ID: ____________________ Name: ____________________

Discrete Mathematics

# Midterm Exam T3 2017

**Instruction**

- Write your name
- Read the questions carefully.
- You have 4 hours to finish the exam.
- There are 6 problems. Each problem worths 100 points. 600 points in total. You only need to get 540 points to get full score.
- Attempt all problems, state your reasons *clearly* and *legibly*, because partial credits will be given.

| Question | Full Score | Your Score |
|---|---|---|
| 1 | 100 | |
| 2 | 100 | |
| 3 | 100 | |
| 4 | 100 | |
| 5 | 100 | |
| 6 | 100 | |
| Bonus | 30 | |

Total: ______ /540

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# Useful Formula and Definitions

## Asymptotics

| Definiton | Definition | | | Intuition |
|---|---|---|---|---|
| Asym. Equal | $f \sim g$ | iff | $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ | $f \underbrace{\equiv}_{x\to\infty} g$ |
| Big Oh | $f \in O(g)$ | iff | $\lim_{x\to\infty} \frac{f(x)}{g(x)} < \infty$ | $f \underbrace{\leq}_{x\to\infty} g$ |
| Little Oh | $f \in o(g)$ | iff | $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$ | $f \underbrace{<}_{x\to\infty} g$ |
| Little Omega | $f \in \omega(g)$ | iff | $\lim_{x\to\infty} \frac{f(x)}{g(x)} \to \infty$ | $f \underbrace{>}_{x\to\infty} g$ |
| Big Omega | $f \in \Omega(g)$ | iff | $\lim_{x\to\infty} \frac{f(x)}{g(x)} > 0$ | $f \underbrace{\geq}_{x\to\infty} g$ |
| Theta | $f \in \Theta(g)$ | iff | $\lim_{x\to\infty} \frac{f(x)}{g(x)} = c, c \neq 0$ | $f \underbrace{=}_{x\to\infty} g$ |

## Sum

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$
$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$
$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

## Integral

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} \qquad \text{if } n \neq -1$$
$$\int \frac{1}{x} \, dx = \ln(x)$$

## Quadratic

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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1. Easy stuff(100 points. 20 each.)

   (a) Draw the truth table for $(P \to Q) \land (Q \lor P)$

   (b) Which of these asymptotic symbols ($\{\sim, o, O, \omega, \Omega, \Theta\}$) are applicable for

   $$f(x) = \_\_ \big(g(x)\big)$$

      i. $f(x) = (x)^2, g(x) = 2x^2 + 1$

      ii. $f(x) = n \log n, g(x) = n^2 \log n$

   (c) Find the closed form formula for the following sum/product:

      i. $\displaystyle\sum_{i=1}^{n} \sum_{j=1}^{m} (i + j)$

      ii. $\displaystyle\prod_{i=1}^{n} \prod_{j=1}^{m} 2^{i-j}$

(d) **Disprove** the following proposition.

Product of irrational numbers is irrational.

(e) Use integral bound to find the *upperbound* for the following sum. You may leave your answer as integral. But make sure you specify the limits.

$$\sum_{x=1}^{n} \sqrt[4]{x}$$

2. Easy Proof.

   (a) (50 points) Show that for $x \in R$. $|x+2| - |x-3| \leq 5$.

   (b) (50 points) Show that if we pick 6 distinct numbers from $\{1, 2, 3, \ldots, 10\}$. There is at least one pair that adds up to 11.

   Hint: 11 = 1+10 = 2+9 = 3+8 = 4+7 = 5+6

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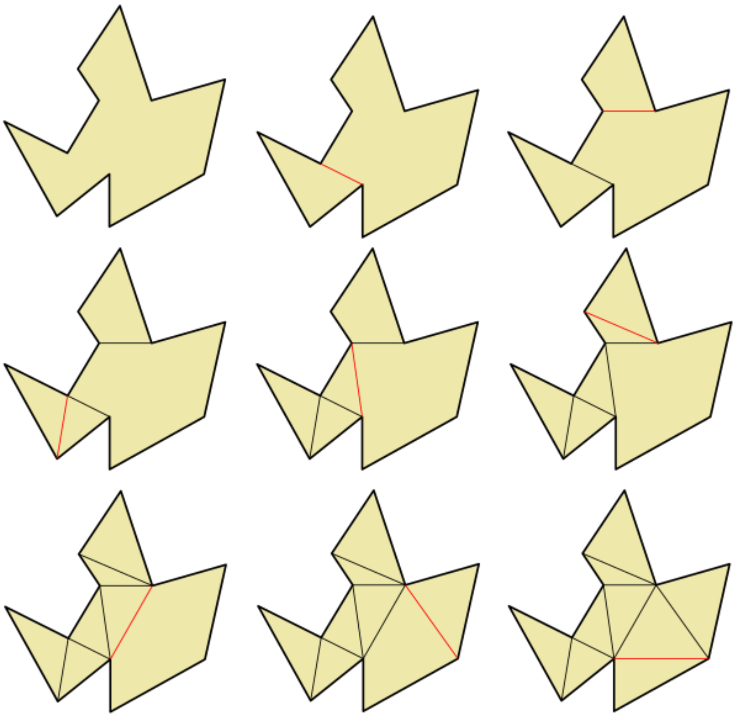
3. (100 points)Pick one. Indicate the one you pick. If you don't, I'll pick the one you got less score. Doing two won't give you more score. Be sure to indicate where you use inductive hypothesis.

