Homework 4 Solution

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Problem 1 Let F_n be the n-th Fibbonacci number. That is

$$F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$$

Show that F_n is given by

$$F_n = \frac{1}{\sqrt{5}}(a^n - b^n) \text{ where } a = \frac{1+\sqrt{5}}{2} \text{ and } b = \frac{1-\sqrt{5}}{2}$$

Proof: See lecture notes for recurrence.

Problem 2 Suppose that I bring 5 Cola yoyo and 6 Grape yoyo to the class and there are 10 students. Here is the rule for deciding which color each student get. The students get in line and pick out two yoyo randomly at a time.

a) If both yoyo are grape yoyo, the student get to eat a grape yoyo and then put the other one back into the bag.

b) If both yoyo are cola yoyo, the student get to eat a cola yoyo and then put the other one back into the bag.

c) If the yoyo are of different color, the student get to eat grape yoyo and put the cola one back into the bag.

The last yoyo in the bag belong to AJ. Of course, I plan this deliberately to get the flavor I want. Which flavor of yoyo do I want? Prove that the rule always get me that flavor.

Invariant: There is at least 1 cola left in the bag.

Def: Let us define the state with the number of yoyo left in the bag (C, G) where there C is the number of cola in the bag and G is the number of grape left in the bag.

Theorem: If we start in the state (C_n, G_n) where $C_n \geq 1$, then after the yoyo picking the yoyo left in the bag (C_{n+1}, G_{n+1}) will also satisfy $C_{n+1} \geq 1$.

Proof: Assume that we start in the state (C_n, G_n) where $C_n \geq 1$. There are three things you can do

2.1) Pick 2 grapes return one. $(C_n, G_n) \to (C_n, G_n - 1)$. This means $C_{n+1} = C_n$. Since $C_n \geq 1$, $C_{n+1} \geq 1$.

2.2) Pick 1 cola 1 grape return cola. $(C_n, G_n) \to (C_n, G_n - 1)$. This means $C_{n+1} = C_n$. Since $C_n \geq 1$, $C_{n+1} \geq 1$.

2.3) Pick 2 cola return cola. This is only possible if $C_n \geq 2$, otherwise you can't pick two cola. $(C_n, G_n) \to (C_n - 1, G_n)$. This means $C_{n+1} = C_n - 1$. Since $C_n \geq 2$, $C_{n+1} \geq 1$.

□

So, there is always 1 cola left in the bag. That means the last yoyo must be cola yoyo.

Problem 3 Consider the numbers 1,2,3,4,5,6. The game is played by picking any two numbers from the list (a,b). Then, we replace the two number with $|a-b|$. For example, if you pick 4 and 3, you erase the number 4 and 3 then add another 1 to the list(which means you will have 1,2,5,6,1). The process is then repeat until there is one number left. Prove that the last number is odd no matter how you pick it.

Invariant: The sum of all number is odd.

Theorem: If we start with in a state $[a_1, a_2, \dots, a_n]$ where the sum of all numbers is odd then after picking the number and put back, the sum of the number $[b_1, b_2, \dots, b_n]$ is odd.

Proof: Assume that we start in a state where the sum of the number in the list is odd. There are three transition we can do

3.1) If we pick two odd number, this means we put back an even number. So the new sum

$$\text{new sum} = \text{old sum} - \text{odd} - \text{odd} + \text{even}$$

Since the old sum is odd by the hypothesis, the

$$\text{new sum} = \text{odd} - \text{odd} - \text{odd} + \text{even} = \text{odd}$$

3.2) If we pick two even number, this means we put back an even number. So the new sum

$$\text{new sum} = \text{old sum} - \text{even} - \text{even} + \text{even}$$

Since the old sum is odd by the hypothesis, the

$$\text{new sum} = \text{odd} - \text{even} - \text{even} + \text{even} = \text{odd}$$

