Homework 6 Solution

Last updated: Thursday 24th November, 2016 14:43

There is no need to get numerical value for the answer. But you must put a short note on where each term comes from.

Problem 1 Poker hands. Let us count how many poker hands there are. This is an excellent exercise. It is important that you do this alone first before checking it with others.

A deck of card consists of 52 cards. There are 13 ranks $\{2, 3, 4, \ldots, 9, 10, J, Q, K, A\}$ and each of them comes with 4 suits: Spade(♠), Heart(♥), Diamond(♦) and club(♣).

The game is played by dealing 5 cards to a player.

1.1) How many 5 cards hands are there?

Answer: $\binom{52}{5}$. Choosing 5 cards from 52 cards.

1.2) A *royal straight flush* happens when the player have $10, J, Q, K, A$ all of the same suit. How many royal straight flush are there?
Ex: 10♥, J♥, Q♥, K♥, A♥

Answer: There is 1 for each suit. So, 4.

1.3) 4-of-a-kind happens when 4 of the 5 cards are of the same rank. How many 4-of-a-kind are there?
Ex: 3♠, 3♥, 3♥, 3♣, 7♥

Answer:

$$\underbrace{13}_{\text{Pick rank}} \times \overbrace{48}^{\text{Pick the lone card}}$$

1.4) Full house happens when 3 of the 5 cards form 3 of a kind and the other 2 form a pair. How many full house hands are there?
Ex: J♥, J♠, J♦, 9♣, 9♥

Answer:

$$\underbrace{13\binom{4}{3}}_{\text{rank and suit for 3 of a kind}} \times \overbrace{12\binom{4}{2}}^{\text{suit and rank for the pair}}$$

1.5) Straight Flush happens when you have 5 consecutive cards of the same suit that is not a Royal Straight Flush. (Note that $A, 2, 3, 4, 5$ is not considered consecutive.) How many straight flush hands are there?
Ex: 2♥, 3♥, 4♥, 5♥, 6♥

Answer:

$$\overset{\substack{\text{only 8 non royal straight}\\ \downarrow}}{8} \times \underset{\substack{\uparrow\\ \text{suit}}}{4}$$

1.6) A flush happens when the 5 cards you get are of the same suit but not staright flush or a royal straight flush. How many of a flush hands are there?

Ex: $5\diamondsuit, 3\diamondsuit, 8\diamondsuit, 10\diamondsuit, K\diamondsuit$

Answer:

$$\underbrace{\left(\binom{13}{5} - 9\right)}_{\text{Choosing 5 non-consectuive no}} \quad \times\overset{\substack{\text{Assign the suit.}\\ \downarrow}}{(4)}$$

1.7) A straight happens when all 5 cards are of consecutive rank but not a straight flush or royal straight flush. How many straight hands are there?

Ex: $7\clubsuit, 8\clubsuit, 9\heartsuit, 10\spadesuit, J\heartsuit$

Answer:

$$\underset{\substack{\uparrow\\ \text{Choose the starting card}}}{9} \times \overset{\substack{\text{Assign the suit.}\\ \downarrow}}{(4^5 - 4)}$$

1.8) 3-of-a-kind happens when exactly 3 of 5 cards are of the same rank but not a full house. How many 3-of-a-kind hands are there?

Ex: $A\spadesuit, A\heartsuit, A\diamondsuit, 5\spadesuit, 7\heartsuit$

Answer:

$$13\overset{\substack{\text{Choose 1 rank for 3 of a kind and the suit.}\\ \downarrow}}{\binom{4}{3}} \times \underset{\substack{\uparrow\\ \text{Choose 2 card of different rank.}}}{\binom{12}{2}} 4^2$$

1.9) 2-pair happens when you have exactly 2 pairs in the hand. How many 2 pairs hand are there? Be careful about double counting this one.

Ex: $3\spadesuit, 3\heartsuit, J\spadesuit, J\diamondsuit, A\spadesuit$

Answer:

$$\underset{\substack{\uparrow\\ \text{Choose 2 rank for the two pairs}}}{\binom{13}{2}} \overset{\substack{\text{Suit for the 2 paris}\\ \downarrow}}{\binom{4}{2}\binom{4}{2}} \underset{\substack{\uparrow\\ \text{Leftover cards}}}{(52 - 8)}$$

1.10) 1-pair hands happens when you have exactly 1 pair in your hand but not a full house.

