ID: ____________________ Name: ___________________________________

Discrete Mathematics

Midterm Exam T1 2016

Instruction

- Write your name
- Read the questions carefully.
- You have 4 hours to finish the exam.
- There are 6 problems. Each problem worths 100 points. 600 points in total. You only need to get 540 points to get full score.
- Attempt all problems, state your reasons *clearly* and *legibly*, because partial credits will be given.

Question	Full Score	Your Score
1	100	
2	100	
3	100	
4	100	
5	100	
6	100	
Bonus	30	

Total: ______ /540

ID: ____________________ Name: ___________________________________

Useful Formula and Definitions

Asymptotics

Definiton	Definition			Intuition
Asym. Equal	$f \sim g$	iff	$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$	$f \underbrace{\equiv}_{x\to\infty} g$
Big Oh	$f \in O(g)$	iff	$\lim_{x\to\infty} \frac{f(x)}{g(x)} < \infty$	$f \underbrace{\leq}_{x\to\infty} g$
Little Oh	$f \in o(g)$	iff	$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$	$f \underbrace{<}_{x\to\infty} g$
Little Omega	$f \in \omega(g)$	iff	$\lim_{x\to\infty} \frac{f(x)}{g(x)} \to \infty$	$f \underbrace{>}_{x\to\infty} g$
Big Omega	$f \in \Omega(g)$	iff	$\lim_{x\to\infty} \frac{f(x)}{g(x)} > 0$	$f \underbrace{\geq}_{x\to\infty} g$
Theta	$f \in \Theta(g)$	iff	$\lim_{x\to\infty} \frac{f(x)}{g(x)} = c, c \neq 0$	$f \underbrace{=}_{x\to\infty} g$

Sum

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$
$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$
$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Integral

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} \qquad \text{if } n \neq -1$$
$$\int \frac{1}{x} \, dx = \ln(x)$$

Quadratic

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

ID: ____________________ Name: ____________________________________

1. Easy stuff(100 points. 20 each.)

   (a) Draw the truth table for $(P \lor Q) \land \neg P$

   (b) Which of these asymptotic symbols ($\{\sim, o, O, \omega, \Omega, \Theta\}$) are applicable for

   $$f(x) = \_\_ \big(g(x)\big)$$

   i. $f(x) = (x+1)^2, g(x) = x^2$

   ii. $f(x) = 2^x, g(x) = 3^x$

   (c) Find the closed form formula for the following sum/product:

   i. $\sum_{i=1}^{n} \sum_{j=1}^{m} (i+2)(j)$

   ii. $\prod_{i=1}^{n} \prod_{j=1}^{m} 2^{i+j}$

(d) Prove or Disprove the following proposition.(Read the question very carefully)

$$\text{If } 2|ab \text{ then } 2|a \textbf{ and } 2|b\ \forall a, b \in I.$$

(e) Use integral bound to find the *lowerbound* for the following sum. You may leave your answer as integral.

$$\sum_{x=1}^{n} \sqrt[4]{x}$$

ID: ____________________ Name: ___________________________________

2. Easy Proof.

   (a) (50 points) For every integer n, $n^2 - n + 6$ is even

   (b) (50 points) Let $a, b \in I$. If $a + b \geq 23$ then $a \geq 12$ **or** $b \geq 12$.

ID: ____________________ Name: ___________________________________

3. (100 points)Pick one. Indicate the one you pick. If you don't, I'll pick the one you got less score. Doing two won't give you more score. Be sure to indicate where you use inductive hypothesis.

   (a) Fibonacci numbers is defined as

   $$F_0 = 1, F_1 = 1, F_2 = 2, F_{n+1} = F_n + F_{n-1}$$

   The first few terms are

   $$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

   Prove the following identity:

   $$F_2 + F_4 + F_6 + \ldots + F_{2n} = F_{2n+1} - 1$$

   (b) Show that every positive integer is a product of a *power of two* and an *odd integer*

Blank page for problem 3

4. Pick one. Indicate the one you pick. If you don't, I'll pick the one you got less score. Doing two won't give you more score.

   (a) Prove by induction that for all integers $n \geq 2$.

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n} \tag{1}$$

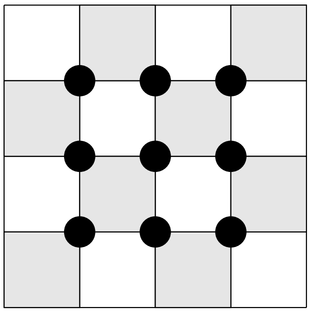
   (b) Consider the following game consisting of a coin stack played by two players(A and B) alternatively picking up coins. A picks the coin first. The winner is the one who pick the last coin. During each turn a player can only pick 1,2,3 coin.

      i. If there are 4 coins at the beginning, who has the guarantee winning strategy and how?

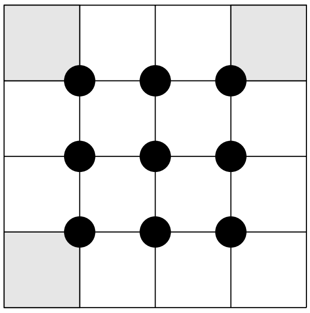
      ii. Prove that that same person has winning strategy for $4n$ coins for any integer n. (Describe the strategy and prove it)

Blank page for problem 4.

5. (100 points)Consider the following 4×4 checkerboard. Each turn you pick and inner intersection indicate by a circles in the picture below and flip the color of the four neighboring cell.



Is it possible to keep flipping and end up with a board that looks like this? Prove it either way.



Blank page for problem 5.

ID: ____________________ Name: ____________________________________

6. (100 points)Solve the following recurrence. Just find the solution. No need to verify it using induction.:

   (a) (30 points)$T(n) = T(n-1) + 2n; T(1) = 1$

   (b) (30 points)$T(n) = 2T\left(\frac{n}{2}\right) + 7 \quad \text{where } T(1) = 1$

(c) (40 points)$T(n) = -2T(n-1) + 15T(n-2); T(0) = 3, T(1) = 1$

7. Bonus.(30 points) No partial credit for this one. Prove that for any lossless compression algorithm(zip, gzip, rar, 7z etc.) if there is one file that gets smaller after compression, then there is at least one file that the size gets bigger after compression.