Homework 3 Solution

Last updated: Thursday 4^{th} June, 2015 01:46

Problem 1 We discussed in class about breaking a bar of chocolate. Now consider the following game of you and your friend:

1) Your friend get the pick the size of the chocolate.(Ex: 5×2 or 11×11)
2) After your friend choose the chocolate size, you get to pick whether you or your friend get to do the first break.
3) Each player then take turns to break the chocolate bar until all the pieces are 1×1 block.
4) Whoever break the last piece wins the game.

Find the winning strategy. Explain why it is the winning strategy.

Bonus: Play this with your friends not in this class. Math should not be used for evil purposes.

Proof: This one is too easy. You can do it yourself.

Problem 2 Consider n-piece jigsaw puzzle. You solve it by, first, trying to find two pieces that fit together. Then, for the subsequent step, you piece together tow blocks each made of one or more jigsaw piece that have been assembled in previous steps. Prove that the number of steps required to put all the n pieces together is given by $n-1$.

Proof: We will prove this by induction.

Inductive Predicate:

$$P(s) = \text{Assembling an } s\text{-piece jigsaw require } s-1 \text{ steps.}$$

Base Case: Combining 1 piece jigsaw requires 0 step. So $P(1)$ is true.

Inductive Step:

Let us assume that i-piece jigsaw takes $i-1$ step to assemble for $i = 1, 2, 3, \ldots, k-1$.

We want to show that assembling k piece jigsaw takes $k-1$ steps.

- To assemble k-piece jigsaw, we need to put p-piece jigsaw and q-piece jigsaw together where
$$p + q = k$$
- So the total number of steps required is
$$\text{\#steps required} = \text{\#steps for p} + \text{\#steps for q} + \overset{\substack{\text{putting them together}\\ \downarrow}}{1}$$
- By IH, the number of steps required to put p-piece together($p < k$) and the number of steps required to put q-piece together($q < k$) is $p-1$ and $q-1$ accordingly.
- Thus, the total number of steps required to put k-piece jigsaw together is

$$\begin{aligned}\text{\#steps required} &= (p-1) + (q-1) + 1\\ &= p + q - 1\\ &= k - 1\end{aligned}$$

Therefore, by mathematical induction putting together n-piece jigsaw requires $n-1$ steps. □

Problem 3 Suppose Brew&Bev issues gift certificate of two types: 7 Baht and 4 Baht. Assume that you have infinitely many of the two types. Prove that we can by, with *exact* change, everything Brew&Bev sells with price $p \geq 18$. Ex: We can use 2 7Baht and 1 4Baht coupon for an 18Baht croissant.

Hint: How do you pay for 22 and 25 Baht?

Proof: Shown in class.

Problem 4 Show that every positive integer is a product of a *power of two* and an *odd integer*. (Previous Midterm).

Proof: We will show this by strong induction.

Inductive Predicate:

$$P(s) = s \text{ can be written as a product of a power of two and an odd integer.}$$

Base Case: $1 = 2^0 \times 1$ which is a product of a power of two and an odd integer ✓;

Inductive Step:

Let us assume that integer i can be written as a product of a power of two and an odd integer for $i = 1, 2, 3, \ldots, k-1$.

We want to show that k can be written as a product of a power of two and an odd integer.

Let us consider the integer k. There are two cases for k.

Case 1 k is odd.

- This case is trivial since $k = 2^0 \times k$ which is a product of a power of two and an odd integer.

Case 2 k is even.

- Since k is even, we know that $\dfrac{k}{2} = m$ is an integer. $\exists m \in \mathbb{I}$

$$k = 2m$$

- Since $m < k$, we know that m can be written as a product of an a power of two and an odd integer that is $\exists n \in \mathbb{I}$ and $\exists o \in \mathbb{O}$,

$$m = 2^n \times o$$

- This means that

$$k = 2 \times m = 2 \times (2^n \times o) = 2^{n+1} \times o$$

which is a product of a power of two and an odd integer.

□

Therefore, by mathematical induction, any positive integer n can be written as a product of power of two and an odd integer.

