ID: ____________________ Name: ____________________________________

Discrete Mathematics

Midterm Exam T3 2014

Instruction

- Write your name
- Read the questions carefully.
- You have 4 hours to finish the exam.
- There are 6 problems. Each problem worths 100 points. 600 points in total. You only need to get 540 points to get full score.
- Attempt all problems, state your reasons *clearly* and *legibly*, because partial credits will be given.
- 2 A4 formula sheet is allowed.

Question	Full Score	Your Score
1	100	
2	100	
3	100	
4	100	
5	100	
6	100	
Bonus	30	

Total: ______ /540

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Useful Formula and Definitions

Asymptotics

Definiton	Definition			Intuition
Asym. Equal	$f \sim g$	iff	$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$	$f \underbrace{\equiv}_{x\to\infty} g$
Big Oh	$f \in O(g)$	iff	$\lim_{x\to\infty} \frac{f(x)}{g(x)} < \infty$	$f \underbrace{\leq}_{x\to\infty} g$
Little Oh	$f \in o(g)$	iff	$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$	$f \underbrace{<}_{x\to\infty} g$
Little Omega	$f \in \omega(g)$	iff	$\lim_{x\to\infty} \frac{f(x)}{g(x)} \to \infty$	$f \underbrace{>}_{x\to\infty} g$
Big Omega	$f \in \Omega(g)$	iff	$\lim_{x\to\infty} \frac{f(x)}{g(x)} > 0$	$f \underbrace{\geq}_{x\to\infty} g$
Theta	$f \in \Theta(g)$	iff	$\lim_{x\to\infty} \frac{f(x)}{g(x)} = c, c \neq 0$	$f \underbrace{=}_{x\to\infty} g$

Sum

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$
$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$
$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Integral

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} \qquad \text{if } n \neq -1$$
$$\int \frac{1}{x} \, dx = \ln(x)$$

Quadratic

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1. Easy stuff(100 points. 20 each.)

   (a) Draw the truth table for $P \wedge (\neg Q \vee P)$

   (b) Which of these asymptotic symbols ($\{\sim, o, O, \omega, \Omega, \Theta\}$) are applicable for

   $$f(x) = \underline{\quad}\big(g(x)\big)$$

      i. $f(x) = 2x^2 + 3x + 1, g(x) = x^2$

      ii. $f(x) = 2^x, g(x) = 3^x$

   (c) Find the closed form formula for the following sum/product:

      i. $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{m}(i+j)^2$

      ii. $\displaystyle\prod_{i=1}^{n}\sum_{j=1}^{m} ij$

(d) Prove or Disprove the following proposition.

Sum of **any** 5 *consecutive* integers is divisible by 3.

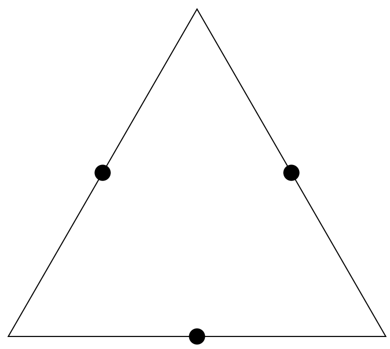
(e) Use integral bound to find the *upperbound* for the following sum.

$$\sum_{x=1}^{n} x^5$$

2. (100 points)Consider an equilateral triangle whose all sides are of length 1. Suppose that we pick 5 points within the triangle, there is at least a pair of point such that distance between the two less than or equal to 1/2.

   Hint: Connect the dots. You may use the fact that all sides of the triangle are of the same length if an only if the three angles are all 60° .

   Hint: You may also use the fact that the angle at the base of isoceles triangle(triangle whose two side are equal) are equal.



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3. (100 points)Pick one. Indicate the one you pick. If you don't I'll pick the one you got less score. Doing two won't give you more score.

   (a) Let F_n be nth fibbonacci number defined by.

   $$F_1 = 1, F_2 = 1, F_{n+1} = F_n + F_{n-1}$$

   Show that

   $$F_1F_2 + F_2F_3 + \cdots + F_{2n-1}F_{2n} = F_{2n}^2$$

   (b) Suppose that MUIC decides to open a new class called group programming. The class will involve splitting students into multiple group of 9 people or 4 people.

   For example, If we have 17 students, the students can be splitted in to 1 9-people group and 2 4-people group with no one left behind.

   What is the minimum number of student(p) such that class of n people $\forall n \geq p$ can be splitted into 9-people group and 4-people group with no one left behind? (Assume no one drops in the middle of the term.) Prove it too.

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4. Pick one. Indicate the one you pick. If you don't I'll pick the one you got less score. Doing two won't give you more score.

   (a) Use induction to show that $n^3 \leq 3^n$ for all $n \geq 0$. You may want to start your induction at $n = 3$. Then fill in the rest explicitly.

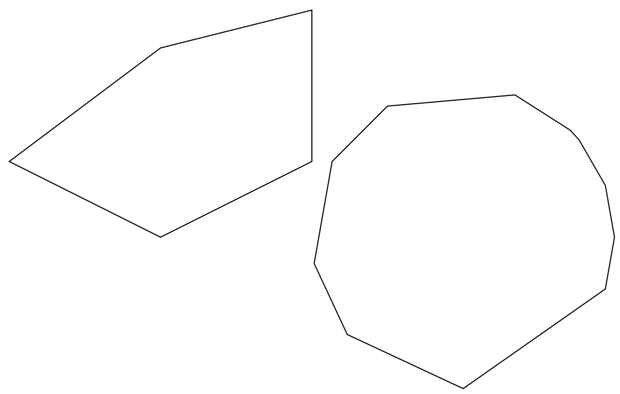
   (b) A convex polygon is a polygon whose all of the angles inside is less than 180 degree(individually).

   Show that the sum of all internal angle for n-sided ($n \geq 3$) convex polygon is

$$(n-2) \times 180^\circ.$$

   For example, sum of all internal angle of a triangle(3-sided convex polygon) is $180^\circ = (3-2) \times 180^\circ$. (You may use this as a fact.) Sum of all internal angle of a rectangle(4-sided convex polygon) is $(4-2) \times 180^\circ = 360^\circ$

   Below is an example of 5 sided and 10 sided convex polygon. It has 5 sides and none of the angle is greater than 180°



Hint: Try draw a straight line connecting two corner of the polygon.

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5. (100 points)The fact that there are 3 AC and 1 remote in our class always bothers me.

   Suppose that when we enter the room, all the *three* AC are all at $23°C$ while the LCD on the remote reads $24°C$.

   Suppose that there are only 2 buttons on the remote: Up and Down. Since the remote is super short range we can select exactly one AC and press a button. Once the button is pressed *both* the selected AC and the remote will go up/down by $1°C$

   Show that we cannot make all the three AC *and* the remote agree. (Ex: 3 AC at $32°C$ temperature and the remote at $32°C$) Be sure to state the invariant.

   Hint: Add them up and play the game.

6. (100 points)Solve the following recurrence. Just find the solution. No need to verify it using induction.:

   (a) (30 points)$T(n) = T(n-1) + 2n^2; T(1) = 1$

   (b) (30 points)$T(n) = 2T\left(\frac{n}{2}\right) + 3; T(1) = 1$

(c) (40 points)$T(n) = 8T(n-1) - 15T(n-2); T(0) = 1, T(1) = 1$

7. Bonus.(30 points) No partial credit for this one. Don't bother attempting this if you aren't done with the others. You need to be a bit creative for this one.

   If we place 9 rugs(of any shape) each of which has area 1 in a room of area 5(of any shape). Prove that there are two rugs which the area overlap by at least 1/9.