   (a) 3D Chocolate. In the class we found some fact about the number of chocolate pieces and the number of splits needed to make them all singles. Lets us consider a 3 dimension chocolate of dimension $10 \times 20$. How many splits do we need? Find a general formula. Prove it by induction and apply it to your problem.

   (b) Triangulation.

   Triangulation is a process that divide a polygon into multiple triangles. It is very important basis for computer graphics processing.



   Let us consider a simple polygon(no hole) of $n$ corners. After triangulation how many triangles would we end up with?

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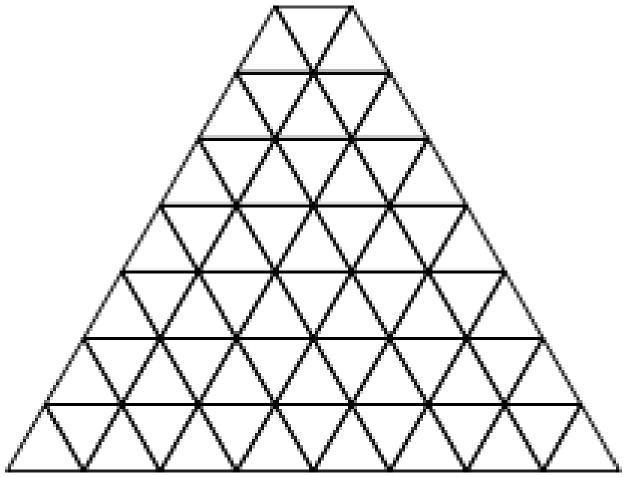
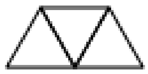
4. Pick one. Indicate the one you pick. If you don't, I'll pick the one you got less score. Doing two won't give you more score.

   (a) Consider triangular grid where the length of each side is $2^n$ where $n \in I+$ with the top cut off[1]. Show that the triangular grid can be fully tiled with 3 triangle piece shown below.

   Hint: With the right placement you can cut off 1 corner from each 3 triangle in one go.

   (b) Lucas Number. Make sure you write down the exact resoning for each step.
   Recall Fibbonacci number:

$$F_{n+2} = F_n + F_{n+1} \tag{1}$$

   with $F_0 = 0, F_1 = 1$
   Lucas number is exactly the same recurrence but with different initial condition.

$$L_{n+2} = L_n + L_{n+1} \tag{2}$$

   with $L_0 = 2, L_1 = 1$
   Show that

$$L_{n-1} + L_{n+1} = 5F_n \tag{3}$$

[1] $2^{n-1}$ if you discount the top that is cut off

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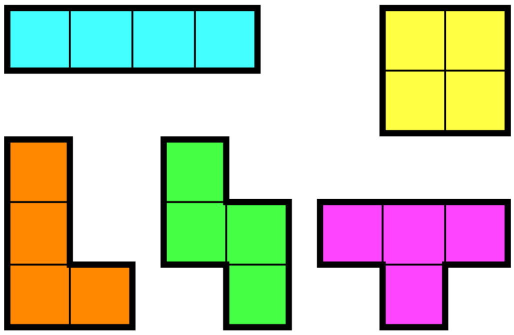
5. Tetris is a falling block puzzle game in which many students got addicted to.

   The game board is 20 cells high[2] and 10 cells wide. The player is given tetromino(shown below) one at a time. The player can then rotate and place it at the bottom of the board.



   A row is cleared if the player is able to fill out the all 10 cells in a row.

   Let us consider a common strategy which is to place tetrimino in 9 consecutive column.

   Can we set up(place down pieces + clear lines) tetrominoes such that it forms a $9 \times 3$ rectangle at the bottom? Prove it.

   Hint: consider all actions that change the number of filled cells on the board.

[2]There are actually 22 but the top two is hidden which is why sometimes rotating I piece at the very top got kicked down and screw your setup.

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6. (100 points)Solve the following recurrence. Just find the solution. No need to verify it using induction.:

   (a) (30 points)$T(n) = T(n-1) + 3n^2; T(1) = 2$ You may find a formula on the front page useful.

   (b) (30 points)$T(n) = T\left(\frac{n}{2}\right) + n \quad \text{where } T(1) = 1$

(c) (40 points)$T(n) = 9T(n-1) - 20T(n-2); T(0) = 3, T(1) = 13$

7. Bonus.(30 points) No partial credit for this one. This is just to keep you sit there for a while. Prove the following:

   Consider a very large rectangular grid. Each intersection is either red green or blue. Show that we can always find a right angle rectangle where all the 4 corners are of the same color. (Taken from USA Mathematical Talent Search problem given to children age around 12.)

   Hint: 81