Solution 1

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Problem 1 Register on Canvas `https://canvas.instructure.com/enroll/LF8PF9`

Problem 2 Basic Stuff.

2.1) Write down the set of integers that is less than 8 and greater than 2.

Answer: $\{3, 4, 5, 6, 7, 8\}$

2.2) Write down the members of $\{x \in I | x^2 < 10\}$.

Answer: $\{-3, -2, -1, 0, 1, 2, 3\}$

2.3) Write down the members of

$$\{x \in I^+ | x^2 < 10\} \cup \{x \in I^+ | 2 < x < 8\}.$$

Answer: $\{-3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7\}$

2.4) Write down the members of

$$\{x \in I^+ | x^2 < 10\} \cap \{x \in I^+ | 2 < x < 8\}.$$

Answer: $\{3\}$

Problem 3 Notation Exercise.

- Let M be the set of all MUIC students.
- Let $P(x)$ be the predicate for student x takes Discrete Math class this term.
- Let $Q(x)$ be the predicate for student x taks Data Structure this term.
- Let $F(x, y)$ be the predicate for student x and student y are friends on facebook.
- Let $\mathbb{A} = \{x \in M | P(x)\}$
- Let $\mathbb{B} = \{x \in M | Q(x)\}$

Write the following propositions in mathematical form using $\cap$, $\cup$, $\rightarrow$, $\forall$, $\exists$ and all those set notations.

3.1) There are some students at MUIC who take Discrete Math is term. (Ex: $\exists x \in M$ such that $P(x)$)

Answer: $\exists x \in M$ such that $P(x)$

3.2) There are some students who take both Discrete Math and Data Structure this term.

Answer: $\exists x \in M$ such that $P(x) \wedge Q(x)$

3.3) There exists some students who take Discrete math but not Data Structure.

Answer: $\exists x \in M$ such that $P(x) \land \neg Q(x)$

3.4) Everyone takes Discrete Math.

Answer: $\forall x \in M, P(x)$

3.5) No one takes Data Structure.

Answer: $\forall x \in M, \neg Q(x)$

3.6) All students who takes discrete math this term also take Data structure. (Ex: $\forall x \in \mathbb{A}, Q(x)$ or you can use $\forall x \in M, P(x) \to Q(x)$)

Answer: $\forall x \in \mathbb{A}, Q(x)$

3.7) There exists some students who take Data Structure but not Discrete Math.

Answer: $\exists x \in \mathbb{B}$ such that $\neg P(x)$

3.8) There exists a student in Discrete Math who is friend to every one in Data structure.

Answer: $\exists x \in \mathbb{A}$ such that $F(x, y) \forall y \in \mathbb{B}$

3.9) Everyone in Data Structure has at least one friend in Discrete Math.

Answer: $\forall x \in \mathbb{B} \exists y \in \mathbb{A} F(x, y)$

3.10) There exists a student in Discrete Math who is friend to no one in Data Structure.

Answer: $\exists x \in \mathbb{A}$ such that $\forall y \in \mathbb{B} \neg F(x, y)$

Problem 4 Write down truth table for the following statements:

4.1) $P \implies (\neg Q \lor P)$

4.2) $P \implies (P \land Q)$

Answer: Too easy. Do it yourself.

Problem 5 Fun With Quantifiers.

5.1) Are the following two propositions equivalent?

A) $\forall x \in I, \exists y \in I$ such that $x + y = 23$

B) $\exists y \in I$ such that $x + y = 23$ $\forall x \in I$

Answer: No. See lecture notes.

5.2) Are the follwing propositions true for all predicate P. Explain/Prove/Disprove it.

A) $\forall x, \exists y$ such that $P(x, y) \implies \exists y$ such that $P(x, y)$ $\forall x$

Answer: No. Counter example can be found in the previous problem. For every number x we can find y by doing $23 - x$ to add to x and gives 23. But, there is no number y that no matter which number we add to y it will give 23.

B) $\exists y$ such that $P(x,y) \forall x \implies \forall x, \exists y$ such that $P(x,y)$

Answer: Since there is at least one y that make $P(x,y)$ true. Let that special y be denoted by y'.
Given any x we want to show that we can find a y that makes $P(x,y)$ true.
This is easy we can just pick y' since we know that $P(x,y')$ is true.
Thus the statement is true.

Problem 6 Recall the card trick I showed you in class. Now it's your turn to write a proof for it:

Suppose that we have a deck of 52 cards(26 Reds and 26 Blacks). We then draw 2 cards at a time until we ran out of cards while counting the number of black pair, red pair, and different color pair. Show that the number of red pair and number of black pair are always equal at the end.

Answer: Suppose that at the end of the game we have $2n$ from some $n \in I$ cards. We know that it is an even number since we put 2 cards at a time in the mixed pile.

Since each time we put 1 red card and 1 black card into the mixed pile, there are n red cards and n black cards in the pile.

The number of red cards in the red pile is then $26 - n$ since there are 26 red cards in total and n of them is in the mixed color pile.

The number of black cards in the black pile is then $26 - n$ since there are 26 black cards in total and n of them is in the mixed color pile.

Thus, the number of pair in the red pile and in the black pile are equal; they are both $26 - n$.

