ID: ____________________ Name: ____________________________________

DISCRETE MATHEMATICS

Midterm Exam T2 2015

Instruction

- Write your name
- Read the questions carefully.
- You have 4 hours to finish the exam.
- There are 6 problems. Each problem worths 100 points. 600 points in total. You only need to get 540 points to get full score.
- Attempt all problems, state your reasons *clearly* and *legibly*, because partial credits will be given.

Question	Full Score	Your Score
1	100	
2	100	
3	100	
4	100	
5	100	
6	100	
Bonus	30	

Total: ______ /540

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Useful Formula and Definitions

Asymptotics

Definiton	Definition			Intuition
Asym. Equal	$f \sim g$	iff	$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 1$	$f \underbrace{\equiv}_{x\to\infty} g$
Big Oh	$f \in O(g)$	iff	$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} < \infty$	$f \underbrace{\leq}_{x\to\infty} g$
Little Oh	$f \in o(g)$	iff	$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 0$	$f \underbrace{<}_{x\to\infty} g$
Little Omega	$f \in \omega(g)$	iff	$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} \to \infty$	$f \underbrace{>}_{x\to\infty} g$
Big Omega	$f \in \Omega(g)$	iff	$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} > 0$	$f \underbrace{\geq}_{x\to\infty} g$
Theta	$f \in \Theta(g)$	iff	$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = c, c \neq 0$	$f \underbrace{=}_{x\to\infty} g$

Sum

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$
$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$
$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Integral

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} \qquad \text{if } n \neq -1$$
$$\int \frac{1}{x} \, dx = \ln(x)$$

Quadratic

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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1. Easy stuff(100 points. 20 each.)

   (a) Draw the truth table for $P \vee (\neg Q \wedge P)$

   (b) Which of these asymptotic symbols ($\{\sim, o, O, \omega, \Omega, \Theta\}$) are applicable for

   $$f(x) = \_\_\big(g(x)\big)$$

      i. $f(x) = x^3, g(x) = x^2 + 10^{100}$

      ii. $f(x) = 5^x, g(x) = 3^x$

   (c) Find the closed form formula for the following sum/product:

      i. $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{m}(i+j)(i-j)$

      ii. $\displaystyle\prod_{i=1}^{n}\sum_{j=1}^{m} 2^i 3^j$

(d) Prove or Disprove the following proposition.

If a and b are rational then a^b is rational.

(e) Use integral bound to find the *lowerbound* for the following sum.

$$\sum_{x=1}^{n} \frac{1}{x^2}$$

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2. Easy Proof.

   (a) (50 points) Let us define

   $$n\%2 = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

   Show that

   $$(a\%2 + b\%2)\%2 = (a + b)\%2$$

   for all integer a and b.

   (b) (50 points) If x is irrational then $\sqrt{x}$ is irrational.

3. (100 points)Pick one. Indicate the one you pick. If you don't, I'll pick the one you got less score. Doing two won't give you more score.

   (a) Let F_n be fibbonacci number such that $F_1 = 1, F_2 = 1$ and

   $$F_{n+1} = F_n + F_{n-1}$$

   Show that

   $$1F_1 + 2F_2 + 3F_3 + 4F_4 + \ldots nF_n = nF_{n+2} - F_{n+3} + 2$$

   (b) Let $m! = m \times (m-1) \times (m-2) \times \ldots \times 1$ (ex: $5! = 5 \times 4 \times 3 \times 2 \times 1$)
   Show that

   $$1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + n \times n! = (n+1)! - 1$$

   for all $n \geq 1$

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4. Pick one. Indicate the one you pick. If you don't, I'll pick the one you got less score. Doing two won't give you more score.

   (a) During ninja war. MUIC empire controls **two** castles: Castle of Discrete and Castle of Data. The two castles have *equal* number of guards (n each) in the beginning. Two famous ninja: Ninja Wit and Ninja Taro, are tasked to destroy MUIC empire. Since the guards are so easy to kill, the two ninjas came up with a competition to make the job more fun.

   The two ninjas decided to take turn. *Ninja Wit starts first.*

   For each night/turn,

   i. The ninja on that turn decide which non-empty castle to go. (Ninja doesn't waste time going to empty castle.)
   ii. The ninja must kill *one or more* guard in that castle on that night. (The ninja cannot kill anyone in another castle)

   The winning ninja is the one who **kill the last guard**.

   Let us look at an example: Suppose that each castle has 1 guard.

   (# of guard in Castle of Discrete, # of guard in Castle of Data) = (1,1)

   i. On the first night, Ninja Wit go to Castle of Discrete and he has only one choice: kill the 1 guard in that castle. This means that the number of guards will become (0, 1)
   ii. On the second night, Ninja Taro go to Castle of Data and kill that last guard and win the competition.

   Show that Ninja Taro, starting second, always has a strategy that guarantees his win (no matter what Ninja Wit does) for all number of guards at the beginning in each castle, n.

   Hint: If each castle has 2 guards and Ninja Wit kill 1 guard in one castle, leaving Ninja Taro with (1,2). What should Ninja Taro do?

   Hint: If each castle has 5 guards and Ninja Wit kill 3 guards in one castle, leaving Ninja Taro with (2,5). What should Ninja Taro do?

   Hint: Use the example as base case and do induction on the number of guards in each castle. Ninja Taro should always try to make sure he go back to smaller problem which he knows how to win for sure.

   (b) Show that $7^{n+2} + 8^{2n+1}$ is divisible by 57. (Less fun though and I think the the top problem is easier.)

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5. (100 points)Consider the following grid of 4×4 with numbers on it. Each turn you can pick a row or a column and add 1 or subtract 1 to/from every cell in that row or column. Can you make all the numbers the same? If it's not possible, prove that it is not possible.

   Hint: Circle problem.

13	14	15	16
9	10	11	12
5	6	7	8
0	2	3	4

6. (100 points)Solve the following recurrence. Just find the solution. No need to verify it using induction.:

   (a) (30 points)$T(n) = T(n-1) + 3n^3; T(1) = 1$

   (b) (30 points)$T(n) = 2T\left(\frac{n}{2}\right) + 2n; T(1) = 1$

(c) (40 points)$T(n) = T(n-1) + 2T(n-2); T(0) = 8, T(1) = 1$

7. Bonus.(30 points) No partial credit for this one. Don't bother attempting this if you aren't done with the others. You will need to be a bit creative for this one.

   Consider number 1 to 10 written in a circle(in any order). Show that there is a three consecutive number that adds up to at least 17 no matter how you order them around the circle. (There might be more than 1 3 consecutive number).

   Ex: $1, 3, 5, 7, 9, 2, 4, 6, 8, 10 \ldots 1$. Then, each of (5,9,7), (7,9,2), (4,6,8), (6,8,10), (8,10,1) triplet adds up to 17 or more.