Discrete Mathematics

Homework 8

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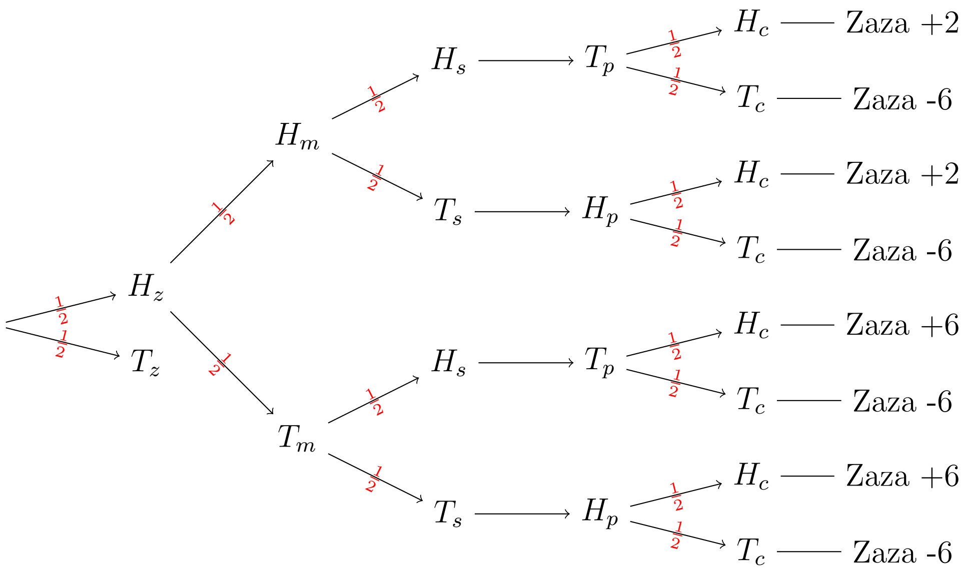
Problem 1 In the class we discussed how to cheat a seemingly fair coin toss game with 3 players. Sam and AJ Piti collaborated to win against May by betting the opposite everytime. Since Zaza missed the class on that day, May suggested the idea to form a team of 3 people then try to take Zaza's money.

So, Sam, May and AJ Piti approach Zaza and ask her to join a spilt-the-pool coin toss game. This time everyone bets 6 Baht a turn.

1.1) Evil Plan 1: The gang repeats the same strategy with Sam and AJ Piti bet the opposite every turn while May bet at random. What would be the expected value of Zaza's profit/loss each turn?

Answer: Subscripts are z:Zaza, s:Sam, p: Piti and m for May and c for coin.



Thus, (multiply by two for the lower half)

$$\mathbb{E}[\text{Zaza}] = \frac{-8}{8} = -1$$

1.2) Use linearity of expected value to calculate the the sum of the return for (Sam + May + AJ Piti). This should be a one line answer.

Answer:

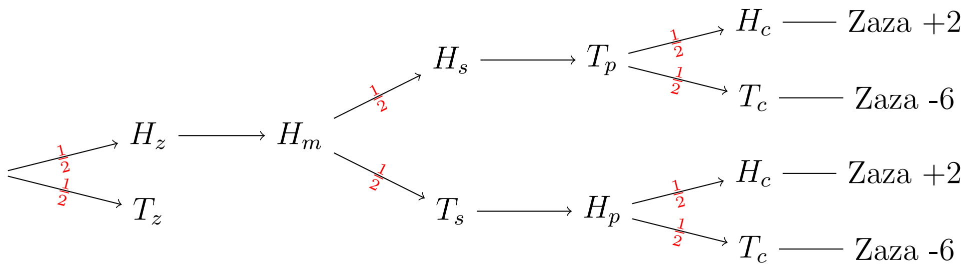
$$\mathbb{E}[\text{Zaza}] + \mathbb{E}[\text{Team}] = 0$$

Thus,

$$\mathbb{E}[\text{Team}] = 1$$

1.3) Evil Plan 2: Suppose May convinces Zaza to be the first one to bet every turn. Find a better stratey than Evil Plan1. What would be Zaza's profit/loss each turn?

Answer: Have May bet the same thing as Zaza all the time so dilute Zaza returns when she wins.