Ex: $5\clubsuit, 5\heartsuit, A\heartsuit, 2\diamondsuit, 10\clubsuit$

Answer:

$$13\overset{\substack{\text{Suit and rank for pair}\\ \downarrow}}{\binom{4}{2}} \underset{\substack{\uparrow\\ \text{Suit and rank for the 3 cards.}}}{\binom{12}{3}} 4^3$$

1.11) High card happens when you do not have anything special(all of the above) in your hand. How many how many high card hands are there? There is an easy way to do this and there is a cool way to do this? Try figure out both ways.

Ex: $2\clubsuit, 3\clubsuit, 10\spadesuit, J\spadesuit, 5\heartsuit$

Answer:

$$\left(\underbrace{\binom{13}{5}}_{\text{5 different non-straight ranks}} - 9\right)(\overbrace{4^5 - 4}^{\text{4 suit not the same.}})$$

Problem 2 How many numbers in $1, 2, 3, \ldots, 1000$ are there that is divisible by 2 or 5 or 7?

Answer: Inclusion-exclusion

$$\left\lfloor \frac{1000}{2} \right\rfloor + \left\lfloor \frac{1000}{5} \right\rfloor + \left\lfloor \frac{1000}{7} \right\rfloor - \left\lfloor \frac{1000}{2 \cdot 5} \right\rfloor - \left\lfloor \frac{1000}{2 \cdot 7} \right\rfloor - \left\lfloor \frac{1000}{5 \cdot 7} \right\rfloor + \left\lfloor \frac{1000}{2 \cdot 5 \cdot 7} \right\rfloor = 657$$

Problem 3 Read up the lecture notes on injective, surjective, and bijective function and decide whether the following functions $A \to B$ is injective, surjective, or surjective.

3.1) $A = \{a, b\}$, $B = \{1, 2\}$, $f(a) = 1, f(b) = 1$.

3.2) $A = \{a, b\}$, $B = \{1, 2, 3\}$, $g(a) = 1, g(b) = 2$.

3.3) $A = \{a, b, c\}$, $B = \{1, 2\}$, $h(a) = 1, h(b) = 2, h(c) = 1$.

3.4) $A = \{a, b\}$, $B = \{1, 2\}$, $q(a) = 1, q(b) = 2$.

Answer: Too easy read up lecture notes.

Problem 4 Combinatoric proof. Find a combinatoric proof of the following identity.

$$\sum_{i=0}^{n} \binom{n}{i} \binom{n}{n-i} = \binom{2n}{n}$$

If you have problem with this, read up lecture notes. This is very similar.

Answer: Read lecture notes.

Problem 5 Best, Boss, Tow+ and Majeed love to play a card game called Jubmoo so much. Ask them for how to play. The games begins with dealing cards to 4 players. Each player will get exactly 13 cards.

5.1) How many ways are there to deal 52 cards to 4 players such that each player get exactly 13 cards?

Answer:

$$\underbrace{\binom{52}{13}}_{\text{Tow+}} \overbrace{\binom{39}{13}}^{\text{Best}} \underbrace{\binom{26}{13}}_{\text{Boss}} \overbrace{\binom{13}{13}}^{\text{Majeed}}$$

However, if it happens that at least 1 person has no ♥ in his/her hand the card will have to be redealt.

5.2) How many ways are there such that Tow+ gets no ♥? (The other three may get no ♥ too.)

Answer:

$$\underset{\substack{\uparrow\\ \text{Tow+}}}{\binom{39}{13}}\overset{\substack{\text{Best}\\ \downarrow}}{\binom{39}{13}}\underset{\substack{\uparrow\\ \text{Boss}}}{\binom{26}{13}}\overset{\substack{\text{Majeed}\\ \downarrow}}{\binom{13}{13}}$$

5.3) How many ways are there such that Tow+ and Boss gets no ♥? (The other two may get no ♥ too.)

Answer:

$$\underset{\substack{\uparrow\\ \text{Tow+}}}{\binom{39}{13}}\overset{\substack{\text{Best}\\ \downarrow}}{\binom{26}{13}}\underset{\substack{\uparrow\\ \text{Boss}}}{\binom{26}{13}}\overset{\substack{\text{Majeed}\\ \downarrow}}{\binom{13}{13}}$$

5.4) How many ways are there such that Tow+, Boss and Majeed gets no ♥?

