ID: ____________________ Name: ___________________________________

DISCRETE MATHEMATICS

Midterm Exam T3 2021

Instruction

- Write your name
- Read the questions carefully.
- This exam is open book, open notes and open internet. You are however not allowed to get help from other human being on exam via any means. (Ex: Talking to a cat for moral support is allowed.)
- Upload the PDF on canvas by the deadline specified on canvas.
- There are 6 problems. Each problem worths 100 points. 600 points in total. You only need to get 540 points to get full score.
- Attempt all problems, state your reasons *clearly* and *legibly*, because partial credits will be given.

Question	Full Score	Your Score
1	100	
2	100	
3	100	
4	100	
5	100	
6	100	

Total: ______ /540

Useful Formula and Definitions

Asymptotics

Definiton	Definition			Intuition
Asym. Equal	$f \sim g$	iff	$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 1$	$f \underbrace{\equiv}_{x\to\infty} g$
Big Oh	$f \in O(g)$	iff	$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} < \infty$	$f \underbrace{\leq}_{x\to\infty} g$
Little Oh	$f \in o(g)$	iff	$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 0$	$f \underbrace{<}_{x\to\infty} g$
Little Omega	$f \in \omega(g)$	iff	$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} \to \infty$	$f \underbrace{>}_{x\to\infty} g$
Big Omega	$f \in \Omega(g)$	iff	$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} > 0$	$f \underbrace{\geq}_{x\to\infty} g$
Theta	$f \in \Theta(g)$	iff	$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = c, c \neq 0$	$f \underbrace{=}_{x\to\infty} g$

Sum

$$
\begin{aligned}
1 + 2 + 3 + \ldots + n &= \frac{n(n+1)}{2} \\
1^2 + 2^2 + 3^2 + \ldots + n^2 &= \frac{n(n+1)(2n+1)}{6} \\
1^3 + 2^3 + 3^3 + \ldots + n^3 &= \left(\frac{n(n+1)}{2}\right)^2
\end{aligned}
$$

Integral

$$
\int x^n \, dx = \frac{1}{n+1} x^{n+1} \qquad \text{if } n \neq -1
$$

$$
\int \frac{1}{x} \, dx = \ln(x)
$$

Quadratic

$$
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
$$

1. Easy stuff(100 points. 20 each.)

   (a) Draw truth table for
   $$(P \to \sim R) \land (P \lor R)$$

   (b) Asymptotic Behavior. $f(x) = x^3 + 3x^2 + 1$, $g(x) = 2x^3$. Indicate whether the following statements are true or false.

      i. $f \in \Theta(g)$

      ii. $f \in O(g)$

   (c) Find the closed form formula for the following sum:
   $$\sum_{i=0}^{n} (x^i + 5y^{i+1})$$
   where x, y are constants.

(d) Disprove the following theorem.

$$x^2 > x \text{ for all } x \text{ in positive integer.}$$

(e) Use integral bound to find the *lowerbound* for the following sum.

$$\sum_{x=1}^{n} \frac{1}{x^3}$$

2. Easy Proof

   (a) (50 points) For any $a, b \in I$, $a^2b + ab^2$ is always even.

   (b) (50 points) Show that If $a|b$ and $b|c$ then $a|c$.
   Ex: a=3 b=54 c=108. 54 is divisible by 3 and 108 is divisible by 54 then 3 divides 108

3. (100 points) Show that $\forall n \geq 2$

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\ldots\left(1-\frac{1}{n}\right)=\frac{1}{n}$$

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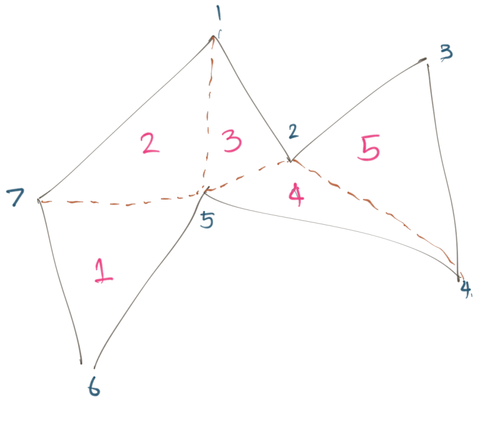
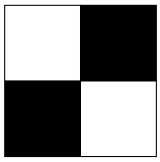
4. Any n-gon(loose definition is a closed shape with n corners) can be triangulate into n-2 triangles.

   Ex: The figure below shows a 7-gon triangulated in to 5 triangles and a 12-gon triangulated into 10 triangles.



5. (100 points) Consider a 2×2 black and white board show below.



Each turn you can pick a **side**(left, right, top, bottom) then flip color of the two squares on that side. Ex: if you pick left then the two squares on the left will be flipped.

Show that no matter what sequence of move you choose, you can't get it to have exactly 1 white square left.

6. (100 points)Solve the following recurrence. Just find the solution. No need to prove it using induction.:

   (a) (30 points)$T(n) = T(n-1) + n; T(1) = 2$

   (b) (30 points)$T(n) = 3T\left(\frac{n}{2}\right) + 3; T(1) = 3$

(c) (40 points)$T(n) = 18T(n-1) - 77T(n-2); T(0) = 5, T(1) = 47$