Thus,

$$\mathbb{E}[\text{Zaza}] = \frac{-16}{8} = -2$$

Problem 2 Consider the following game: you pick a number from 1-6 then you roll 3 fair and independent dice.

- If the number you picked never come up, you lose 1 Baht.
- If the number you picked come up once, you win 1 Baht.
- If the number you picked come up twice, you win 2 Baht.
- If the number you picked come up three times, you win 4 Baht.

Here come the questions.

2.1) What is the expected value of profit/loss? Should you play this game?

Answer:

$$\mathbb{E}(\text{Gain}) = -1 \times \left(\frac{5}{6}\right)^3 + 1 \times 3 \times \frac{1}{6} \times \left(\frac{5}{6}\right)^2 + 2 \times 3 \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right) + 4\left(\frac{1}{6}\right)^3$$

2.2) What is the variance of your profit/loss?

Answer:

$$\mathbb{E}(\text{Gain}^2) = (-1)^2 \times \left(\frac{5}{6}\right)^3 + 1^2 \times 3 \times \frac{1}{6} \times \left(\frac{5}{6}\right)^2 + 2^2 \times 3 \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right) + 4^2\left(\frac{1}{6}\right)^3$$

So, the variance is

$$\text{Var}(\text{Gain}) = \mathbb{E}(\text{Gain}^2) - \mathbb{E}(Gain)^2$$

2.3) Now we replace the three and independent dice with three dice taped together such that all three will come up the same number everytime. What is the expected value of profit/loss?

Answer: You can either get -1 or 4. So

$$\mathbb{E}(\text{Gain}) = -1\left(\frac{5}{6}\right) + 4\left(\frac{1}{6}\right)$$

2.4) What would be the variance of profit/loss for the taped-together dice case?

Answer:

$$\mathbb{E}(\text{Gain}^2) = (-1)^2 \left(\frac{5}{6}\right) + 4^2 \left(\frac{1}{6}\right)$$

Thus, the variance is

$$\text{Var}(\text{Gain}) = \mathbb{E}(\text{Gain}^2) - \mathbb{E}(\text{Gain})^2$$

Problem 3 Consider a Monopoly game. You roll two independent and fair dice with a catch.

- First, you advance by the sum of two dice. If the dice you have is not of the same number(double), then your turn ends.
- If the first time you roll, you get a double then you get to roll again. You then advance further by the sum of the dice on the second time you roll. If the second time you roll, you do not get a double then your turn ends.
- If on the second time you roll, you get a double again then you get to roll the third time. If the third time you roll, you get a double again then you will go to jail(the total advance becomes 0). If not, then you advance further by the sum of the dice and your turn ends.

3.1) Given that you do not get a double on the first roll, what is your expected advance? **Answer:**

$$\mathbb{E}[A] = \left(3 \times \underset{\overset{\uparrow}{3}}{2} + 4 \times \underset{\overset{\uparrow}{4}}{2} + 5 \times \underset{\overset{\uparrow}{5}}{4} + 6 \times \underset{\overset{\uparrow}{6}}{4} + 7 \times \underset{\overset{\uparrow}{7}}{6} + 8 \times \underset{\overset{\uparrow}{8}}{4} + 9 \times \underset{\overset{\uparrow}{9}}{4} + 10 \times \underset{\overset{\uparrow}{10}}{2} + 11 \times \underset{\overset{\uparrow}{11}}{2}\right) \times \frac{1}{30}$$
$$=7$$

3.2) Given that you do get a double on the first roll but not on the second roll, what is your expected advance? (Don't forget to add the advance from the first roll).

Answer: Expected advance for the first double is 7. The expected advance of the second roll is also 7. Thus

$$E[A_2] = 7 + 7 = 14$$

3.3) Given that you get a double on the first two roll but not the third roll, what is your expected advance?

Answer:

$$E[A_2] = 7 + 7 + 7 = 21$$

3.4) What is the expected value of advance for a turn? Hint: Law of total expectation.

