Discrete Maths - Homework 1 Answers

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Question 1

Assume that both x and y are perfect squares. This means that we can say

$$x = m^2, y = n^2 \quad \exists m, n \in \mathbb{Z}.$$

This means that we can write xy as

$$xy = m^2 n^2 = (mn)^2.$$

Since m and n are positive integers, so will the product of the two. This means we are able to write xy as a square of the integer mn, hence must be a perfect square. Boom.

Question 2

We will prove this problem by contradiction. Assume that $a > \sqrt{n}$ and $b > \sqrt{n}$. This means that $ab > n$, which means that under our assumption, $ab \neq n$, which does not match what we are trying to prove. This means that if $ab = n^2$, one of our assumption must bee false, or that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$. Boom.

Question 3

For a number to be rational, it must be able to be written in the form $x = a/b$ where $a, b \in \mathbb{Z}$. So we will assume that

$$x = \frac{a}{b}, y = \frac{c}{d} \quad a, b, c, d \in \mathbb{Z}.$$

This means that

$$x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

Since both $ad + bc$ and bd are both integers, it means that $x + y$ is a rational number since it is an integer divided by another integer. Boom.

Question 4

A number x is irrational if

$$x \neq \frac{a}{b} \quad \forall a, b \in \mathbb{Z}.$$

So assume x is irrational by the definition above. Then $1/x$ is given by

$$\frac{1}{x} \neq \frac{b}{a} \quad \forall a, b \in \mathbb{Z}.$$

This means that you cannot write $1/x$ as an integer divided by another integer. This therefore means that $1/x$ is an irrational number as well. Boom.

Question 5

Assume that the number of pairs with one red card and one black card that will appear is p. Also assume that the number of pairs with only red cards and only black cards are r and b respectively.

In a standard deck of cards, there are 26 black cards and 26 red cards. The 26 black cards will either have been dealt out with the red-black pair (where one black card would have been dealt per such pair), or dealt out with the black-black pair (where two black cards would have been dealt out per pair). We can write the expression for the distribution of black cards as

$$26 = 2b + p.$$

Similarly, the 26 red cards will either have been dealt out with the red-black pair (where one red card would have been dealt per such pair), or dealt out with the red-red pair (where two red cards would have been dealt out per pair). We can write a similar expression for the distribution of red cards as

$$26 = 2r + p.$$

If we rearrange the above two equations we will get

$$2r + p = 2b + p \quad \Rightarrow \quad b = r$$

which means that there will be the same number of red-red pairs and black-black pairs. Boom.

Question 6

We have to prove that

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|} \quad \forall a, b \in \mathbb{R}, b \neq 0.$$

We will consider four cases of the possible value of a and b.

- For $a \geq 0, b > 0$: $\left|\frac{a}{b}\right| = \frac{a}{b}, \frac{|a|}{|b|} = \frac{a}{b}$
- For $a \geq 0, b < 0$: $\left|\frac{a}{b}\right| = -\frac{a}{b}, \frac{|a|}{|b|} = \frac{a}{-b} = -\frac{a}{b}$
- For $a \leq 0, b > 0$: $\left|\frac{a}{b}\right| = -\frac{a}{b}, \frac{|a|}{|b|} = \frac{-a}{b} = -\frac{a}{b}$
- For $a \leq 0, b < 0$: $\left|\frac{a}{b}\right| = \frac{a}{b}, \frac{|a|}{|b|} = \frac{-a}{-b} = \frac{a}{b}$

Since what we want to prove is true for all possible cases, we can be sure that the statement must be true. Boom.

Question 7

Q 7.1

Contrapositive: if $r^{\frac{1}{5}}$ is rational, then r is rational.

To prove the contrapositive, we will assume that $r^{\frac{1}{5}}$ is rational, and can be written in the form of a/b, where a and b are integers. This means that

$$r = \frac{a^5}{b^5}$$

which shows that r is still in the form of an integer divided by another integer. This means that r will also be rational, and hence the contrapositive is true. This will then imply that the original statement is also true. Boom.

