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DISCRETE MATHEMATICS

Final Exam T2 2015

Instruction

- Write your name
- Read the questions carefully.
- You have 4 hours to finish the exam.
- There are 5 problems. 700 points in total. You only need to get 630 points to get full score.
- Attempt all problems, state your reasons *clearly* and *legibly*, because partial credits will be given.
- Open book. Open notes.

Question	Full Score	Your Score
1	100	100
2	100	80
3	200	180
4	200	~~195~~ ~~190~~ 175
5	100	100
Bonus	30	30

Total: ~~688~~ 665 ~~/540~~ 630

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Useful Formulas

Sum

$$
\begin{aligned}
1+2+3+\ldots+n &= \frac{n(n+1)}{2} \\
1^2+2^2+3^2+\ldots+n^2 &= \frac{n(n+1)(2n+1)}{6} \\
1^3+2^3+3^3+\ldots+n^3 &= \left(\frac{n(n+1)}{2}\right)^2 \\
1+3+5+7+\ldots+(2n-1) &= n^2
\end{aligned}
$$

Euler's Formula

$$e+2=v+f$$

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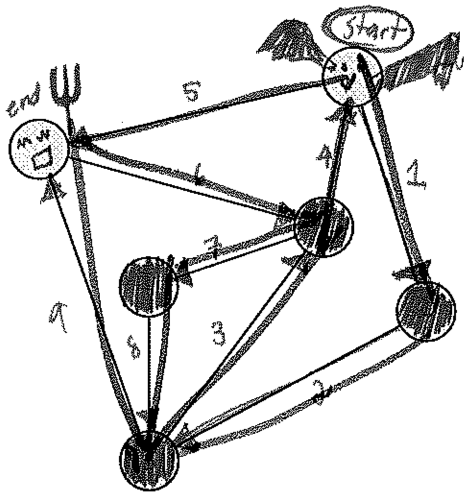
1. Easy Stuff(100 points. 20 each)

   (a) Drawing Graph Draw a graph of 6 node with *no cycle* and at least one ~~edge~~ node of degree exactly 2.

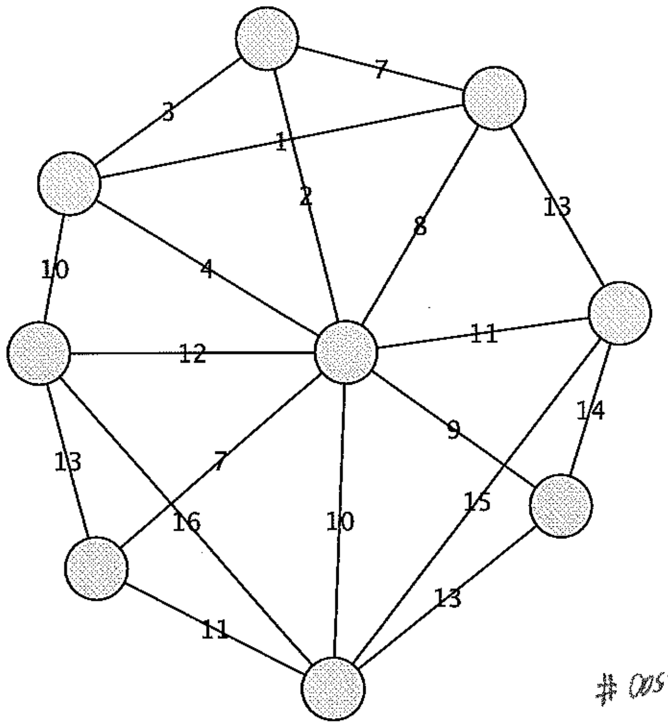
+20

   (b) Find an euler walk for the following graph. Label the edges with *numbers* so I can follow.

