# Homework 1

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**Problem 1** A number $x$ is a perfect square if and only if

$$\exists m \in I \text{ such that } x = m^2$$

Prove that if $x$ and $y$ are both perfect squares, then $xy$ is also a perfect square.

**Problem 2** If $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

**Problem 3** Show that if $x$ is a rational number and $y$ is a rational number then, $x + y$ is a rational number.

**Problem 4** Show that if $x$ is irrational then $1/x$ is also irrational.

**Problem 5** Recall the card trick I showed you in class `https://youtu.be/UmlZa0Np-_I?t=3671`. Now it's your turn to write a proof for it:

Suppose that we have a deck of 52 cards(26 Reds and 26 Blacks). We then draw 2 cards at a time until we ran out of cards while counting the number of black pair, red pair, and different color pair. Show that the number of red pair and number of black pair are always equal at the end.

**Problem 6** Recall the absolute function

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Prove the following statement:

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|} \; \forall a, b \in \mathbb{R}, b \neq 0$$

**Problem 7** **7.1)** State the contrapositive and prove the following proposition:

If $r$ is *irrational*, then $r^{1/5}$ is *irrational*.

**7.2)** Is the following statement true? Prove it either way.

If $r$ is *rational*, then $r^{1/5}$ is *rational*.

**7.3)** How about the following statement? Prove it either way.

If $r^{1/5}$ is *irrational*, then $r$ is *irrational*.

**Problem 8** Consider the following game. Supposed there are 10,000 people in the game.

1 The game starts with everyone having 1 million Baht.

2 In each round, a time-serie graph of a stock is shown to everyone.

3 Each person has 4 choices: to bet 1000 Baht on whether the price of the stock will go up, go down or stay the same exactly 1 hour from now. Each player also have the choice to opt out from this round.

4 After one hour, we get the result from the stock. All the bet money will then be splitted equally among those who bet on the correct answer.

For example, if there are 10 people bet on stock going up, 20 people bet on stock going down and 5 people bet on stock staying the same (the rest opt-out). If after one hour, the stock really go up. The 10 people who bet on the stock going up will each get

$$\frac{(10+20+5)\times 1,000}{10} = 3,500\text{Baht}$$

5 The game is then repeated from step 2.

Your technical analysis friend then come and offer everyone a program which claims that it can read in the graph then tell you exactly what to bet. He claim that if *everyone* use this program then *everyone will make a profit* after 20,000 round.

Your job is to prove that your technical analysis friend's program cannot work as he claimed. (Hint: Zero-sum)

**Problem 9** Consider a room of area 5 $\text{m}^2$ where you can pick a shape. Billy told you that he wants to place 9 rugs, each of area 1 $\text{m}^2$ again you can pick the shape of all the rugs too. Then, since Intouch hates having the rugs intersecting each other, he demands that no pair rugs should have an area of intersection $\geq 1/9\text{m}^2$.

After Several tries, you found that picking shapes of the room and rugs to satisfy both Billy's and Intouch's requirement is just impossible. So, you went to tell them that this is impossible. However, since they are pretty good at mathematical reasoning. They both demand a proof that it is impossible.

Your task for this problem is to provide them a proof that this is impossible.

Hint: contradiction.

**Problem 10** Let $\mathbb{O}$ be the set of odd integers. Proof the following statement:

$$\forall x \in \mathbb{O}\ \ \exists p, q \in I \ \text{ such that} x = p^2 - q^2$$

in other words, every odd integer can be written as a difference of two squares.