Discrete Mathematics 2

ANSWERS (taken from a past student)

Problem 1:

Theorem: $1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \quad \forall n \in \mathbb{Z}_{>0}$

Inductive Predicate: $P(i) \equiv 1^2 + 2^2 + 3^2 + 4^2 + \ldots + i^2 = \frac{i(i+1)(2i+1)}{6}$

Base case: $P(1)$

LHS: $1^2 = 1$

RHS: $\frac{1(1+1)(2\times 1+1}{6} = \frac{1(2)(3)}{6} = 1$

LHS = RHS, $P(1)$ is true.

Inductive step:

Let us assume that $\exists k \in \mathbb{Z}_{>1}$

$$1^2 + 2^2 + 3^2 + 4^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

We want to show that

$$\begin{aligned} 1^2 + 2^2 + 3^2 + 4^2 + \ldots + k^2 + (k+1)^2 &= \frac{(k+1)[(k+1)+1][2(k+1)+1]}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \\ &= \frac{(k+1)(2k^2+7k+6)}{6} \end{aligned}$$

Consider the left hand side.

$$\begin{aligned} \underbrace{1^2 + 2^2 + 3^2 + 4^2 + \ldots + k^2}_{\text{IH}} + (k+1)^2 &= \underbrace{\frac{k(k+1)(2k+1)}{6}}_{\text{IH}} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1) + 6(k+1)(k+1)}{6} \\ &= \frac{(k+1)[k(2k+1) + 6k + 6]}{6} \\ &= \frac{(k+1)(2k^2+7k+6)}{6} \\ &= \text{RHS} \end{aligned}$$

Therefore, by mathematical induction,

$$1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \quad \forall n \in \mathbb{Z}_{>0}$$

$\square$

Problem 2:

Theorem: $1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2 \quad \forall n \in \mathbb{Z}_{>0}$

Definition:

(1) $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \quad \forall n \in \mathbb{Z}_{>0}$

(2) $(a + b)^2 = a^2 + 2ab + b^2$

Inductive Predicate: $P(i) \equiv 1^3 + 2^3 + 3^3 + \ldots + i^3 = (1 + 2 + 3 + \ldots + i)^2$

Base case: $P(1)$

LHS: $1^3 = 1$

RHS: $1^2 = 1$

LHS = RHS, $P(1)$ is true.

Inductive step:

Let us assume that $\exists k \in \mathbb{Z}_{>1}$

$$1^3 + 2^3 + 3^3 + \ldots + k^3 = (1 + 2 + 3 + \ldots + k)^2$$

We want to show that

$$\begin{aligned}
1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 &= [\underbrace{1 + 2 + 3 + \ldots + k}_{\text{Definition (1)}} + (k+1)]^2 \\
&= \underbrace{[\overbrace{\frac{k(k+1)}{2}}^{\text{Definition (1)}} + (k+1)]^2}_{\text{Definition (2)}} \\
&= \underbrace{[\frac{k(k+1)}{2}]^2 + 2[\frac{k(k+1)}{2}](k+1) + (k+1)^2}_{\text{Definition (2)}} \\
&= [\frac{k(k+1)}{2}]^2 + k(k+1)^2 + (k+1)^2 \\
&= [\frac{k(k+1)}{2}]^2 + (k+1)^2(k+1) \\
&= [\frac{k(k+1)}{2}]^2 + (k+1)^3
\end{aligned}$$

Consider the left hand side.

$$\begin{aligned}
\underbrace{1^3 + 2^3 + 3^3 + \ldots + k^3}_{\text{IH}} + (k+1)^3 &= \underbrace{(1+2+3+\ldots+k)^2}_{\text{IH / Definition (1)}} + (k+1)^3 \\
&= \underbrace{[\frac{k(k+1)}{2}]^2}_{\text{Definition (1)}} + (k+1)^3 \\
&= \text{RHS}
\end{aligned}$$

Therefore, by mathematical induction,

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = (1+2+3+\ldots+n)^2 \quad \forall n \in \mathbb{Z}_{>0}$$

$\square$

Problem 3:

Definition: $F_0 = 1, F_1 = 1, F_2 = 2, F_{n+1} = F_n + F_{n-1}$

Theorem: $F_2 + F_4 + F_6 + \ldots + F_{2n} = F_{2n+1} - 1$

Inductive Predicate: $P(i) \equiv F_2 + F_4 + F_6 + \ldots + F_{2i} = F_{2i+1} - 1$

Base case: $P(1) \equiv F_2 = F_3 - 1$

LHS: $F_2 = 2$

RHS: $F_3 - 1 = F_2 + F_{2-1} - 1 = 2 + 1 - 1 = 2$

LHS = RHS, $P(1)$ is true.

