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Discrete Mathematics

Midterm Exam T1 2019

Instruction

- Write your name
- Read the questions carefully.
- There are 6 problems. Each problem worths 100 points. 600 points in total. You only need to get 540 points to get full score.
- Attempt all problems, state your reasons *clearly* and *legibly*, because partial credits will be given.

Question	Full Score	Your Score
1	100	
2	100	
3	100	
4	100	
5	100	
6	100	
Bonus	30	

Total: $\boxed{}$ /540

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Useful Formula and Definitions

Asymptotics

Definiton	Definition			Intuition
Asym. Equal	$f \sim g$	iff	$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 1$	$f \underbrace{\equiv}_{x\to\infty} g$
Big Oh	$f \in O(g)$	iff	$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} < \infty$	$f \underbrace{\leq}_{x\to\infty} g$
Little Oh	$f \in o(g)$	iff	$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 0$	$f \underbrace{<}_{x\to\infty} g$
Little Omega	$f \in \omega(g)$	iff	$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} \to \infty$	$f \underbrace{>}_{x\to\infty} g$
Big Omega	$f \in \Omega(g)$	iff	$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} > 0$	$f \underbrace{\geq}_{x\to\infty} g$
Theta	$f \in \Theta(g)$	iff	$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = c, c \neq 0$	$f \underbrace{=}_{x\to\infty} g$

Sum

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Integral

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} \qquad \text{if } n \neq -1$$

$$\int \frac{1}{x} \, dx = \ln(x)$$

Quadratic

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

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1. Easy stuff(100 points. 20 each.)

   (a) Draw the truth table for $Q \wedge (\neg P \vee Q)$

   (b) Which of these asymptotic symbols ($\{\sim, o, O, \omega, \Omega, \Theta\}$) are applicable for

   $$f(x) = \_\_ \big(g(x)\big)$$

      i. $f(x) = n^3 + 3, g(x) = n^2 + 9999$

      ii. $f(x) = 3^x, g(x) = 5^x$

   (c) Find the closed form formula for the following sum/product:

      i. $\displaystyle\sum_{i=1}^{n}\prod_{j=1}^{m} i^2$

(d) Disprove the following proposition.

If p is even, then $p^2 - 1$ is a prime number.

(e) Use integral bound to find the *lowerbound* for the following sum.

$$\sum_{x=1}^{n} x^{1/3}$$

2. (a) (50 points)Given a positive integer,if the sum of all digits of that number is divisible by 9 then the the number itself is divisible by 9.

   Ex: Consider 7362. The sum of all digits is $7+3+6+2=18$ which is divisible by 9. Also, $7362 = 9 \times 818$

   Hint:

   - $7362 = 7 \times 10^3 + 3 \times 10^2 + 6 \times 10^1 + 2 \times 10^0$
   - $10^3 - 1 = 999$ is divisible by 9(You may take this fact as given that $99999\ldots9999$ is divisible by 9)
   - This may sound stupid but $10^3 - 1 + 1 = 10^3$

(b) (50 points) Show that choosing distinct 6 numbers from $\{1, 2, 3, \ldots, 10\}$ always ends up with at least 2 consecutive number.

Ex: $\{1, 3, 5, 6, 8, 10\}$. 5,6 are consecutive.

Hint: 12 34 56 78 910

3. (100 points) Recall matrix multiplication.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix} \tag{1}$$

and Fibbonacci sequence defined by $F_{n+2} = F_n + F_{n+1}$ with $F_0 = 0$ and $F_1 = 1$
Show that if

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \tag{2}$$

Then

$$A^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} \tag{3}$$

for all integer $n \geq 1$. (A^n is just muliplying matrix A together n times.)

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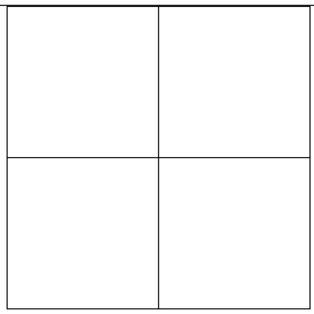
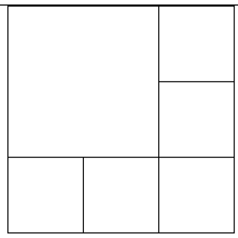
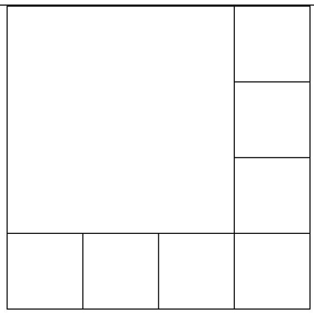
4. A square is a rectangle whose all sides are equal and all angles are right angle. We could split a square in to many squares of not necessarily equal area. For example:

   We could split it in to squares of varying number.

4 squares	
6 squares	
8 squares	

   Your task: Show that a square can be splitted in to any number of squares greater than or equal to 6. Make sure you proof has enough base cases and assuming the correct inductive hypothesis.

   Hint: Find out how to do 7 from 4 split. Then, how do we do 9 from 6?

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Blank page for problem 4.

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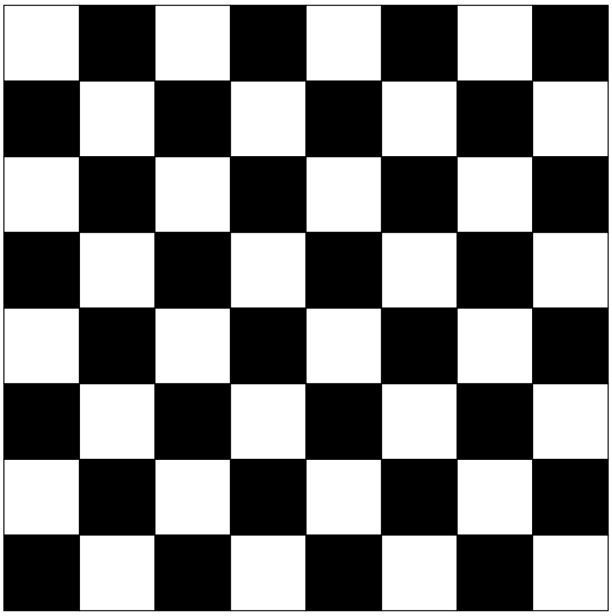
5. (100 points) Consider an 8×8 black and white board show below.



Consider a variant of the problem you did in the homework. Each turn you can pick 2 squares (not necessarily adjacent) and flip both of them. Show that you can't reach grid with exactly 1 black square left.

6. (100 points)Solve the following recurrence. Just find the solution. No need to prove it using induction.:

   (a) (30 points)$T(n) = T(n-1) + n + 2; T(1) = 1$

   (b) (30 points)$T(n) = 2T\left(\frac{n}{2}\right) + 1; T(1) = 1$

(c) (40 points)$T(n) = 8 \times T(n-1) - 15T(n-2); T(0) = 6, T(1) = 26$

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7. Bonus.(30 points) No partial credit for this one. This is one of the easiest bonus ever.

   Show that for any number divisible by 9. If we switch any two digits, it is still divisible by 9.

   Ex:

   18 is divisible by 9. So is 81.

   $117 = 9 \times 13$. Both 171 and 711 are divisible by 9.