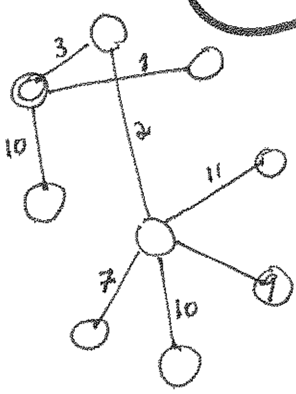


+20

   (c) The graph belows shows the cost of connecting a road from one city to another. Find the way to connect every city with minimum cost.



+20



cost = 44

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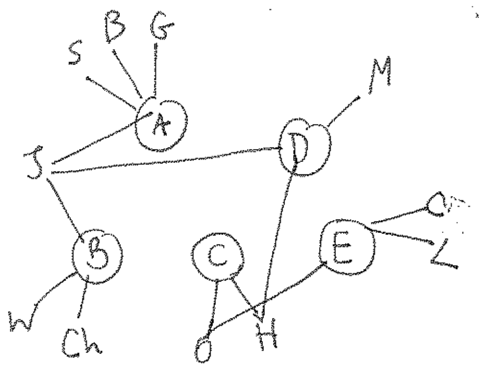
(d) Suppose we want to hold couple meetings among executives.

Meeting	Members
A	{Smith, Jones, Brown, Green}
B	{Jones, Wagner, Chase}
C	{Harris, Oliver}
D	{Harris, Jones, Mason}
E	{Oliver, Cummings, Larson}

Each meeting takes an hour. What is the minimum amount of time needed to hold all the meeting. Multiple meeting can run in parallel as long as there is no common person in the two meeting.

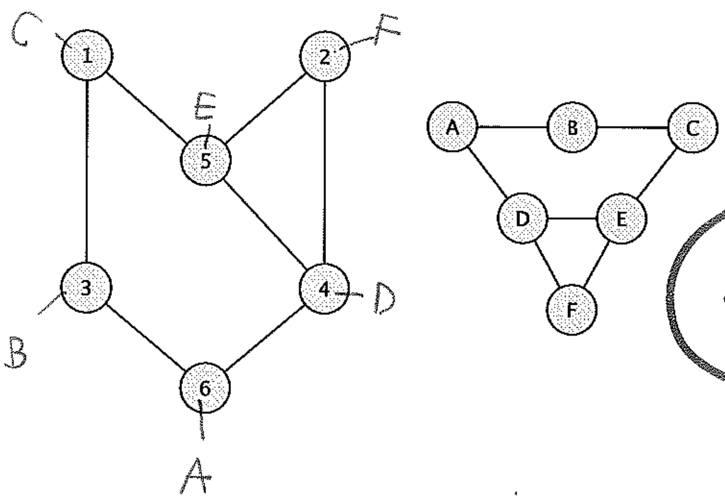


Hour	
1	A C
2	B E
3	D

3 hours

+20

(e) Find an isomorphism between these two graphs.



+20

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2. (100 points) 10 point each. Billy and Nat are starting iPhone case business. Their shop is called stoner_case. Each of their case is sold for 200 Baht. (Actual data. I just asked them.)

Here is what Billy does.

- Billy approach 200 people.
- For each person, Billy approach there is a 10% chance that the person will buy 1 case and 90% chance of not buying anything.
- Each person makes decision independently.

Nat is more business-oriented. So he goes for wholesales.

- Nat approach exactly 2 shop for trying to sell 100 case to each of them.
- Each shop has 10% chance of buying all 100 cases and 90% of not buying anything.
- The two shops make decision independently.

Furthermore, the shops and each person decision are mutually independent.