Problem 7 Recall the absolute function

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Prove the following statement:

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|} \ \forall a, b \in \mathbb{R}, b \neq 0$$

Answer:

Case 1: $a > 0$ and $b > 0$.

$$\frac{|a|}{|b|} = \frac{a}{b}$$

Since $\frac{a}{b} > 0$. Thus.

$$\left|\frac{a}{b}\right| = \frac{a}{b}$$

Thus,

$$\frac{|a|}{|b|} = \frac{a}{b} = \left|\frac{a}{b}\right|$$

Case 2: $a < 0$ and $b > 0$.

$$\frac{|a|}{|b|} = \frac{-a}{b}$$

Since $\frac{a}{b} < 0$. Thus.

$$\left|\frac{a}{b}\right| = -\frac{a}{b}$$

Thus,

$$\frac{|a|}{|b|} = -\frac{a}{b} = \left|\frac{a}{b}\right|$$

Case 3: $a < 0$ and $b < 0$.

$$\frac{|a|}{|b|} = \frac{-a}{-b} = \frac{a}{b}$$

Since $\frac{a}{b} > 0$. Thus.

$$\left|\frac{a}{b}\right| = \frac{a}{b}$$

Thus,

$$\frac{|a|}{|b|} = \frac{a}{b} = \left|\frac{a}{b}\right|$$

Case 4: $a > 0$ and $b < 0$.

$$\frac{|a|}{|b|} = \frac{a}{-b} = -\frac{a}{b}$$

Since $\frac{a}{b} < 0$. Thus.

$$\left|\frac{a}{b}\right| = -\frac{a}{b}$$

Thus,

$$\frac{|a|}{|b|} = -\frac{a}{b} = \left|\frac{a}{b}\right|$$

Since the theorem is true in all cases.

$$\frac{|a|}{|b|} = \left|\frac{a}{b}\right|$$

Problem 8 8.1) State the contrapositive and prove the following proposition:

If r is *irrational*, then $r^{1/5}$ is *irrational*.

Answer: Contrapositive is

If $r^{1/5}$ is rational, then r is rational.

Proof:

Since $r^{1/5}$ is rational we know that

$$r^{1/5} = \frac{p}{q} \;\; \exists p, q \in I$$

Raise it to power 5 on both sides.

$$r = \frac{p^5}{q^5}$$

This is a ratio of integers. Thus, r is rational.$\blacksquare$

8.2) Is the following statement true? Prove it either way.

If r is *rational*, then $r^{1/5}$ is *rational.*

Answer: No. Counter example is $r = 5$ is rational but $\sqrt[5]{5}$ is not.

8.3) How about the following statement? Prove it either way.

If $r^{1/5}$ is *irrational*, then r is *irrational.*

Answer: False. Counter example $\sqrt[5]{5}$ is irrational but 5 is rational.

Problem 9 Consider a room of area 5 m^2 where you can pick a shape. Billy told you that he wants to place 9 rugs, each of area 1 m^2 again you can pick the shape of all the rugs too. Then, since Intouch hates having the rugs intersecting each other, he demands that no pair rugs should have an area of intersection $\geq 1/9 m^2$.

After Several tries, you found that picking shapes of the room and rugs to satisfy both Billy's and Intouch's requirement is just impossible. So, you went to tell them that this is impossible. However, since they are pretty good at mathematical reasoning. They both demand a proof that it is impossible.

Your task for this problem is to provide them a proof that this is impossible.

Hint: contradiction.

Answer: Let us assume for the sake of contradiction that such algorithm exists.

- There are at most 36 pair-wise intersections.
- Since each intersection takes away $< 1/9$ from the total area.
- This means that the total area

$$\text{Total area} = 9 - 36x$$

  where $x < 1/9$.
- Thus,

$$\text{Total area} > 9 - \frac{36}{9}$$

  which means

$$\text{Total area} > 5$$

  which contradicts the fact that the total area must be 5.$\blacksquare$.

Problem 10 Using the convention I showed in class, If I lay the following cards on the table, what is the hidden card?

$$3\clubsuit, 2\heartsuit, 7\spadesuit, Q\heartsuit$$

Note for the rule (L=Low, M=Medium, H=High)

- LMH = 1
- LHM = 2
- MLH = 3
- MHL = 4
- HLM = 5
- HML = 6

Answer: 4$\clubsuit$

Problem 11 Let $\mathbb{O}$ be the set of odd integers. Proof the following statement:

$$\forall x \in \mathbb{O} \;\; \exists p, q \in I \;\text{ such that} x = p^2 - q^2$$

in other words, every odd integer can be written as a difference of two squares.

Answer: A lot of you got this one backward. The question ask whether every odd integer can be written as a difference of square or not. It DOES NOT ask whether the sume of square is even or odd. For those of you who got this one wrong, read this argument carefully.

Proof:

Since x is odd, we know that

$$x = 2m + 1 \ \ \exists m \in I$$

Adding and subtracting m^2 on the right hand side.

$$x = m^2 + 2m + 1 - m^2$$

This gives

$$\begin{aligned} x &= (m+1)^2 - m^2 && (1) \\ &= p^2 - q^2 && (2) \end{aligned}$$

where $p = m + 1$ and $q = m$.

Which is a a difference of integer square. (We know each one of them exists since we know that m exists from the first line.)■