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## DISCRETE MATHEMATICS

# Final Exam T3 2014

**Instruction**

- Write your name
- Read the questions carefully.
- You have 4 hours to finish the exam.
- There are 7 problems. 600 points in total. You only need to get 540 points to get full score.
- Attempt all problems, state your reasons *clearly* and *legibly*, because partial credits will be given.
- Open book. Open notes.

Question	Full Score	Your Score
1	100+20	
2	100	
3	100	
4	100	
5	50	
6	50	
7	100	

Total:

/540

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## Useful Formulas

### Sum

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \left( \frac{n(n+1)}{2} \right)^2$$

$$1 + 3 + 5 + 7 + \dots + (2n-1) = n^2$$

### Euler's Formula

$$e + 2 = v + f$$

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1. Counting. Please don't miss any of this. You do not have to get numerical value for your results.

(a) (20) How many binary strings of length 10 are there? Ex: 0011001100.

(b) (20) If we draw 6 cards from a standard deck of card (13 rank, 4 suits). How many 3 pair hands are there? Ex:  $2\spadesuit, 2\clubsuit, 3\heartsuit, 3\clubsuit, 4\heartsuit, 4\diamondsuit$ .

(c) (20) An American lottery is played by drawing 6 balls with number from 00-99. No numbers can appear twice in the list of 6 numbers and the order does *not* matter. Ex: 00, 63, 43, 12, 90, 05. How many lottery outcomes are there?

(d) (20) With the American lottery described above, we pick 3 different numbers: 12, 34, 56. If at least one of the numbers we pick is in the outcome, then we win the lottery. Guessing two numbers correctly still counts as one winning outcome. How many lottery outcomes are there that we win?

- (e) (20) Everyone in discrete math class (17 people) goes to a Somtum shop. The shop only has 5 items on the menu.

- Roasted Porkneck.
- Somtum.
- BBQ Chicken.
- Spicy Chicken Soup.
- Bamboo Shoot Salad.

If everyone orders exactly 1 item from the list, then we share the meal afterward. How many ways can we make our order? For example, we can order 5 Somtum, 5 BBQ Chicken, 7 Bamboo Shoot Salad, 0 Spicy Chicken Soup and 0 Roasted pork. This counts as one no matter who order which item.

- (f) (Bonus)(20) If you spend more than 5 minutes on this, just skip and come back later. Since Zaza is a pescetarian. She only eats vegetable and fish. So, she only has 2 choices in the menu: Somtum or Bamboo Shoot Salad, which she has to order one. With this constraint, how many ways we can make our order?

**Hint:** 5 Spicy Chicken Soup, 5 BBQ Chicken, 7 Bamboo Shoot Salad will never happen since Zaza will order either Somtum or Bamboo Shoot.

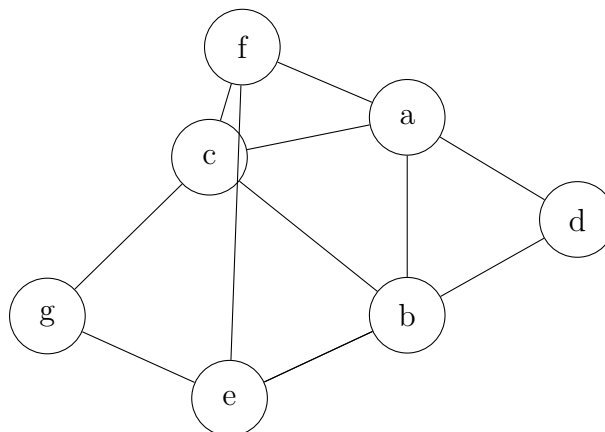
## 2. Easy Prob and Graph

(a) (20) Suppose we throw two fair dice. What is the probability that the *product* of the two dice is even? Ex: If we get 2 and 3 then the product is  $2 \times 3$  which is even.