Answer:

$$\underset{\substack{\uparrow\\ \text{Tow+}}}{\binom{39}{13}}\overset{\substack{\text{Best}\\ \downarrow}}{\binom{26}{13}}\underset{\substack{\uparrow\\ \text{Boss}}}{\binom{13}{13}}\overset{\substack{\text{Majeed}\\ \downarrow}}{\binom{13}{13}}$$

5.5) How many ways are there such that four people has no ♥? (Trick question.)
Answer: Impossible. 0.

5.6) How many ways are there such that *at least* one person has no ♥? (Hint: Inclusion-Exclusion)

Answer:

$$4 \times A_{5.2} - \binom{4}{2} \times A_{5.3} + \binom{4}{3} \times A_{5.4} - 0$$

5.7) What is the probability that they have to redeal the cards? Give me numerical answer. Use computer or something to do it. (I got around 5%.)

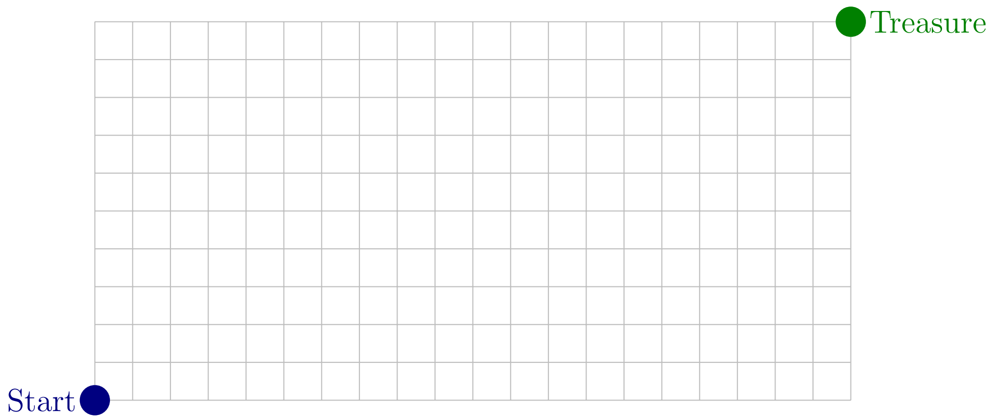
Answer:$\approx 5\%$

Problem 6 Mixed up deck. There are three decks of card in 1409 they got all mixed up so we have a pile of (156 cards). How many ways are there to arrange them? Assume that the card of the same rank and same suit from the three decks looks identical.

Answer:

$$\frac{\overset{\substack{\text{Arranging } 52 \times 3 \text{ cards}\\ \downarrow}}{156!}}{\underset{\substack{\uparrow\\ \text{each of the 52 card got overcounted 3! times}}}{(3!)^{52}}}$$

Problem 7 Our computer concept students are doing a new robot competition. Grid walking robot. The robot must walk along the grid lines. The robot can *only go up or right.* The robot starts at (0,0). The treasure is placed at (20,10).

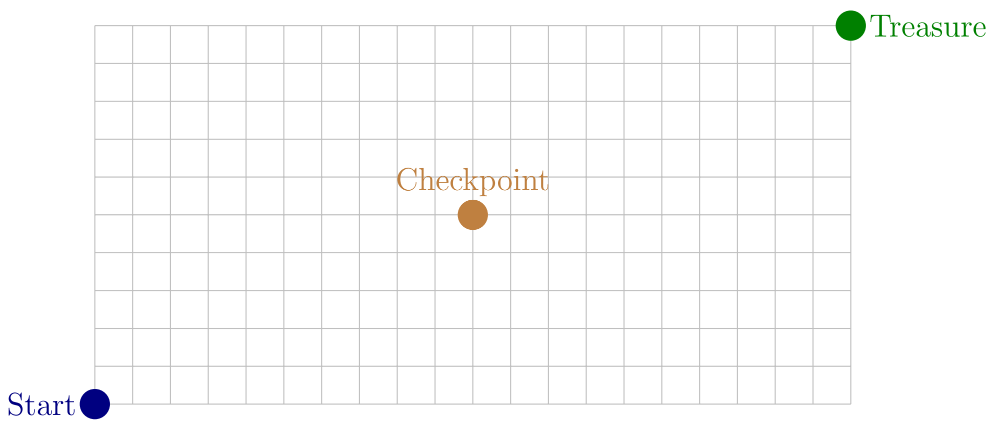


7.1) How many ways are there for the robot to reach the goal?