3.3) If we pick one odd number and one even number, this means we put back an odd number. So the new sum

$$\text{new sum} = \text{old sum} - \text{odd} - \text{even} + \text{odd}$$

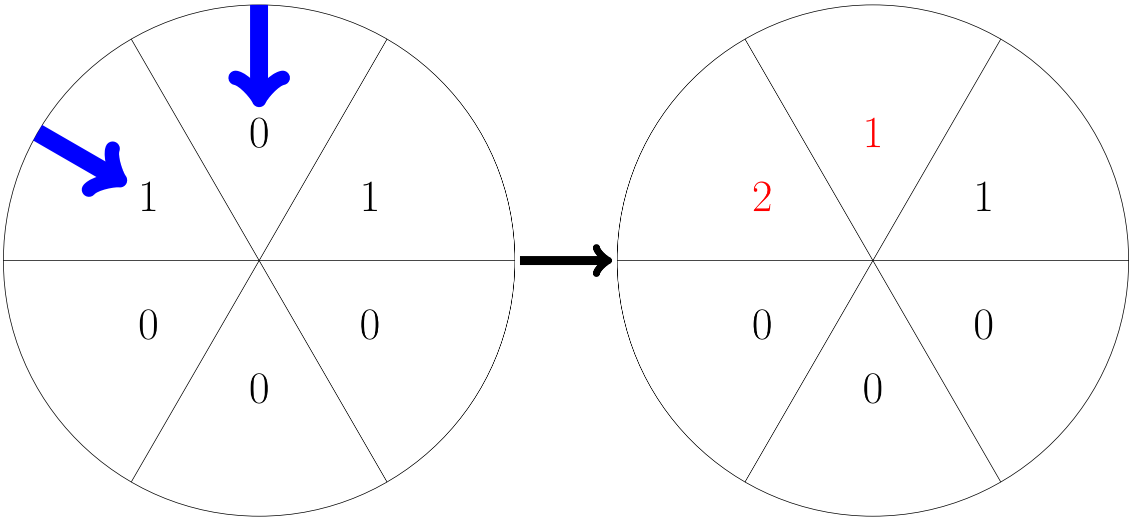
Since the old sum is odd by the hypothesis, the

$$\text{new sum} = \text{odd} - \text{odd} - \text{even} + \text{odd} = \text{odd}$$

So the sum of the list is odd is a preserved invairant. □

Since the sume of all of number in the list is odd, the last number is odd otherwise the sum of all number in the list is even.

Problem 4 Let us play a game. A circle is divided into 6 sectors. The number 1,0,1,0,0,0 are written on each sector. You may pick two neighboring sectors and increase the number on both of them by 1. Can you make all the number in the circle equal? (If you want a hint, find me with at least 6 tries at the game.)



Theorem: Let the numbers on the sector on the circle(start somewhere then go clockwise) be $a_1, a_2, a_3, \ldots, a_6$. Then,

$$S = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 = 2$$

is a preserved invariant.

Proof: We need to show that we start in a state where $S_n = 2$ then after a move then we end up in a state where $S_{n+1} = 2$.

For all pairs of sectors we pick, we add one to S and subtract one from S. This means

$$S_{n+1} = S_n + 1 - 1 = S_n = 2$$

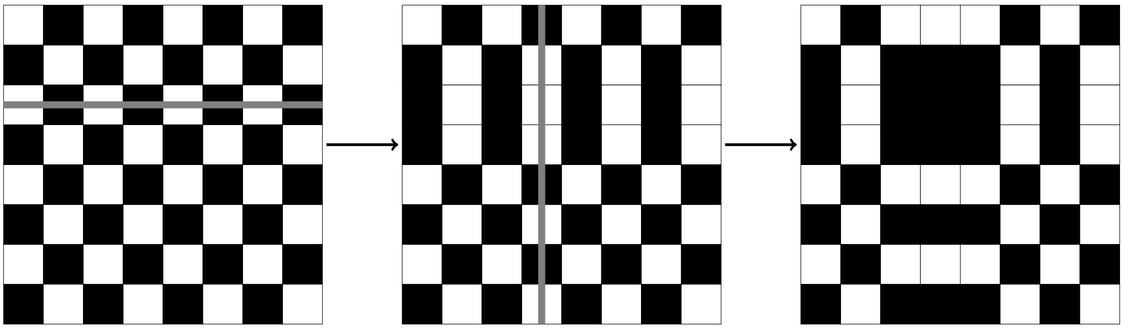
So,$a_1 - a_2 + a_3 - a_4 + a_5 - a_6 = 2$ is a preserved invariant.