(a) What is expected value of the total revenue from **Billy**?

$$E[R_{Billy}] = E_1 + E_2 + \dots + E_{200}$$
$$= \frac{1}{10}[\underbrace{200 + 200 + \dots + 200}_{200}]$$
$$= \frac{1}{10} \times 40000 \quad = 4{,}000 \text{ Baht}$$

+10

(b) What is the variance of the total revenue from **Billy**?

$$Var[p] = E[p^2] - (E[p])^2 \quad \text{or} \quad \frac{1}{10}(200-20)^2 + \frac{9}{10}(0-20)^2$$
$$= \frac{1}{10}(200^2) - 20^2 \qquad = 3{,}600$$
$$= 3{,}600$$

+5

$$Var[200p] = 40000\, Var[p]$$
$$= 40{,}000 \times 3{,}600 = 1.44 \times 10^8$$

(c) What is expected value of the total revenue from **Nat**?

$$E[R_{Nat}] = \underbrace{\frac{1}{10}[\overset{200\times100}{20000}]}_{shop\ 1} + \underbrace{\frac{1}{10}[20000]}_{shop\ 2}$$
$$= 4{,}000 \text{ Baht}$$

+10

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(d) What is the variance of the total revenue from **Nat**? each shop

$$Var[S] = E[S^2] - (E[S])^2$$
$$= \frac{1}{10}[20000^2] - (2,000)^2$$
$$= 36 \times 10^6$$

+5

$$Var[2S] = 4\,Var[S]$$
$$= 4 \times 36 \times 10^6$$
$$= 1.44 \times 10^8$$

(e) What is the expected value of the total revenue of **stoner_case**?

$$E[R_{Total}] = E[R_{Billy}] + E[R_{Nat}]$$
$$= [4,000] + [4,000]$$
$$= 8,000$$

+10

(f) What is the variance of the total revenue of **stoner_case**?

independent

$$Var[R_1 + R_2] = Var[R_1] + Var[R_2] + 0$$
$$= 2 \times 1.44 \times 10^8$$

+10

(g) If the price got raised to 250 Bath per case, what is expected value of the total revenue from **Billy**?

$$E[p] = \frac{1}{10}[250] = 25$$

$$E[R_{new}] = E[200p] = 200\,E[p] = 5,000 \text{ baht}$$

+10

(h) If the price got raised to 250 Bath per case what is the variance of the total revenue from **Billy**?

~~Var[p+50] = Var[p]~~

~~= same as b)~~

$E[p]$

$$Var[p] = \frac{1}{10}[250^2] - 25^2$$
$$= 5,625$$

+5

$$Var[200p] = 40000\,Var[p]$$
$$= 40000 \times 5625$$
$$= 2.25 \times 10^8$$

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(i) If the price got raised to 250 Bath per case, what is expected value of the total revenue from **Nat**?

$E[Shop] = \frac{1}{10}[250 \times 100] = 2500$

$E[R_{new}] = 2E[Shop] = 5,000$ baht

+10

(j) If the price got raised to 250 Bath per case, what is the variance of the total revenue from **Nat**?

~~Var[S + 50] = Var[S]~~

~~∴ same as d~~

$Var[S] = \frac{1}{10}[25000^2] - 2500^2$

$= 5.625 \times 10^7$

$Var[2S] = 4 Var[S]$

$= 2.25 \times 10^8$

+5

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3. (200 points. 20 Each) The ninja world is in peace and war cycle. After each war the ninja world is torn apart and the ninja land becomes free for all.

After Ninja War I (Ninja Wit VS Ninja Taro from the midterm):

- 6 legendary ninjas: Ninja A, Ninja B, Ninja C, Ninja D, Ninja E and Ninja F survive the war.
- There are 10 different castles left over.
- There are 12 ninja dogs left.
- There are 20 legendary swords left.
- There are 100 legendary kunais(throwing dagger) left. All kunais are the *different.*
- There are 1000 shuriken(ninja star) are left. All shurikens are the *indistinguishable.*

Ninja senate at ninja parliament is now on the job of distributing the left over weapons to ninja and keep peace between the ninja. The asterisk(*) indicate the difficulty. If you can't do it in 5 minutes, skip and comeback later.

(a) Each of 10 castle can accomodate infinite amount of ninja, how many different ninja accomodations are there? Each of 6 legendary ninja has to live in a castle.

10^6 +20

(b) None of 12 ninja dogs get along with each other. If a ninja owns more than one dog they will get jealous ans start fighting each other. So, each of 6 ninja will get exactly 1 dog. How many ways are there to pair up ninja dog and ninja?