Answer: Since you need to go 20 right and 10 up. All you need to choose is which turn to go right.

$$\binom{30}{10}$$

Since the game seems to be too hard. AJ Boonyanit decide to place a checkpoint at (10,10)

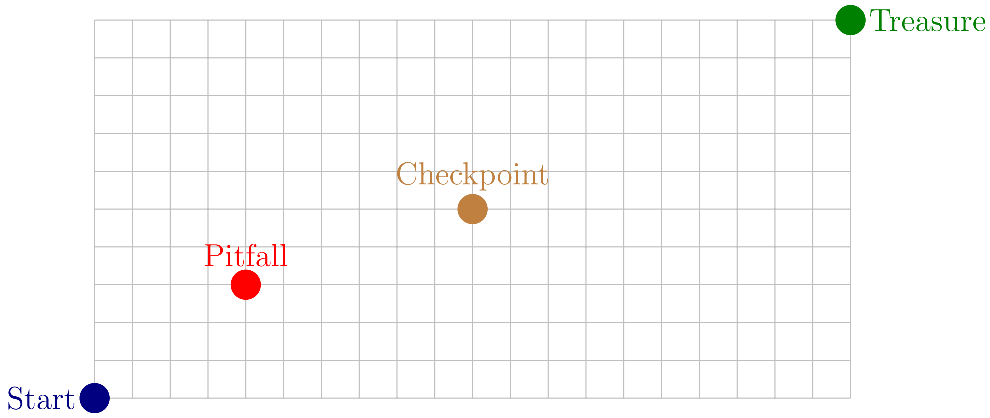


7.2) How many ways are there for the robot to reach the checkpoint then the goal?

Answer:

$$\underset{\substack{\uparrow \\ \text{Start to checkpoint}}}{\binom{15}{5}}\overset{\substack{\text{Checkpoint to Goal} \\ \downarrow}}{\binom{15}{5}}$$

AJ Piti feels that the competition is too easy so he places a pitfall at (4,3).



7.3) How many ways are there for the robot to reach the checkpoint then the goal while avoiding the pitfall?

Answer:

$$\left[\underset{\substack{\uparrow\\\text{Start to CP}}}{\binom{15}{5}} - \overset{\substack{\text{Start to pitfall}\\\downarrow}}{\binom{7}{3}}\underset{\substack{\uparrow\\\text{Pitfall to goal}}}{\binom{8}{2}}\right]\overset{\substack{\text{Checkpoint to Goal}\\\downarrow}}{\binom{15}{5}}$$

Problem 8 Consider Tower Hanoi with 3 poles and 10 disks of distinct sizes. A disk can only be place on the bottom level or on the bigger disk.

8.1) How many valid configurations are there if the three poles are distinct?

Answer: Each disk has 3 choices for the pole. Plus, given a set of disk in a pole there is only one way to arrage them on that pole. So, the answer is

$$3^{10}$$

8.2) If the three poles are identical, how many valid configurations are there? Be careful on this one and make sure your answer is integer.

Answer: It is tempting to just divide by 3! from the answer in the previous part. But you will notice that the result would not be an integer. We need to be a bit careful in dividing out the duplicate. There are two cases for this

- If exactly one pole has disc we want to divide by 3 since there are two identical empty pole. There are 3 of this cases.
- If two or three pole has disc then we want to divide it by 3!. There are $3^{10} - 3$ of these.

So, the answer is

$$\frac{3^{10}-3}{3!} + \frac{3}{3}$$

Problem 9 How many ways can n books be placed on k distinguishable shelves

9.1) if the books are indistinguishable copies of the same title?

Answer: Stick and stone.

$$\binom{n+k-1}{k-1}$$

9.2) if no two books are the same, and the positions of the books on the shelves matter?

Answer: For each of what we found in the previous part there are $n!$ way to shuffle the book

$$\binom{n+k-1}{k-1} \times n!$$