Answer:

$$\mathbb{E}[\text{total}] = 7 \times \frac{5}{6} + 14 \times \frac{1}{6} \times \frac{5}{6} + 21 \times \left(\frac{1}{6}\right)^2 \times \left(\frac{5}{6}\right) = 8.26$$

Problem 4 Suppose I give you a uniform random number generator which gives you an integer, X, from 1 to 10 inclusively. Moreover, I give you another uniform random number generator which is independent from the first one. The second generator give you an integer Y, ranges from 201 to 210 inclusively. Answer the following questions. Be smart. Don't do it the tedious way.

4.1) What is $\mathbb{E}(X)$??

Answer:5.5

4.2) What is $\text{Var}(X)$?

Answer:$38.5 - 5.5^2 = 8.25$

4.3) What is $\mathbb{E}(200 + X)$?

Answer:$\mathbb{E}(200 + X) = \mathbb{E}[200] + \mathbb{E}[X] = 200 + 5.5$

4.4) What is $\mathbb{E}(Y)$?

Answer:$\mathbb{E}(Y) = 200 + 5.5 = 205.5$

4.5) What is $\text{Var}(200 + X)$?

Answer:$\text{Var}(200 + X) = \text{Var}(X) = 8.25$

4.6) Use the first two questions to find $\mathbb{E}(X^2)$.

Answer:$\mathbb{E}[X^2] = \text{Var}[X] + E[X]^2 = 38.5$

4.7) What is $\mathbb{E}(Y^2)$?

Answer:$\mathbb{E}[Y^2] = \text{Var}[Y] + E[Y]^2 = 42238.5$

4.8) What is $\mathbb{E}((200 + X)^2)$?

Answer:$E[(200 + X)^2] = E[200^2 + 400X + X^2] = E[200^2] + 400\mathbb{E}[X] + \mathbb{E}[X^2]$

4.9) What is $\mathbb{E}((200 + X) \times Y)$? Why is it not the same as $\mathbb{E}(Y^2)$? Hint: independence.

Answer:$\mathbb{E}((200 + X) \times Y) = \mathbb{E}[200Y + XY] = 200\mathbb{E}[Y] + \underset{\substack{\uparrow \\ \text{Independence}}}{\mathbb{E}[X]\mathbb{E}[Y]}$

4.10) What is $\mathbb{E}(2X + Y + 10)$?

Answer:$\mathbb{E}(2X + Y + 10) = 2\mathbb{E}[X] + \mathbb{E}[Y] + \mathbb{E}[10])$

4.11) What is $\text{Var}(2X + Y + 10)$?

Answer:$\text{Var}(2X + Y + 10) = 4\,\text{Var}(X) + \text{Var}(Y) + 0$

Problem 5 Consider two stocks: Yoyo stock(Y) and Sugus(S) stock. Both of the stock is now 1 Baht. The price of the stocks 1 year from now has the following probability:

$$Y = \begin{cases} 0.8\text{Baht} & \text{with probability } 0.25. \\ 1.2\text{Baht} & \text{with probability } 0.75. \end{cases}$$

$$S = \begin{cases} 0.9\text{Baht} & \text{with probability } 0.5. \\ 1.3\text{Baht} & \text{with probability } 0.5. \end{cases}$$

5.1) What is expected value of for price of Y and S after 1 year?

Answer:1.1 for both.

5.2) What is the variance of Y?

Answer:

$$\text{Var}[Y] = (1.1 - 0.8)^2 \times 0.25 + (1.1 - 1.2)^2 \times 0.75 = 0.03$$

5.3) What is the variance of S?

Answer:

$$\text{Var}[S] = (1.1 - 0.9)^2 \times 0.25 + (1.1 - 1.3)^2 \times 0.75 = 0.04$$

5.4) If we use variance to quantify the risk, which one is more risky?

Answer: The one with more variance.

5.5) Suppose you have 100 Baht now. You bought 70 Y and 30 S for your portfolio. What is the expected value of your portfolio 1 year from now?

Answer:$70 \times 1.1 + 30 \times 1.1 = 110$

5.6) Now supposed that the correlation between the two stock is -1. This means if one stock go up the other will go down. What is the variance of the value of your portfolio? Is it now more or less risky than buying just S or just Y?