Q 7.2

The contrapositive to the statement given is that "if $r^{\frac{1}{5}}$ is irrational, then r is irrational".

We can disprove the contrapositive with a counterexample. First we have to establish that $2^{\frac{1}{5}}$ is irrational. We can show this by contradiction, by assuming it is rational and can be written in the form of a/b, where a and b are coprimes (by definition of rational numbers). This would also mean that a and b cannot both be even numbers. In equation, this becomes

$$2^{\frac{1}{5}} = \frac{a}{b} \quad \Rightarrow \quad 2 = \frac{a^5}{b^5} \quad \Rightarrow \quad 2b^5 = a^5.$$

This means that a^5 must be even, hence a is even. So let $a = 2c$. Now we have the equation $2b^5 = 32c^5$ or $b^5 = 16c^5$. This would imply b^5 is even, hence b is even. But we have said that both a and b cannot be even, hence a contradiction. This means $2^{\frac{1}{5}}$ is irrational. But this proof isn't over.

When $r^{\frac{1}{5}} = 2^{\frac{1}{5}}$ or an irrational number, $r = 2$, which is a rational number. This disproves the contrapositive, which means the original statement is also false. Boom.

Q 7.3

The statement here is just the contrapositive of the statement above. This has been proven in the previous question. Boom.

Question 8

We will prove this by contradiction.

Assume that everyone has made a profit. Meaning that all of their current money will be more than what they start with, or that it equals to $(1 \times 10^6) + \Delta x_i$ baht, where Δx_i is the extra money gained by player i. Because all of the players have made a profit, they must all have a positive difference in money, or $\Delta x_i > 0$ for all i.

The total amount of money after 20,000 rounds, T, will be given by

$$T = \sum_i [(1 \times 10^6) + \Delta x_i] = (1 \times 10^{10}) + \sum_i \Delta x_i$$

Since all the money that was gambled came from the ten thousand people playing, all the money flow will only be amongst the players. There should be no money that leaves or enters the system after whichever round. The amount of money at the end therefore should be equal to the amount of money that was in the system to begin with. The initial amount of money at the beginning is 1×10^{10} baht, meaning that $T = 1 \times 10^{10}$.

We can equate the two expressions for T, and get

$$(1 \times 10^{10}) + \sum_i \Delta x_i = 1 \times 10^{10},$$

$$\sum_i \Delta x_i = 0.$$

Given that $\Delta x_i > 0$, this will not be possible since for the sum to be possible, either $\Delta x_i = 0$ for all i, or $\Delta x_i < 0$ for some i. This means there is a contradiction, meaning that it is impossible for everyone to make a profit in the game. Boom.

Question 9

We will, again, prove this by contradiction.

Assume there is such a configuration which allows all the rugs to fit within a 5 metre-squared area.

If all the rugs are laid out without overlap, then the total area taken will be 9m^2. Since there are 9 rugs, the total number of combinations of rugs overlapping each other is $(9 \times 8)/2 = 36$. Assume that all the rugs will overlap with each other. Since the area of the overlap is restricted to be $< 1/9\text{m}^2$, the area of the overlap, L, will be given by

$$L < \left(\frac{1}{9} \times 36\right) = 4.$$

The area of the rug arrangement, A, will equal to the total area taken up by the rugs minus the amount of area saved by the overlappings. This means that the possible area of the rug arrangement is given by

$$A > 9 - L \quad \Rightarrow \quad A > 5$$

which means it is not possible to have $A = 5$ as wanted by the question. Boom.

Question 10

All odd numbers can be written in the form of $2n + 1$ where n is a positive integer. Let x be an odd number, which can be written in the form $x = 2n + 1$ for some positive integer n.

Assume that we can write x as $p^2 - q^2$ where p and q is some positive integer. Since the choice for p and q is up to us, we can let $p = q + 1$. This would mean that

$$x = (q + 1)^2 - q^2 = 2q + 1.$$

To get any odd number $x = 2n + 1$, we would just have to let $q = n$ (and $p = n + 1$). This means that we can write all odd numbers as a difference of two squares. Boom.