(b) (20) If you drop a smartphone, there is a probability of 0.2 that the screen will crack. On average how many times a smartphone can be dropped before the screen cracks? (Assume that if the screen doesn't crack then it also does *not* make the screen easier to crack next time.)

(c) (20) Draw a simple graph of 5 vertex with exactly one node of degree 4. Just draw one.

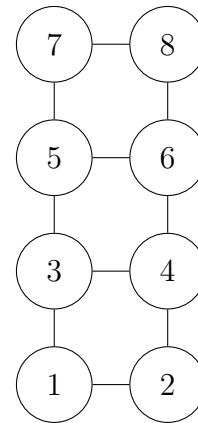
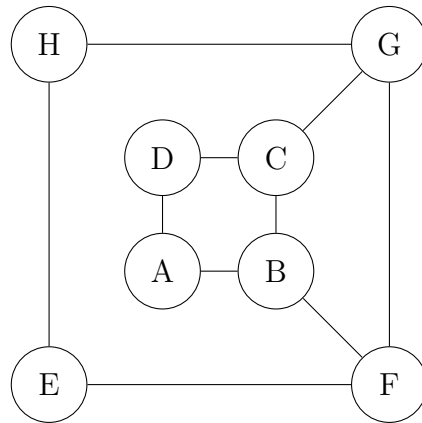
(d) (20) Euler walk. Find an Euler walk for the following graph. Label each edge with number so I can follow.



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(e) (20) Find an isomorphism between these two graphs.



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3. (100) Consider a fair coin with number 1 on one side and 2 on the other side. Let us consider a 2-player game with the following rule:

- The first player(A) throws the coin twice. The product of the two coins toss is the score for the first player. For example, if the first player gets 2 and 1 then his score would be  $2 \times 1$ .
- The second player(B) throws the coin once. The score would then be the value of the coin square. For example, if the second player gets a 2, then his score is  $2^2 = 4$ . If the second player gets a 1, then his score is  $1^2 = 1$ .
- Whoever has more score wins.

(a) (20)What is the probability that A wins? What is the probability that they will draw? What is the probability that B wins?

(b) (20)What is the probability that the A get 1 on the first coin toss *and* A wins?

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(c) (20) What is the probability that A wins given that A gets 1 on the first coin toss?

(d) (20) What is the probability that A gets 1 on the first coin toss given that A win?

(e) (20) Are the event that A gets 1 on the first coin toss and the event that A win independent?



4. (100)AJ Piti usually buy bags of candies for his class. He noticed that for a bag of yoyo.

- Most of the time there are 15 Yoyo in a bag.
- 20% of the time there are 14 Yoyo in a bag
- 20% of the time there are 16 Yoyo in a bag.

For a Sugus bag:

- Most of the time there are 30 Sugus in a bag.
- 20% of the time there are 29 Sugus in a bag.
- 20% of the time there are 31 Sugus in a bag.

If we quantify quality control( $Q$ ) of packaging by the *ratio* of the standard deviation of number of candies devided in a bag by the expected number of candies in a bag. For example, for Sugus factory the quality control factor  $Q_S$  is given by the ratio of the standard deviation of the number Sugus in a bag  $\sigma_S$  and expected number of Sugus in a bag  $\mathbb{E}[S]$ .

$$Q_S = \frac{\sigma_S}{\mathbb{E}[S]}$$

(a) (15)Find the quality control factor  $Q_S$  for Sugus factory.

(b) (15)Find the quality control factor  $Q_Y$  for Yoyo factory.

(c) (10)Which factory has better quality control?

(d) (10)A Box of Sugus has 30 bags. What is expected number of Sugus candies in a box?