Inductive step:

Let us assume that $\exists k \in \mathbb{Z}_{>1}$

$$F_2 + F_4 + F_6 + \ldots + F_{2k} = F_{2k+1} - 1$$

We want to show that

$$\begin{aligned}
\underbrace{F_2 + F_4 + F_6 + \ldots + F_{2k}}_{\text{IH}} + F_{2(k+1)} &= F_{2(k+1)+1} - 1 \\
\underbrace{F_{2k+1} - 1}_{\text{IH}} + F_{2(k+1)} &= F_{2(k+1)+1} - 1 \\
F_{2k+1} + F_{2k+2} &= F_{2k+3} \\
\underbrace{F_{2k+2} + F_{2k+1}}_{\text{Definition}} &= F_{2k+3} \\
\underbrace{F_{2k+3}}_{\text{Definition}} &= F_{2k+3}
\end{aligned}$$

Therefore, by mathematical induction $F_2 + F_4 + F_6 + \ldots + F_{2n} = F_{2n+1} - 1$ $\square$

Problem 4:

Definition: a is *divisible* by b if and only if $\exists n \in \mathbb{Z}$ such that $a = nb$. We also denote this by $b|a$ which reads b divides a.

(4.1)

Theorem: $8^n - 3^n$ is divisible by 5 $\forall n \in \mathbb{Z}_{>0}$

Inductive Predicate: $P(i) \equiv 8^i - 3^i$ is divisible by 5

Base case: $P(1) \equiv 8^1 - 3^1$ is divisible by 5

$\frac{8-5}{5} = \frac{5}{5} = 1$ $P(1)$ is true.

Inductive step:

Let us assume that $\exists k \in \mathbb{Z}_{>1}$ $\exists p \in \mathbb{Z}$

$$8^k - 3^k = 5p$$

We want to show that $\exists q \in \mathbb{Z}$

$$8^{k+1} - 3^{k+1} = 5q$$

Consider the left hand side.

$$\begin{aligned}
8^{k+1} - 3^{k+1} &= 8 \cdot 8^k - 3 \cdot 3^k - 5 \cdot 3^k + 5 \cdot 3^k \\
&= 8 \cdot 8^k - 8 \cdot 3^k + 5 \cdot 3^k \\
&= 8\underbrace{(8^k - 3^k)}_{\text{IH}} + 5 \cdot 3^k \\
&= 8\underbrace{(5p)}_{\text{IH}} + 5 \cdot 3^k \\
&= 5(8p + 3^k) \\
&= 5q
\end{aligned}$$

Therefore, by mathematical induction,

$$8^n - 3^n \text{ is divisible by } 5\ \forall n \in \mathbb{Z}_{>0}$$

□

(4.2)

Theorem: $\forall n \in \mathbb{Z}_{>0}$ $\forall a, b \in \mathbb{Z}$ $a - b | a^n - b^n$

Inductive Predicate: $P(i) \equiv a^i - b^i$ is divisible by $a - b$

Base case: $P(1) \equiv a^1 - b^1$ is divisible by $a - b$

$\frac{a-b}{a-b} = 1$ $P(1)$ is true.

Inductive step:

Let us assume that $\exists k \in \mathbb{Z}_{>1} \quad \exists p \in \mathbb{Z}$

$$a^k - b^k = (a-b)p$$

We want to show that $\exists q \in \mathbb{Z}$

$$a^{k+1} - b^{k+1} = (a-b)q$$

Consider the left hand side.

$$\begin{aligned}
a^{k+1} - b^{k+1} &= a \cdot a^k - b \cdot b^k - a \cdot b^k + a \cdot b^k \\
&= a \cdot a^k - a \cdot b^k + a \cdot b^k - b \cdot b^k \\
&= a \underbrace{(a^k - b^k)}_{\text{IH}} + b^k(a-b) \\
&= a \underbrace{(a-b)p}_{\text{IH}} + b^k(a-b) \\
&= (a-b)(ap + b^k) \\
&= (a-b)q
\end{aligned}$$

Therefore, by mathematical induction,

$$\forall n \in \mathbb{Z}_{>0} \quad \forall a, b \in \mathbb{Z} \quad a - b | a^n - b^n$$

$\square$

(4.3)

Theorem: If n is odd then $a + b | a^n + b^n$

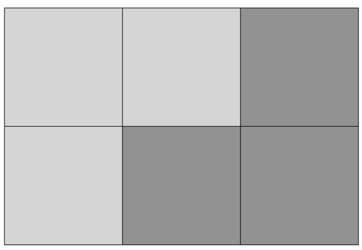
Proof: If a negative number is raised to an odd power, the result will be negative. Therefore, if n is odd, $a^n + b^n = a^n - (-b)^n$ which is proven in (4.2) to be divisible by $a - (-b) = a + b$. $\square$

Problem 5:

Theorem: $2n \times 3m$ checkerboard can be covered with L shape triminoes.