□

Since we start in a state where $S = 2$, and $S = 2$ is a preserved invariant, the state where all the number are equal($S = 0$) is not reachable by any series of moves.

Problem 5 Consider an 8×8 chessboard with standard coloring. For each move you can switch all the color of *one row* or *one column* at a time. You can then repeat this as many times as you like and wherever you like. Can you reach the board with exacly 1 black square left?

Hint: Play it on 4×4 grid with the playing cards lying around in 1409.



If you are up for a challenge, look at the bonus problem of the previous midterm. Only Majeed got it last term.

Theorem: Let (B, W) be the number of black squares and white square on the board. Then, if we start in the state (B_n, W_n) where B is even then after a move then we will end up in a state (B_{n+1}, W_{n+1}) where B_{n+1} is even.

Proof: Let us assume that we start in a state where the number of black is even. There are actually two cases for the moves we can do.

5.1) If the row/colum we pick has even number of black

- The selected row must have even number of white since the number of black and white in the row/column we pick must add up to 8.
- After we flip the color in the row/col we pick the total number black in the board will become

$$B_{n+1} = B_n - even + even$$

- From the hypothesis, we start in the state where B_n is even.

$$B_{n+1} = even - even + even = even.$$

5.2) If the row/colum we pick has odd number of black

- The selected row must have odd number of white since the number of black and white in the row/column we pick must add up to 8.
- After we flip the color in the row/col we pick the total number black in the board will become

$$B_{n+1} = B_n - odd + odd$$

- From the hypothesis, we start in the state where B_n is even.

$$B_{n+1} = even - odd + odd = even.$$

So, the total number of black on the board is even is a preserved invaraint. □

Since we start in the state where there are even number of black(64) and the total number of black on the board is even is a preserved invaraint. We cannot reach the state where the number of black is 1, which is odd.

Problem 6 Consider an island with 3 types of pokemon: blue pokemons, red pokemons and green pokemons.

a) At the beginning, there are 13 blue pokemons, 15 red pokemons, 17 green pokemons.

b) Whenever two pokemons meet, if the two pokemon are of different color, they will both transformed to the other color. For example, if a blue pokemon and a red pokemon meets, they both transform to a green pokemon resulting in 1 less blue pokemon, 1 less red pokemon and 2 more green pokemons.

c) Those that already transformed can meet and transform again.

Show that it is impossible for the island to be left with exactly one type of pokemon(ex: all blue) with any series of meeting and transformation.

Hint: Modulo 3

Theorem: Let (a, b, c) be the modulo 3 of the number blue, green, red pokemon accordingly. Then, if we start in a state (a_n, b_n, c_n) where a, b, c are distinct, then after a move we will end up in a state $(a_{n+1}, b_{n+1}, c_{n+1})$ where a, b, c are distinct.

Proof: Without loss of generality, let us start in a state where the modulo 3 of the three color are $(0, 1, 2)$. There are three cases:

6.1) Have the color that was of modulo 1 meet the color of modulo 0, this adds the color that was of modulo 2 by 2. This means the new modulo is

$$(0, 1, 2) \rightarrow (2, 0, 1)$$

6.2) Have the color that was of modulo 1 meet the color of modulo 2, this adds the color that was of modulo 0 by 2. This means the new modulo is

$$(0, 1, 2) \rightarrow (2, 0, 1)$$

6.3) Have the color that was of modulo 2 meet the color of modulo 0, this adds the color that was of modulo 1 by 2. This means the new modulo is

$$(0, 1, 2) \rightarrow (2, 0, 1)$$

This means the modulo 3 of all colors are distinct is a preserved invariant. $\square$.

Since we start in a state where the modulo 3 of all colors are distinct and the fact that the modulo 3 of all color are distinct is a preserved invraint. We cannot reach the state where there is 1 color left since in that state the modulo of the three color are all 0.