Answer:

$$\sigma^2_{70Y+30S} = 70^2\sigma_Y^2 + 30^2\sigma_S^2 + 2(-1) \times 70 \times 30\sigma_Y\sigma_S$$

5.7) Now supposed that the correlation between the two stock is 1. This means if one stock go up the other will go up. What is the variance of the value of your portfolio? Is it now more or less risky than buying just S or just Y?

Answer:

$$\sigma^2_{70Y+30S} = 70^2\sigma_Y^2 + 30^2\sigma_S^2 + 2(+1) \times 70 \times 30\sigma_Y\sigma_S$$

5.8) Suppose that you now have 100 Baht and the correlation of the two stock is -1. How many Y and how many S should you buy such that you would have no risk at all? What would be the expected return in such case?

Answer: Solve for a

$$0 = (a)^2\sigma_Y^2 + (100 - a)^2\sigma_S^2 + 2(+1)(a)(1 - a)\sigma_Y\sigma_S$$

5.9) What is the minimum variance for your portfolio if the correlation of the two stocks is ρ? (Suppose you can own negative amount of stock. This is called shorting a stock.)

Answer:

$$Risk(a) = (a)^2\sigma_Y^2 + (100-a)^2\sigma_S^2 + 2\rho(a)(1-a)\sigma_Y\sigma_S$$

Differentiate with respect to a set it to zero. Then plug it back in.

Problem 6 Your computer consists of many components. Suppose that at the end of each hour a component has equal probability of failing. The probability depends on which component. Consider a computer consisting of the following components.

- One harddrive is rated at 300,000hr MTBF(Mean time between failure).
- One graphics card rated at 120,000hr MTBF.
- One power supply rated at 100,000hr MTBF.

A computer is considered to be fail if at least one of the component above fail.

6.1) Find the probability, p_h, that the harddrive will fail within an hour from now.

Answer:

$$p_h = \frac{1}{300,000}$$

6.2) Find the probability, p_g, that the graphics card will fail within an hour from now.

Answer:

$$p_g = \frac{1}{120,000}$$

6.3) Find the probability, p_s, that the power supply will fail within an hour from now.

Answer:

$$p_s = \frac{1}{100,000}$$

6.4) What is the probability that all the component will still work 5 hr from now?

Answer: The probability that the component works 1 hour from now is

$$p_w = (1-p_h)(1-p_g)(1-p_s)$$

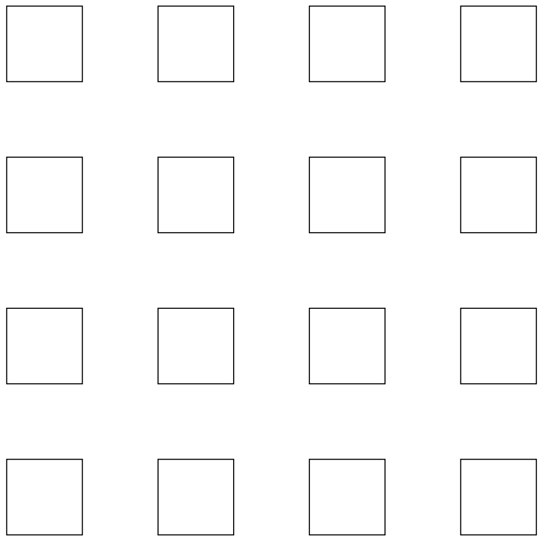
Thus, the probability that it works 5 hour from now is p_w^5

6.5) What is the MTBF of this computer?

Answer: Probability that the computer fail in 1 hour is $1-p_w$. Thus MTBF of the computer is

$$\text{MTBF for computer} = \frac{1}{1-p_w}$$

Problem 7 Given a class room where the desk are arrange in a 4×4 square.



Each desk is assigned randomly and independently to a boy or a girl. If a boy found a girl sitting right next to him (up, down left, right), then the boy will start flirting with that girl. Each boy can flirt with multiple girls next to him. For example, the boy at the corner can flirt up to 2 girls at a time. While the boy in the center can flirt up to 4 girls around him. What is the expected number of flirtation?

Hint: Split expectation into a sum.

Answer: Each edge between the desk has probability of $\frac{2}{4}$ of happening. There are 24 edges. So the expected number of flirtation is $24/2 = 12$.