$\binom{12}{6} \times 6!$

+20

(c) Each ninja needs a whole bunch of kunais. How many ways are there to distribute 100 legendary kunais to all ninja such that every ninja get *at least* one kunais and all kunais are distributed?

$\binom{100-1}{6-1} = \binom{99}{5}$

* free +20

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(d) *The senate want to distribute the 20 swords. 3 out of 6 legendary ninjas know double swords skill. So these 3 people will get two legendary swords while the other three will only get 1 sword. How many ways are there to distribute legendary swords to ninjas?

get 9 swords

$$\binom{20}{9} \times \underbrace{\binom{9}{2}\times\binom{7}{2}\times\binom{5}{2}}_{\text{double swords}} \times \underbrace{3!}_{\text{the rest}}$$

+20

(e) To make the ninja world peace long last. The senate decide to throw a vacation. All of the ninja will stay at a hotel. The hotel has 6 rooms. Room 1, Room 2, ..., Room 6. The rooms are in linear order. Each room can accomodate exactly 1 person. How many ways are there to assign the 6 ninjas to the rooms?

$6!$

+20

(f) However, Ninja A and Ninja B are a very good friends. They love chatting with each other. If these two ninjas stay right next to each other (eg: Ninja A at room 5 and Ninja B at room 4), the noise level will be too high and disturb others. This might create another ninja war. How many ways are there to assign the ninjas to the rooms such that Ninja A and Ninja B are **not** next to each other?

Hint: count the one that they are next to each other.

1 unit

2!

$= P[\text{All}] - P[\text{A and B next to each other}]$

$= 6! - 5!2!$

+20

(g) *** After one night, the senate discover that Ninja B and Ninja C are also a good friends. This means that Ninja B cannot be next to Ninja A or Ninja ~~B~~ C. How many ways are there to assign the ninjas to the rooms such that Ninja B are not next to Ninja A or Ninja C?

~~Hint:~~ This means that the following arrangements are not allowed. AB, BA, BC, CB, ABC, CBA. All others are ok.

+20

consider arrangement of 5 Ninjas (Not B) $\to 5!$

plug B

144

Case 1 B & C are apart $= (5! - 4!2!) \times 2$

plug B

144

Case 2 B & C are next to each other $= 4! \times 2! \times 3$

+

288

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(h) At the end of vacation, the senate distribute *all* shurikens to the ninjas. Some ninja **may not** get any shuriken. How many way are there to distribute the shuriken to ninja?

$$\binom{1000+5}{6-1} = \binom{1005}{5}$$

+20

(i) * After 10 years eveyone feels that Ninja A overpowered the rest of the ninja. So to balance the power, the ninja senate decide to limit the number of shuriken Ninja A have must be ≤ 50. The senate then confiscate all 1000 shurikens from ninjas and redistribute.

How many ways are there to distribute shuriken to all the ninja such that Ninja A get ≤ 50 shurikens? Remember that all shuriken are distributed and some ninja may not get any shuriken.

Hint: Count the number of ways that break the law.

50

let A have it

Calculate the rest

$$= 1 - P(A \text{ get} \geq 50)$$

$$= 1 - \frac{\binom{1000-50+5}{5}}{\binom{1005}{5}}$$

why prob....

5

(j) After 30 years, the peace comes to an end. The 6 ninjas engange in a 3 VS 3 war. How many different combinations of 3 VS 3 match are there?

Clarification: ABC VS DEF, CBA VS FED, DEF VS ABC are all the same match up.

$$\binom{6}{3} \times 1$$

~~choose~~ first 3 — the rest together

15

~~+10~~

2.

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4. (200 points 20 each)Let's consider MUIC club fair day. Suppose that there are 6 clubs: Club A, Club B, Club C, Club D, Club E and Club F.

   Furthermore, there are 6 students: Student 1, Student 2, Student 3, ... and Student 6.