Inductive Predicate: $P(i, j) \equiv 2i \times 3j$ checkerboard can be covered with L shape triminoes.

Base case: $P(1, 1) \equiv 2(1) \times 3(1)$ checkerboard can be covered with L shape triminoes



Inductive step: Let us assume that $\exists n \in \mathbb{Z}_{>1}$ such that $2n \times 3$ checkerboard can be covered with L shape triminoes.

We want to show that $2(n+1) \times 3$ checkerboard can be covered with L shape triminoes.

$$\begin{aligned} 2(n+1) \times 3 &= (2n+2) \times 3 \\ &= \underbrace{(2n \times 3)}_{\text{IH}} + \underbrace{(2 \times 3)}_{\text{base case}} \end{aligned}$$

Since both $2n \times 3$ and 2×3 board can be covered with L shape triminoes, $2(n+1) \times 3$ board can be covered with L shape triminoes.
$\therefore \forall n \geq 1, \quad 2n \times 3$ checkerboard can be covered with L shape triminoes.

Let us assume that $\forall n \geq 1 \quad \exists m \in \mathbb{Z}_{>1}$ such that $2n \times 3m$ checkerboard can be covered with L shape triminoes.

We want to show that $2n \times 3(m+1)$ checkerboard can be covered with L shape triminoes.

$$\begin{aligned} 2n \times 3(m+1) &= 2n \times (3m+3) \\ &= \underbrace{(2n \times 3m)}_{\text{IH}} + \underbrace{(2n \times 3)}_{\text{proven}} \end{aligned}$$

Therefore, by mathematical induction, $2n \times 3m$ checkerboard can be covered with L shape triminoes. $\square$

Problem 6:

Theorem: $n^3 \leq 3^n \quad \forall n \geq 1$

Inductive Predicate: $P(i) \equiv i^3 \leq 3^i$

Base case:

$$\begin{array}{lll} P(1) \equiv 1^3 \leq 3^1 & 1 < 3 & P(1) \text{ is true.} \\ P(2) \equiv 2^3 \leq 3^2 & 8 < 9 & P(2) \text{ is true.} \\ P(3) \equiv 3^3 \leq 3^3 & 9 = 9 & P(3) \text{ is true.} \end{array}$$

Inductive step:

Let us assume that $\exists k \geq 4$

$$k^3 \leq 3^k$$

We want to show that

$$\begin{aligned} (k+1)^3 &\leq 3^{(k+1)} \\ k^3 + 3k^2 + 3k + 1 &\leq 3^k \times 3 \\ &\leq 3^k + 3^k + 3^k \end{aligned}$$

If we can prove the following, we can prove that $(k+1)^3 \leq 3^{(k+1)}$

$$\begin{align} k^3 &\leq 3^k \tag{1} \\ 3k^2 &\leq 3^k \tag{2} \\ 3k+1 &\leq 3^k \tag{3} \end{align}$$

(1) By inductive hypothesis, $k^3 \leq 3^k$ is true.

(2) We will prove that $\forall j \geq 3 \quad 3j^2 \leq j^3$.

$$\begin{aligned} 3 &\leq j \\ 3 \times j^2 &\leq j \times j^2 \\ 3j^2 &\leq j^3 \end{aligned}$$

By applying our inductive hypothesis $\exists k \geq 4 \quad k^3 \leq 3^k$, we can conclude that $3k^2 \leq k^3 \leq 3^k$.

(3) Firstly, we will prove that $\forall a \geq 1 \quad 4a \geq 3a+1$.

$$\begin{aligned} a &\geq 1 \\ a+3a &\geq 1+3a \\ 4a &\geq 3a+1 \end{aligned}$$

Secondly, we will prove that $\forall b \geq 4 \quad b^2 \geq 4b$

$$\begin{aligned} b &\geq 4 \\ b \times b &\geq 4 \times b \\ b^2 &\geq 4b \end{aligned}$$

Next, we will prove that $\forall c \geq 1 \quad c^3 \geq c^2$.

$$\begin{aligned} c &\geq 1 \\ c \times c^2 &\geq 1 \times c^2 \\ c^3 &\geq c^2 \end{aligned}$$

By applying our inductive hypothesis $\exists k \geq 4 \quad k^3 \leq 3^k$, we can conclude that $3k+1 \leq 4k \leq k^2 \leq k^3 \leq 3^k$.