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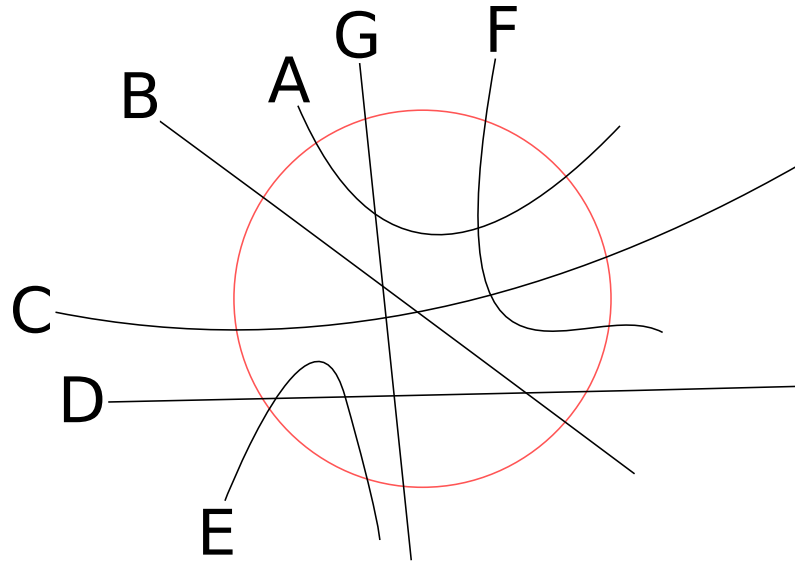
(e) (10) A Box of Yoyo has 30 bags. What is expected number of Yoyo candies in a box?

(f) (10) What is the *standard deviation*,  $\sigma_{S,Box}$ , for the number of Sugus candies in a box assuming that the numbers of candies in each bag are independent.

(g) (10) Yoyo factory however pack the box such that all bags in a box have exactly the same amount of candies. That is either all 14, all 15 or all 16. What is the *standard deviation*  $\sigma_{Y,Box}$  for number of yoyo in a box?

(h) (10) Which factory has better strategy for putting packages in a box? Explain.

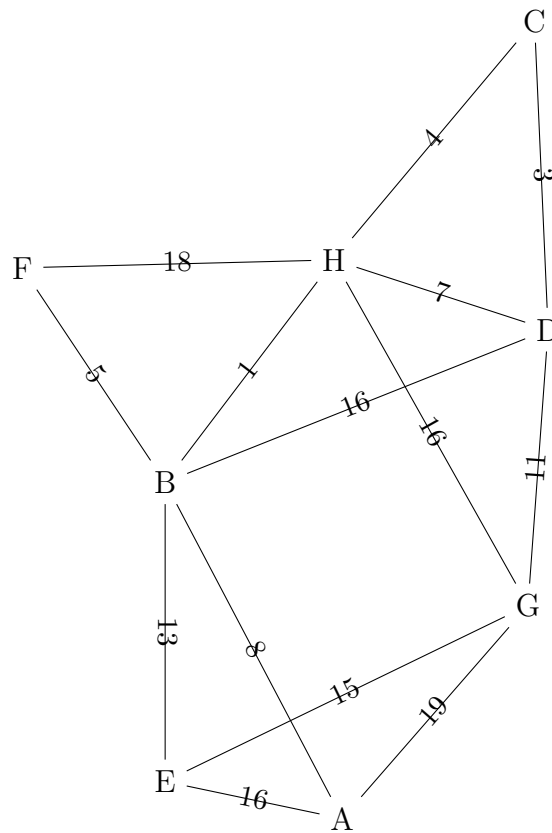
5. (50) Consider the following messy road intersection. You want to let multiple lanes go at the same time. However, you cannot let two lanes run at the same time if they intersect inside the intersection since they will crash. Use graph coloring to solve this problem. What do nodes, edges and colors represent? How would you schedule the traffic light(which lanes go together)?