   Each student must join exactly one club. At the end of the day, any club has no student registered the club will have to be closed.

   Let us consider 2 strategies for the students in choosing the club they join.

   - Strategy A: each student pick the club uniformly and independently. Specifically each student has 1/6 probability of joining any club independently.

     (a) What is the probability that club A will get closed?

     $\left(\frac{5}{6}\right)^6$

     +20

     (b) What is the probability that club B has 3 students?

     $\binom{6}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^3$

     ~~+20~~ ~~10~~ 15

     (c) What is the probability that club A get closed *given* that club B has 3 students?

     $P(\text{A closed} \mid \text{B has 3}) = \frac{P(\text{A closed} \cap \text{B has 3})}{P(\text{B has 3})}$

     $= \frac{\left(\frac{1}{6}\right)^3\left(\frac{4}{6}\right)^3}{\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^3} = \left(\frac{4}{5}\right)^3$

     3 students go to C, D, E or F only

     +20

     (d) What is the probability that club A get closed *and* club B has 3 students?

     $\binom{6}{3}\left(\frac{1}{6}\right)^3\left(\frac{4}{6}\right)^3$

     ~~+20~~ ~~16.~~ 15

     (e) Are the event that club A get closed and club B has 3 students independent?

     NO

     $P(\text{A closed} \mid \text{B has 3}) \neq P(\text{A closed})$

     +20

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(f) What is the expected number of student that will join club A at the end of day?

$$E[X] = \frac{1}{6}(6) = 1$$

20

probability that club will survive

$$Pr[X] = 1 - \left(\frac{5}{6}\right)^6$$

closed

(g) *What is the expected number of club that will survive the club fair?

$$E[X] = 6 \times \left[1 - \left(\frac{5}{6}\right)^6\right]$$

20

(h) *SMO set up a fee: 1000 Baht for being the first person to join the club and 500 Baht for each subsequent member. What is the expected amount of money SMO will collect at the end of the day?

$$E[X] = \cancel{\frac{1}{6}[1{,}000] + \frac{1}{6}(500)} + \dots$$

5 ~~terms~~

choose SMO

be first person

be 2nd, 3rd, ...

Each person expects to pay $\frac{1}{6} \times \frac{1}{6}[1{,}000] + \frac{1}{6} \times \frac{5}{6}[500]$

$$\therefore E[6p] = 6E[p]$$

$$= \cancel{6} \times \frac{1}{\cancel{6}}\left[\frac{1}{6}(1{,}000) + \frac{5}{6}(500)\right]$$

$$= 583.3 \text{ baht}$$

~~10~~

It should be more than 3k.

free 20

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- Strategy B: the club selection goes as follow.
  - Student 1 goes to club A.
  - Student i select uniformly among all the previous students Student 1, …Student $i-1$ and the choice of joining a new club each with equal probability. If Student i choose a student k then he/she will join the same club as student k. Otherwise, Student i will join a new club.

    For example, Student 3 can choose to join Student 1's club or Student 2's club or he can choose to join a new club; each with probability 1/3.

    Beware that if Student 1 and Student 2 are in the same club, this would mean that there is actually 2/3 chance of Student 3 joining club A and 1/3 chance of him joining a new club.

(a) What is the probablity that none of the club gets closed?

Student 2 | 3 | 4 | 5 | 6

$= \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5} \times \frac{1}{6}$ 20

always chooses to go to new club (student 2, 3, 4, 5, 6)

(b) What is the probability that club A get 6 members?

$= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} \times \frac{5}{6}$ 20

$= \frac{1}{6}$

(c) **What is the expected number of club that survive at the end of the day? Do not leave it as a sum. Just add them up.

