Homework 5

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Problem 1 Find closed form expression of the following sum/products.

1.1) $\sum_{i=1}^{n}(2i+1)$

1.2) $\sum_{i=1}^{n}\sum_{j=1}^{m}(i+j^2)$

1.3) $\sum_{i=1}^{n}\sum_{j=1}^{m}2^{i+2j}$

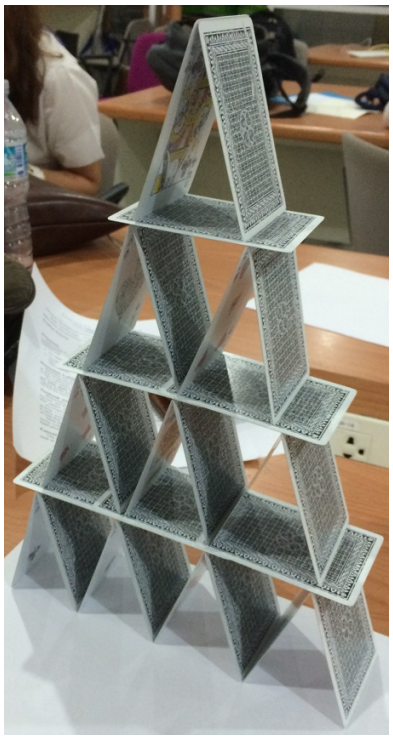
1.4) $\prod_{i=1}^{i=n}\prod_{j=1}^{j=m}2^i 3^j$

Problem 2 Use integral bound to find upper bound and lower bound of the following sum.

2.1) $\sum_{x=1}^{n}\frac{1}{x^2}$

2.2) $\sum_{x=1}^{n}x^{\frac{3}{2}}$

Problem 3 House of Card. Bossy wants to build a tall house of card (classic pyramid one). `http://en.wikipedia.org/wiki/House_of_cards`. He needs your help to figure out the number of cards he needs for his big project. Find the formula of the number of cards needed to build n stories house of card.



How many deck of cards does he need to buy to build a 10 stories tall house of card? Ask Bossy or May to build one if you don't know what it looks like.

Bonus: Build one take a picture upload it on canvas.

Problem 4 Data Structure Cheat Sheet.

4.1) $T(n) = T(n-1) + n$ where $T(1) = 1$. Only for this problem. Verify it using induction.

4.2) $T(n) = T(n-1) + n^3$ where $T(1) = 1$.

4.3) $T(n) = T\left(\frac{n}{2}\right) + 1$ where $T(1) = 1$.

4.4) $T(n) = T\left(\frac{n}{2}\right) + n$ where $T(1) = 1$.

4.5) $T(n) = 2 \times T\left(\frac{n}{2}\right) + n$ where $T(1) = 1$.

4.6) $T(n) = 2 \times T\left(\frac{n}{2}\right) + 1$ where $T(1) = 1$.

4.7) (Optional) $T(n) = 2 \times T\left(\frac{n}{2}\right) + \log_2 n$ where $T(1) = 1$. You will need a funny sum which we learn how to compute before.

Problem 5 Fill in all the symbols $\{\sim, o, O, \omega, \Omega, \Theta\}$ that apply. $f = \boxed{\ ?\ }(g)$

5.1) $f(n) = 30n + 900 \log n, g(n) = n$

5.2) $f(n) = \log n, g(n) = n$

5.3) $f(n) = \log_2 n, g(n) = n$

5.4) $f(n) = \sqrt{(n)}, g(n) = (\log n)^{999}$

5.5) $f(n) = n2^n, g(n) = n$

5.6) $f(n) = n^2, g(n) = 1.0000000001^n$

5.7) $f(n) = 200000, g(n) = 1$

5.8) $f(n) = 2^n, g(n) = 10^n$ Be careful for this one.

Problem 6 Solve the following recurrence.

6.1) $T_{i+1} = 5T_i - 6T_{i-1}; T_0 = 8, T_1 = 17$

6.2) $T_{i+1} = -T_i + 12T_{i-1}; T_0 = 0, T_1 = -7$

Problem 7 Recall the "Roulette" Trick we did in class. Your job is to prove a stronger version of the game.

a) Consider a set of r red cards and b black cards.

b) The betting strategy is to always bet on red.

c) If my guess is correct, I'll get back 2 times the amount I bet. For example, if I bet 10 Baht on red and the card turn out to be red, I'll bet back 20 Baht making 10 Baht profit.

d) The betting amount is calculated with the following algorithm:

   i. If I win a turn I'll bet w times the previous bet on the next turn.

   ii. But, if I lose this turn, I'll bet l times the previous bet.

   For example, if the bet for this turn is 10 Baht, and I win this turn, the betting amount for the next turn would be $10w$. Whereas if I lose this turn, then the betting amount for the next turn will bet 10l.

7.1) Show that if $w + l = 2$ then the result of the game would be independent of the permutation of the card.

7.2) What would be the total gain/lose amount if $w + l = 2$. (Answer in terms of r, b, w, l and m the starting bet.)

7.3) (Optional)If there are 20 black cards and 18 red cards. What should w be to guarantee the maximum gain. (You will may computer for this. Numerical answer is OK.)

Problem 8 Coupon Bond. Coupon bond is a popular financial instrument. Here is the deal:

A coupon bonds of face value F and maturity date N years from now with payment P per year will pay you P Baht per years for N years. Then at last payment you will also get F Baht.

For example, if you buy today a coupon bond of face value $F = 10$Million Baht with maturity date $N = 3$ years from now with payment of $P = 10,000$ Baht. You will get 10K one year from now, then two years from now, then three years after you bought it. Then, at the end of the third year, along with the last 10K payment, you will also get 10Million Baht.

The question is if the (yearly) interest rate is r how much should you pay for a coupon bonds of face value F with maturity date N years from now with payment of P baht per year?