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1. Easy stuff(100 points. 20 each.)

(a) Draw the truth table for $P \vee (\neg Q \wedge P)$

P	Q	¬Q	¬Q∧P	P∨(¬Q∧P)
T	T	F	F	T
T	F	T	T	T
F	T	F	F	F
F	F	T	F	F

+20

(b) Which of these asymptotic symbols ($\{\sim, o, O, \omega, \Omega, \Theta\}$) are applicable for

$$f(x) = \_\_(g(x))$$

i. $f(x) = x^3, g(x) = x^2 + 10^{100}$ lowerbound

$\boxed{\omega, \Omega}$

since $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^3}{x^2+10^{100}} \simeq \infty$

which $\not> 0$ and $\to \infty$

ii. $f(x) = 5^x, g(x) = 3^x$

$\lim_{x\to\infty} \left(\frac{5}{3}\right)^x \to \infty$
> 0

$\boxed{\omega, \Omega}$

+20

(c) Find the closed form formula for the following sum/product:

i. $\sum_{i=1}^{n}\sum_{j=1}^{m}(i+j)(i-j) = \sum_{i=1}^{n}\sum_{j=1}^{m}(i^2-j^2)$

$= (1^2-1^2)+(1^2-2^2)+(1^2-3^2)+\dots(1^2-m^2)$
$+(2^2-1^2)+(2^2-2^2)+(2^2-3^2)+\dots(2^2-m^2)$
$+\dots(n^2-1^2)+(n^2-2^2)+(n^2-3^2)+\dots(n^2-m^2)$
$= m(1^2+2^2+3^2+\dots n^2) - n(1^2+2^2+3^2+\dots m^2)$
$= m\left[\frac{n(n+1)(2n+1)}{6}\right] - n\left[\frac{m(m+1)(2m+1)}{6}\right]$

~~+20~~ 15

+10

ii. $\prod_{i=1}^{n}\sum_{j=1}^{m}2^i3^j = \prod_{i=1}^{n} 2^i \times (3^1+3^2+3^3+\dots 3^m)$

$= (2^1\times2^2\times2^3\times\dots 2^n)(3^1+3^2+3^3+\dots 3^m)^n$

$= 2^{1+2+3+\dots+n} \times \frac{1}{2}[3^{m+1}-3]^n$

$= 2^{\frac{n}{2}(n+1)} \times \frac{1}{2}[3^{m+1}-3]^n$

$S = 3^1+3^2+\dots 3^m$
$3S = 3^2+\dots 3^m+3^{m+1}$
$\therefore 2S = 3^{m+1}-3$

(-5)

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(d) Prove or Disprove the following proposition.

If a and b are rational then a^b is rational.

Disprove by counter example

+20

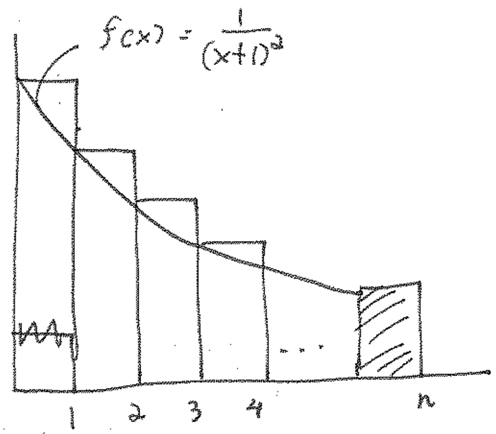
$a = 2,\ b = \frac{1}{2}$

but a^b which is $2^{\frac{1}{2}}$ or $\sqrt{2}$ is irrational

(e) Use integral bound to find the *lowerbound* for the following sum.

$$\sum_{x=1}^{n} \frac{1}{x^2}$$

+20



lowerbound $= \displaystyle\int_{x=0}^{x=n-1} \frac{1}{(x+1)^2}\,dx + \frac{1}{(n+1)^2}$

$= -\frac{1}{x+1}\Big|_0^{n-1} + \frac{1}{(n+1)^2}$

$= -\left[\frac{1}{n\cancel{-1}\cancel{+1}} - 1\right] + \frac{1}{(n+1)^2}$

$= 1 - \frac{1}{n} + \frac{1}{(n+1)^2}$

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2. Easy Proof.

(a) (50 points) Let us define

$$n\%2 = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Show that

$$(a\%2 + b\%2)\%2 = (a+b)\%2$$

for all integer a and b.