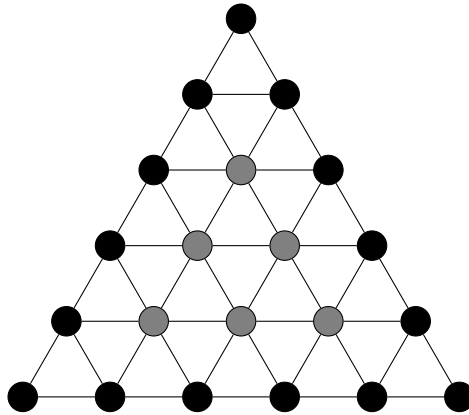
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6. (50) You are in charge of a cable company. You want to figure out a way to connect everyone to the internet with minimum cost. The cost for connecting 2 regions are shown in the following graph. Find the way to connect everyone with the minimum cost.

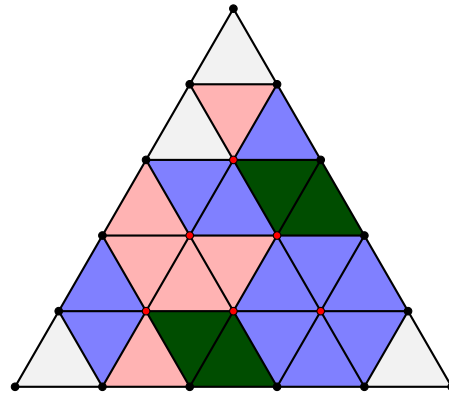


7. (100) For this problem if you need to borrow an answer from previous subquestion which you can't do. Just write it down as variable. You also don't need to simplify your result. Consider a triangular grid with  $n$  level. For example, the figure shown below is of  $n = 6$  level.

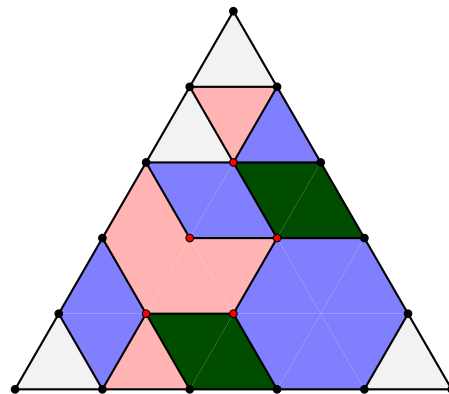


- (a) (10) How many vertex are there for  $n$ -level triangular grid ( $v_{\text{total}}$ )? (Both black and gray dots)
- (b) (10) How many vertex are there outside for  $n$ -level triangular grid ( $v_{\text{out}}$ )? (Just the black dots)
- (c) (10) How many vertex are there inside  $n$ -level triangular grid ( $v_{\text{in}}$ )? (Just the gray dots)
- (d) (10) How many edges are there around the boundary of the triangle ( $e_{\text{out}}$ )? We will call this outside edges. Give your answer in terms of  $n$ .
- (e) (10) How many edges are there inside the triangle ( $e_{\text{in}}$ )? Give your answer in terms of  $n$ .

- (f) (10) How many unit triangles are there? Unit triangle is defined as a triangle that are formed by three vertices that are all adjacent to each other. Give your answer in terms of  $n$ . You may find one of the sum on the formula page useful
- (g) (10) Now we start coloring each unit triangle independently with  $c$  color. Each grid has equal probability of being any color. For example, if  $c = 4$  the grid may look like the following.



Our goal is to draw a that only has edges and vertex that form the boundary. An example of such graph is shown below.



First, let us consider an inside edge. We want to remove an inside edge if the two colors on both side are the same. For example, if two side of an edge are both red. What is the probability of an inside edge being removed? Give your answer in terms of  $c$ .

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- (h) (10) Find the expected number of edge we will have to remove and what is the expected number of edge we have left? Do not forget to count the outside edges which we never remove. Give your answer in terms of  $n$  and  $c$ .
- (i) (10) Now we are going to remove inside vertices too if all 6 triangles around a vertex have the same color. What is the probability of an inside vertex being removed? Give your answer in terms of  $c$ .
- (j) (10) What is the expected number of vertex we will have to remove? How many vertex will we have left? Give your answer in terms of  $n$  and  $c$ .