Problem 5 Let the sequence T_n be defined by $T_1 = T_2 = T_3 = 1$ and

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \text{ for } n \geq 4.$$

Show that

$$T_n < 2^n$$

Proof: We will prove this by induction

Inductive Predicate:

$$P(s) := T(s) < 2^s$$

Base Case:

- $P(1) := 1 < 2^1 \checkmark$
- $P(2) := 1 < 2^2 \checkmark$
- $P(3) := 1 < 2^3 \checkmark$

Inductive Step:

Let as assume that $T(i) < 2^i$ for $i = 1, 2, 3, \ldots, k$ for some $k \geq 3$.

We want to show that $T(k+1) < 2^{k+1}$.

- Consider $T(k+1)$, by definition of T we know that

$$T(k+1) = T(k) + T(k-1) + T(k-2)$$

- By inductive hypothesis we know that
  - $T(k) < 2^k$
  - $T(k-1) < 2^{k-1}$
  - $T(k-2) < 2^{k-2}$

  Therefore,

$$\begin{aligned} T(k+1) &< 2^k + 2^{k-1} + 2^{k-2} \\ &< 2^{k+1}\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) \\ &< \frac{7}{8} 2^{k+1} \\ &< 2^{k+1} \end{aligned}$$

$\square$

Thus, by mathematical induction, $T(n) < 2^n$.

Problem 6 Show that every integer $n \geq 1$ can be written as a sum of *distinct* power of 2. For example,

$$1 = 2^0,\ 21 = 1 + 2^2 + 2^4,\ 100 = 2^2 + 2^5 + 2^6$$

Proof: We will show this by strong induction.

Inductive Predicate:

$P(s) :=$ s can be written as a sum of *distinct* power of 2.

Specifically, this means that is there exists distinct integer $a_1, a_2 \ldots a_n$ such that

$$s = 2^{a_1} + 2^{a_2} + \ldots + 2^{a_n}$$

Base Case: $1 = 2^0$✓

Inductive Step:

Let us assume that integer i can be written as a sum of distinct power of two for $i = 1, 2, 3, \ldots, k-1$.

We want to show that k can be written as a sum of distnct power of two.

There are two cases for k

Case 1 k is odd.

- By algebra $k = (k-1) + 1$
- Since $k - 1 < k$, by IH, we know that $k - 1$ can be written as a sum of distinct power of two.

$$k - 1 = 2^{a_1} + 2^{a_2} + \ldots 2^{a_n}$$

- Furthermore, since k is odd, $k-1$ is even this means that none of $a_1, a_2, \ldots, a_n$ is zero.
- This means that

$$k = (k-1) + 1 = 2^{a_1} + 2^{a_2} + \ldots 2^{a_n} + 2^0$$

is a sum of distinct power of two.

Case 2 k is even.

- Since k is even we know that $m = \frac{k}{2}$ is an integer.

$$k = 2m$$

- Since $m < k$, we know that m can be written as a sum of distinct power of two that is

$$m = 2^{a_1} + 2^{a_2} + \ldots 2^{a_n}$$

- This means

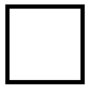
$$\begin{aligned} k = 2m &= 2 \times (2^{a_1} + 2^{a_2} + \ldots 2^{a_n}) \\ &= 2^{a_1+1} + 2^{a_2+1} + \ldots 2^{a_n+1} \end{aligned}$$

- Since $a_1, a_2, \ldots, a_n$ are distinct $a_1 + 1, a_2 + 1, \ldots, a_n + 1$ are distinct.
- This means that k is a sum of distinct power of twos.

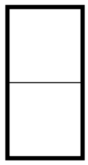
□

Therefore, by mathematical induction, every integer $n \geq 1$ is a sum of distinct power of two.

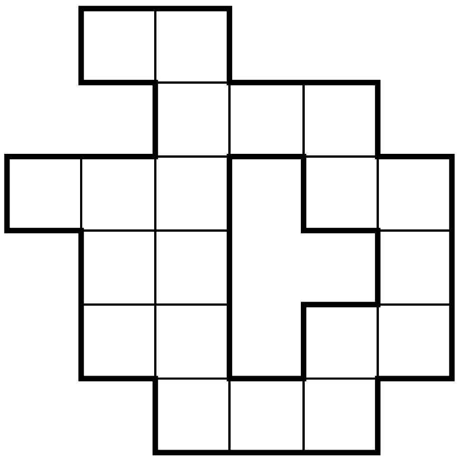
Problem 7 Consider an infinite grid. First we put a unit square on the grid. We will find that the length of the periphery is 4 which is even number. (count the length of thick line).