5

$E[X] =$

A is guaranteed to survive

$= 5 \times Pr[X]$ ≈ 2.84 :P

Student	Probability of not being picked
2	$\frac{1}{2} + \frac{1}{2}\left(\frac{4}{5}\right)$
3	$\frac{2}{3} + \frac{1}{3}\left(\frac{3}{4}\right)$
4	$\frac{3}{4} + \frac{1}{4}\left(\frac{2}{3}\right)$
5	$\frac{4}{5} + \frac{1}{5}\left(\frac{1}{2}\right)$
6	$\frac{5}{6}$

$X = \frac{9}{10} \times \frac{11}{12} \times \frac{11}{12} \times \frac{9}{10} \times \frac{5}{6}$ ≈ 0.567

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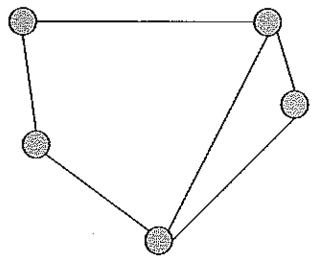
5. Graph Theory.

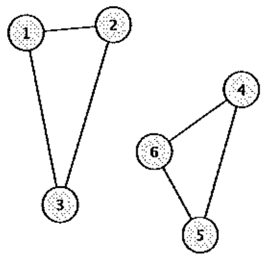
(a) (50)CS Kids and Dinosaur Farm. A group of MUIC CS students loves to play Ark Survival Evolved. A game which allows you to farm Dinosaurs.

To make a containment area, the player will need posts and electric wires. The posts will act as corners for the containment area and the electric will define the boundary. For example, the arrangement below shows 5 posts and 6 electri wires. The electric wire cannot cross each other or it will create a short and all the dinosaur will get lose.



Here is a catch.

- The dinosaurs do not get along with each other. If we keep them in the same containment area, they will start to fight each other. So the arrangement above can hold 2 dinosaurs.
- We want all the containment areas adjacent to another one. So you can't do this.



How many dinosaur can you contain with 100 posts and 250 wires? (All wires and posts must be used).

Hint: Be careful with off by 1 and this is a one-line answer.

$f = e - v + 2$

$= 250 - 100 + 2$

$= 152$

50

ans 151 (face outside is not a containment area)

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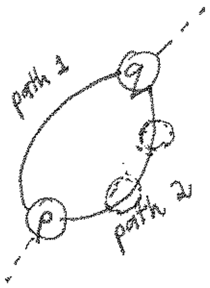
(b) (50) Let T be a tree. Let u and v be two vertices of the graph. Show that there *exists* a *unique* path from u to v.
You must show that a) it exists and b) that it is unique.
Hint: Use definition of a tree.

+50

a) By definition of a tree, T is a connected graph which means there is a path connecting any vertices within T

Since u and v are vertices of T, a path is guaranteed to exist from u to v

b) Let's assume that it is not unique i.e. there is at least 2 paths from u to v



If there is more than one path, there exists vertex p from u to v that has non-unique paths to vertex q.
This generates a cycle which means T is not a tree

contradiction ∎

∴ path from u to v must be unique

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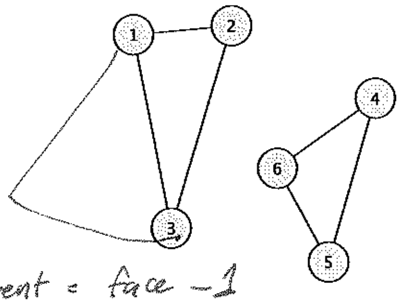
6. (Bonus) 30 points. This is actually not that hard.

   In the dinosaur problem, if we make 10 separate containment areas using all the posts and wires. How many dinosaur can we contain?

   This means that something like this but with 10 connected components and many more posts and wires.



containment = face − 1

$$\left.\begin{aligned} C_1 &= e_1 - v_1 + 1 \\ C_2 &= e_2 - v_2 + 1 \\ C_3 &= e_3 - v_3 + 1 \\ &\vdots \\ C_{10} &= e_{10} - v_{10} + 1 \end{aligned}\right\} \quad \begin{aligned} \Sigma C &= \Sigma e - \Sigma v + 10 \\ &= 250 - 100 + 10 \\ &= 160 \end{aligned}$$

30