+100

Consider 4 cases of a and b ~~let~~

Case 1 a is odd, b is odd

$LHS = (a\%2 + b\%2)\%2$
$= (1+1)\%2$
$= 0$
$RHS = (\underbrace{a+b}_{\text{even}})\%2$
$= 0$

Case 2 a is odd, b is even

$LHS = (a\%2 + b\%2)\%2$
$= (1+0)\%2$
$= 1$
$RHS = (\underbrace{a+b}_{\text{odd}})\%2$
$= 1$

Case 3 a is even, b is odd

$LHS = (0+1)\%2$
$= 1$
$RHS = (\underbrace{a+b}_{\text{odd}})\%2$
$= 1$

Case 4 a is even, b is even

$LHS = (0+0)\%2$
$= 0$
$RHS = (\underbrace{a+b}_{\text{even}})\%2$
$= 0$

For all cases, LHS = RHS
$\therefore (a\%2 + b\%2)\%2 = (a+b)\%2$ is proven ■

(b) (50 points) If x is irrational then $\sqrt{x}$ is irrational.

Consider p be the predicate $\sqrt{x}$ is rational
q be the predicate x is rational

We want to prove that $\sim q \rightarrow \sim p$ which is equivalent to $p \rightarrow q$, thus if we prove that $p \rightarrow q$ is true, we prove the given statement

Proof $\exists\, a, b \in I$ such that $\sqrt{x} = \frac{a}{b}$ since $\sqrt{x}$ is rational

$$(\sqrt{x})^2 = \left(\frac{a}{b}\right)^2$$

$$x = \frac{a^2}{b^2}$$

we can see that $a^2 \in I$ since $a \in I$
and $b^2 \in I$ since $b \in I$

Therefore x is rational by definition

Conclusion we have proven ~~that~~ $p \rightarrow q$, thus we prove $\sim q \rightarrow \sim p$

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3. (100 points)Pick one. Indicate the one you pick. If you don't, I'll pick the one you got less score. Doing two won't give you more score.

   (a) Let F_n be fibbonacci number such that $F_1 = 1, F_2 = 1$ and

   $$F_{n+1} = F_n + F_{n-1}$$

   Show that

   $$1F_1 + 2F_2 + 3F_3 + 4F_4 + \ldots nF_n = nF_{n+2} - F_{n+3} + 2$$

   (b) Let $m! = m \times (m-1) \times (m-2) \times \ldots \times 1$ (ex: $5! = 5 \times 4 \times 3 \times 2 \times 1$) Show that

   $$1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + n \times n! = (n+1)! - 1$$

   for all $n \geq 1$

+100

pick a) Proof by induction

Predicate $P(n) := 1F_1 + 2F_2 + 3F_3 + \ldots + nF_n = nF_{n+2} - F_{n+3} + 2$

Base case $P(2)$

$\boxed{F_3 = 2,\ F_4 = 3,\ F_5 = \boxed{5}}$

$LHS = 1F_1 + 2F_2 = 1(1) + 2(1) = 3$

$RHS = 2F_4 - F_5 + 2 = 2(3) - 5 + 2 = 3$

Since $LHS = RHS$, base case is true ✓

Inductive Step: We want to prove that $P(n+1)$ is true assuming $P(n)$ is true

Consider $P(n+1)$: $1F_1 + 2F_2 + 3F_3 + \ldots + nF_n + (n+1)F_{n+1} = (n+1)F_{n+3} - F_{n+4} + 2$

$$LHS = \underbrace{1F_1 + 2F_2 + 3F_3 + \ldots + nF_n} + (n+1)F_{n+1}$$

$= (nF_{n+2} - F_{n+3} + 2) + (n+1)F_{n+1}$ ← by Inductive hypothesis

$= nF_{n+2} - F_{n+3} + 2 + nF_{n+1} + F_{n+1}$

$= n(F_{n+1} + F_{n+2}) - (F_{n+1} + F_{n+2}) + F_{n+1} + 2$

$= nF_{n+3} - \cancel{F_{n+1}} - F_{n+2} + \cancel{F_{n+1}} + 2$

$= nF_{n+3} - F_{n+2} + 2$

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$$RHS = (n+1)F_{n+3} - F_{n+4} + 2$$

$$= nF_{n+3} + F_{n+3} - (F_{n+2} + F_{n+3}) + 2$$

$$= nF_{n+3} + \cancel{F_{n+3}} - F_{n+2} - \cancel{F_{n+3}} + 2$$

$$= nF_{n+3} - F_{n+2} + 2$$

Since LHS = RHS, $P(n+1)$ is true and $P(n) \rightarrow P(n+1)$

Therefore, by inductive proof, $P(n)$ is true for all cases

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4. Pick one. Indicate the one you pick. If you don't, I'll pick the one you got less score. Doing two won't give you more score.