We then place another unit grid such that at least the new square share at least one grid with the already placed square. We will find again that the length of the periphery is 6. Another even number. What a coincidence!



Then, we keep placing one grid after another such that the next grid share at least one edge with the previous blob. But, no matter how we place it, we always find that the length of periphery is even.



What an amazing discovery! Your job is to prove it.

Theorem: All n square shape has even number of periphery.

Proof: There are two ways to prove this: cool way and not so cool way. The cool way is to count the number of periphery in terms of shared boundaries and number of square. But, we are going to do it in a not so cool way using induction.

Inductive Predicate:

$$P(s) = \text{every shape of } s\text{-square has even number of periphery.}$$

Base Case: One square grid has 4 periphery which is even ✓.

Inductive Step:

Let us assume that all k square shape has even number of periphery.

We want to show that all $k+1$ square shape has even number of periphery.

$k+1$ square shpae can be formed from adding 1 square to k square shape. There are 4 cases of how to add the 1 square to the k square shape.

A) The square we place where 1 side with the previous shape. This means that 1 periphery will be taken away and 3 more will be added. This means

$$\text{periphery}_{k+1} = \text{periphery}_k - 1 + 3 = \text{periphery}_k + 2$$

Since, by IH, the number of periphery of k-square is even, $\text{periphery}_{k+1} = \text{periphery}_k + 2$ is even.

B) The square we place share 2 sides with the previous shape. This means that 2 periphery will be taken away and 2 more will be added. This means

$$\text{periphery}_{k+1} = \text{periphery}_k - 2 + 2 = \text{periphery}_k$$

Since, by IH, the number of periphery of k-square is even, periphery_{k+1} is even.

C) The square we place share 3 sides with the previous shape. This means that 3 periphery will be taken away and 1 more will be added. This means

$$\text{periphery}_{k+1} = \text{periphery}_k - 3 + 1 = \text{periphery}_k - 2$$

Since, by IH, the number of periphery of k-square is even, $\text{periphery}_{k+1} = \text{periphery}_k - 2$ is even.

D) The square we place share 4 sides with the previous shape. This means that 4 periphery will be taken away and 0 more will be added. This means

$$\text{periphery}_{k+1} = \text{periphery}_k - 4 + 0 = \text{periphery}_k - 4$$

Since, by IH, the number of periphery of k-square is even. $\text{periphery}_{k+1} = \text{periphery}_k - 4$ is even.

□

Thus, by mathematical, all n square shape has even number of periphery.

Problem 8 Is the following proposition true? If not, what is wrong with the proof?

Theorem: If we have n straight line in a 2 dimensional plane($n \geq 2$), and none of the line are parallel to each other, then all lines must intersect at exactly one and the same point.

Proof: by induction-ish?.

Inductive Predicate:$P(n) =$ every set of n straight lines intersect at exactly one point.

Base Case: $n = 2$. Every two lines intersect at one point. Therefore, $P(2)$ is true.

Inductive Step:

Assume that every n straight lines intersect at one point, we want to show that every $n + 1$ line intersect at one point.

- Let the set of $n + 1$ be $\{a_1, a_2, a_3, \ldots, a_n, a_{n+1}\}$
- From the inductive hypothesis $\{a_1, a_2, a_3, \ldots, a_n\}$ must intersect at one point since it is a set of n lines. Let us call the intersection point for these lines D.
- Similarly, $\{a_2, a_3, a_4 \ldots, a_n, a_{n+1}\}$ intersect at exactly one point since there are n lines in this set. Let us call this point E.
- Since both D and E are the point where a_2 intersects a_3. D and E are therefore the same point.
- Therefore, $\{a_1, a_2, a_3, \ldots, a_n, a_{n+1}\}$ intersect at exactly one point.■

Solution: This fails when $n = 3$. Try write the the whole proof when $n = 3$ you will see that the two set doesn't have enough intesection to make the claim.