Homework 2

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Problem 1 Use induction to prove the following proposition:

$$1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \;\forall n \in I^+$$

Problem 2 Use induction to prove the following proposition:

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2; \forall n \in I^+$$

Problem 3 Fibonacci numbers is defined as

$$F_0 = 1, F_1 = 1, F_2 = 2, F_{n+1} = F_n + F_{n-1}$$

The first few terms are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

Prove the following identity:

$$F_2 + F_4 + F_6 + \ldots + F_{2n} = F_{2n+1} - 1$$

Problem 4 **Def:** a is *divisible* by b if and only if $\exists n \in I$ such that $a = nb$. We also denotes this by $b|a$, which reads b divides a.

For example, 15 is divisible by 5 since $15 = 3 \times 5$.

4.1) Show that

$$8^n - 3^n \text{ is divisible by } 5 \;\forall n \in I^+$$

4.2) General case of the above problem Given $a, b \in I$ and $n \in I^+$ use induction to show that

$$a - b | a^n - b^n$$

4.3) Use the above to show that if n is odd then

$$a + b | a^n + b^n$$

Using the above relation. This is one of those one line proof.

Problem 5 Show that we can cover $2n \times 3m$ checkerboard with L shape triminoes.

Hint: Use induction twice. Prove by induction that you can do it for $2 \times 3m, \forall m \in I^+$ first. Then use induction again on the other dimension.

Make sure whatever you prove is actually for $2n \times 3m$ $\forall n, m \geq 1$ board.

Problem 6 This problem is much harder than it looks. Use induction to show that

$$n^3 \leq 3^n \;\forall n \geq 1$$

Hint: you may wish to use induction for $n \geq 3$ first then fill in $n = 1$ and $n = 2$ manually.

Problem 7 Use induction to show that

$$\sum_{i=1}^{n} i \times 2^i = (n-1) \times 2^{n+1} + 2$$

Problem 8 Suppose you have infinite amount of green, red, black, yellow and white sock. You are pulling it out from a drawer blindly one at a time. How many socks do you need to pull to guarantee that you have at least a pair?

Problem 9 Suppose S is a set of $n+1$ integer (not necessarily consecutive). Show that there is at least one pair $a, b \in S$ such that $a - b$ is a multiple of n. (Hint: consider every element modulo n.)

Problem 10 Prove that among any 5 points in a square of 1 meter by 1 meter, there is at least a pair of points with distance $\leq \frac{\sqrt{2}}{2}$. (Hint: break the square into 4 squares.)