Therefore, by mathematical induction, $n^3 \leq 3^n \quad \forall n \geq 1$ $\square$

Problem 7:

Theorem:

$$\sum_{i=1}^{n} i \times 2^i = (n-1) \times 2^{n+1} + 2$$

Inductive Predicate: $P(j) \equiv \sum_{i=1}^{j} i \times 2^i = (j-1) \times 2^{j+1} + 2$

Base case: $P(1)$

LHS: $\sum_{i=1}^{1} i \times 2^i = 1 \times 2^1 = 2$

RHS: $= (1-1) \times 2^{1+1} + 2 = 2$

LHS = RHS, $P(1)$ is true.

Inductive step:

Let us assume that $\exists k > 1$

$$\sum_{i=1}^{k} i \times 2^i = (k-1) \times 2^{k+1} + 2$$

We want to show that

$$\begin{aligned}
\sum_{i=1}^{k+1} i \times 2^i &= (k+1-1) \times 2^{k+1+1} + 2 \\
\underbrace{\sum_{i=1}^{k} i \times 2^i}_{\text{IH}} + (k+1) \times 2^{k+1} &= k \times 2^{k+2} + 2 \\
\underbrace{(k-1) \times 2^{k+1} + 2}_{\text{IH}} + (k+1) \times 2^{k+1} &= k \times 2^{k+2} + 2 \\
2^{k+1}(k-1+k+1) &= k \times 2^{k+2} \\
2^{k+1} \times 2k &= k \times 2^{k+2} \\
k \times 2^{k+2} &= k \times 2^{k+2}
\end{aligned}$$

Therefore, by mathematical induction $\sum_{i=1}^{n} i \times 2^i = (n-1) \times 2^{n+1} + 2$ □

Problem 8:

Suppose that you have infinite amount of green, red, black, yellow and white sock. You are pulling it out from a drawer blindly one at a time. How many socks do you need to pull to guarantee that you have at least a pair?

Proof: Let's think about the colors of sock as holes and the number of socks we need to pull out as pigeons. By the pigeon hole principle, if there are more pigeons than hole then at least one hole has more than one pigeons.

In other words, if we pull out more socks than the number of colors then at least two socks are of the same color. Since we have 5 colors of sock, if we pull out 6 socks we are guarantee that at least a pair are of the same color. □

Problem 9:

Suppose S is a set of $n + 1$ integer. Show that there is at least one pair $a, b \in S$ such that $a - b$ is a multiple of n.

Since division of any integer by n will result in remainders ranging from 0,1,. . . ,n-1 there are n possible remainders. We can think of the possible remainders as holes and the numbers in the set S as pigeons. Since we have n holes and n+1 pigeons, by the pigeon hole principle, we are guaranteed that division of at least 2 numbers by n will give the same remainder.

Let two numbers with the same remainder when divided by n be a and b, we will prove that a-b is a multiple of n.

Proof: Let c be the remainder $\exists m_1, m_2 \in \mathbb{Z}$

$$\begin{aligned} a &= m_1 \times n + c \\ b &= m_2 \times n + c \\ a - b &= (m_1 - m_2)n \end{aligned}$$

Therefore, there is at least one pair $a, b \in S$ such that $a - b$ is a multiple of n. □

Problem 10:

Among any 5 points in a $1m \times 1m$ square, there is at least a pair of points with distance $\leq \frac{\sqrt{2}}{2}$

Proof: Firstly, divide the square into four $\frac{1}{2}m \times \frac{1}{2}m$ squares. We can think of each $\frac{1}{2}m \times \frac{1}{2}m$ square as a hole and the 5 dots as pigeons. By the pigeon hole principle, at least one $\frac{1}{2}m \times \frac{1}{2}m$ square has 2 dots in it.

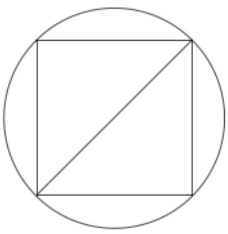
We know that the longest straight line in a square (the longest distance) is the diagonal. By proving that the diagonal of a $\frac{1}{2}m \times \frac{1}{2}m$ square is $\frac{\sqrt{2}}{2}m$, we can prove that 2 dots in the square has a distance $\leq \frac{\sqrt{2}}{2}$.

By the Pythagoras theorem, the length of the diagonal d of a $\frac{1}{2}m \times \frac{1}{2}m$ square is

$$\begin{aligned} d^2 &= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \\ d &= \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \\ &= \sqrt{\frac{1}{2}} \\ &= \frac{1}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2} \end{aligned}$$

Proof that the longest line in a square is the diagonal:

If we draw a circle to cover a square, we can see that the longest line in a square is the diagonal because it is the only line that is equal to the diameter of the circle.



Since there is at least one $\frac{1}{2}m \times \frac{1}{2}m$ square with 2 dots in it and the maximum distance the 2 dots can have is $\leq \frac{\sqrt{2}}{2}$, we have proven the theorem. □