices}} - \text{removed}]$$

$$= \frac{n(n+1)}{2} - E[\text{removed}]$$

$$= \frac{n(n+1)}{2} - \frac{(n-3)(n-2)}{2c^5}$$

+10