+10 0

(a) During ninja war. MUIC empire controls **two** castles: Castle of Discrete and Castle of Data. The two castles have *equal* number of guards (n each) in the beginning. Two famous ninja: Ninja Wit and Ninja Taro, are tasked to destroy MUIC empire. Since the guards are so easy to kill, the two ninjas came up with a competition to make the job more fun.

The two ninjas decided to take turn. *Ninja Wit starts first.*

For each night/turn,

i. The ninja on that turn decide which non-empty castle to go. (Ninja doesn't waste time going to empty castle.)

ii. The ninja must kill *one or more* guard in that castle on that night. (The ninja cannot kill anyone in another castle)

The winning ninja is the one who **kill the last guard**.

Let us look at an example: Suppose that each castle has 1 guard.

(# of guard in Castle of Discrete, # of guard in Castle of Data) = (1,1)

i. On the first night, Ninja Wit go to Castle of Discrete and he has only one choice: kill the 1 guard in that castle. This means that the number of guards will become (0, 1)

ii. On the second night, Ninja Taro go to Castle of Data and kill that last guard and win the competition.

Show that Ninja Taro, starting second, always has a strategy that guarantees his win (no matter what Ninja Wit does) for all number of guards at the beginning in each castle, n.

Hint: If each castle has 2 guards and Ninja Wit kill 1 guard in one castle, leaving Ninja Taro with (1,2). What should Ninja Taro do?

Hint: If each castle has 5 guards and Ninja Wit kill 3 guards in one castle, leaving Ninja Taro with (2,5). What should Ninja Taro do?

Hint: Use the example as base case and do induction on the number of guards in each castle. Ninja Taro should always try to make sure he go back to smaller problem which he knows how to win for sure.

(b) Show that $7^{n+2} + 8^{2n+1}$ is divisible by 57. (Less fun though and I think the the top problem is easier.)

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Pick b) Proof by induction

Predicate : $P(n) := 57 \mid 7^{n+2} + 8^{2n+1}$

Base case : $P(0) := 57 \mid 7^2 + 8^1$ is true since $57 \mid 57$

Inductive step : we want to prove ~~that~~ $P(n+1)$ assuming $P(n)$ is true

Consider $P(n+1)$

$$7^{(n+1)+2} + 8^{2(n+1)+1} = 7^{n+3} + 8^{2n+3}$$

$$= 7 \cdot 7^{n+2} + 64 \cdot 8^{2n+1}$$

$$= \underbrace{7 \cdot 7^{n+2} + 7 \cdot 8^{2n+1}} + 57 \cdot 8^{2n+1}$$

$$= 7\underbrace{(7^{n+2} + 8^{2n+1})}_{\text{By IH divisible by 57}} + \underbrace{57 \cdot 8^{2n+1}}_{\text{divisible by 57}}$$

$\therefore 57 \mid 7^{(n+1)+2} + 8^{2(n+1)+1}$

Conclusion : since we prove that $7^{(n+1)+2} + 8^{2(n+1)+1}$ is divisible by 57 which makes $P(n+1)$ true

From induction where $P(n) \to P(n+1)$, we prove $P(n)$ true for all cases

+100

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5. (100 points)Consider the following grid of 4×4 with numbers on it. Each turn you can pick a row or a column and add 1 or subtract 1 to/from every cell in that row or column. Can you make all the numbers the same? If it's not possible, prove that it is not possible.

Hint: Circle problem.

13	14	15	16
9	10	11	12
5	6	7	8
0	2	3	4

No, impossible

let S be the total sum of all the number in the grid

Invariant : S is odd

- Initial state : S ~~is~~ $0+2+3+4+5+6+7+8+9+10+11+12+13+14+15+16$ ~~13~~

$= \frac{16 \cdot 17}{2} - 1$

$= 135$ is odd

- Each turn : Consider 2 cases of changing state of a particular row or column

Case 1 : add 1 to every cell

The new sum can be computed by the following formula

$$\boxed{\underbrace{S'}_{odd} = \underbrace{S}_{odd} + 4}$$

Since S is odd, S' will be odd

Case 2 : subtract 1 from every cell

$$\boxed{\underbrace{S'}_{odd} = \underbrace{S}_{odd} - 4}$$ S' will still be odd

- Conclusion : Since S will always be odd nomatter what operations we perform, there will never be a state where S is even

Consider the state wheres all numbers are the same (let it be X), which implies the sum to be even since $16(X)$ is even

$\therefore$ this state is unreachable

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6. (100 points)Solve the following recurrence. Just find the solution. No need to verify it using induction.:

(a) (30 points)$T(n) = T(n-1) + 3n^3; T(1) = 1$

let $n=5$

$$T(5) = T(4) + 3(5)^3$$
$$= T(3) + 3(4)^3 + 3(5)^3$$
$$= T(2) + 3(3)^3 + 3(4)^3 + 3(5)^3$$
$$= T(1) + 3(2)^3 + 3(3)^3 + 3(4)^3 + 3(5)^3$$
$$= 1 + 3(2)^3 + 3(3)^3 + 3(4)^3 + 3(5)^3$$

+30

from observation, we conclude that

$$T(n) = 1 + 3[2^3 + 3^3 + \cdots n^3]$$
$$= 1 + 3\left[\left(\frac{n}{2}(n+1)\right)^2 - 1\right]$$
$$= \frac{3}{4}[n^2+n]^2 - 2$$

(b) (30 points)$T(n) = 2T\left(\frac{n}{2}\right) + 2n; T(1) = 1$

let $n = 2^k$, $k = \log_2 n$

30

let $k=5$

$$T(2^5) = 2T(2^4) + 2\cdot 2^5$$
$$= 2[2T(2^3) + 2\cdot 2^4] + 2\cdot 2^5$$
$$= 2[2[2T(2^2) + 2\cdot 2^3] + 2\cdot 2^4] + 2\cdot 2^5$$
$$= 2[2[2[2T(2) + 2\cdot 2^2] + 2\cdot 2^3] + 2\cdot 2^4] + 2\cdot 2^5$$
$$= 2[2[2[2[2T(1) + 2\cdot 2] + 2\cdot 2^2] + 2\cdot 2^3] + 2\cdot 2^4] + 2\cdot 2^5$$
$$= 2^5 + 2^6 + 2^6 + 2^6 + 2^6 + 2^6$$
$$= 2^5 + 5\cdot 2^6$$

from observation, we conclude that

$$T(2^k) = 2^k + k\cdot 2^{k+1} \quad 2\cdot 2^k$$
$$\therefore T(n) = n + 2n\log_2 n$$

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(c) (40 points)$T(n) = T(n-1) + 2T(n-2); T(0) = 8, T(1) = 1$

40

let $T(n)$ be in the form of x^n

$$x^n = x^{n-1} + 2x^{n-2}$$

Assume $\boxed{n=2}$ $\rightarrow$

$$1 = \frac{1}{x} + \frac{2}{x^2}$$

$$x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = -1, 2$$

We know that $T(n) = a(-1)^n + b(2)^n$, find a and b

$T(0) = a + b = 8$ — ①

$T(1) = -a + 2b = 1$ — ②

①+②: $3b = 9 \rightarrow b = 3$

$\therefore a = 5$

Therefore, ~~$T(n) = (-5)^n + 3 \cdot 2^n$~~

$$T(n) = 5(-1)^n + 3(2)^n$$

check $T(2) = 1 + 2(8) = 17$

$T(2) = \cancel{4}5 + \cancel{12} = 17$

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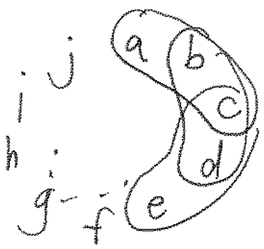
7. Bonus.(30 points) No partial credit for this one. Don't bother attempting this if you aren't done with the others. You will need to be a bit creative for this one.

Consider number 1 to 10 written in a circle(in any order). Show that there is a three consecutive number that adds up to at least 17 no matter how you order them around the circle. (There might be more than 1 3 consecutive number).

Ex: 1, 3, 5, 7, 9, 2, 4, 6, 8, 10 . . . 1. Then, each of (5,9,7), (7,9,2), (4,6,8), (6,8,10), (8,10,1) triplet adds up to 17 or more.

Consider the case where all triplets add up to 16 or less

let a, b, c, d, e, f, g, h, i, j be the distinct number from 1 to 10



each triplet must fulfill the condition ~~that~~ ≤ 16

$a+b+c \leq 16$
$b+c+d \leq 16$
$c+d+e \leq 16$
$d+e+f \leq 16$
$e+f+g \leq 16$
$f+g+h \leq 16$
$g+h+i \leq 16$
$h+i+j \leq 16$
$i+j+a \leq 16$
$j+a+b \leq 16$

+30

The sum of all inequality equation will be

$3(a+b+c+d+e+f+g+h+i+j) \leq 160$

$a+b+c+d+e+f+g+h+i+j \leq \frac{160}{3} \approx 53.3$

which is impossible since the sum of numbers from 1 to 10 must be 55

$55 \not\leq 53$

Thus, by contradiction the ~~given problem~~ must be true ~~the given case that all add up to 16 or less is impossible~~ ~~that must be true~~