Discrete Maths: Assignment 5

These are rough answers for HW5 (some questions found in another file).

Question 4: DS Cheat Sheet

Note: $\lg x$ is equivalent to $\log_2 x$.

(4.1) $T(n) = T(n-1) + n, T(1) = 1$

$$\begin{aligned} T(n) &= T(n-1) + n \\ &= T(n-2) + (n-1) + n \\ &\vdots \\ &= 1 + 2 + \ldots + (n-1) + n = \frac{n(n+1)}{2}. \end{aligned}$$

To prove by induction, the predicate is

$$P(n) \quad \equiv \quad T(n) - \frac{n(n+1)}{2}.$$

Actual proof is left as exerise for reader.

(4.2) $T(n) = T(n-1) + n^3, T(1) = 1$

$$\begin{aligned} T(n) &= T(n-1) + n^3 \\ &= T(n-2) + (n-1) + n^3 \\ &\vdots \\ &= 1^3 + 2^3 + \ldots + (n-1)^3 + n^3 = \left(\frac{n(n+1)}{2}\right)^2. \end{aligned}$$

(4.3) $T(n) = T(n/2) + 1, T(1) = 1$

$$\begin{aligned} T(n) &= T(n/2) + 1 \\ &= T(n/4) + 1 + 1 \\ &\vdots \\ &= \underbrace{1 + 1 + \ldots + 1}_{\lg n + 1 \text{ times}} = \lg n + 1. \end{aligned}$$

(4.4) $T(n) = T(n/2) + n, T(1) = 1$

$$\begin{aligned} T(n) &= T(n/2) + n \\ &= T(n/4) + \frac{n}{2} + n \\ &\vdots \\ &= 1 + 2 + \ldots + \frac{n}{2} + n = 2n - 1. \end{aligned}$$

(4.5) $T(n) = 2 \cdot T(n/2) + n, T(1) = 1$

$$\begin{aligned} T(n) &= 2 \cdot T(n/2) + n \\ &= 2 \cdot T(n/4) + 2 \cdot \frac{n}{2} + n \\ &\vdots \\ &= \underbrace{n + n + \ldots + n}_{\lg n + 1 \text{ times}} = n(\lg n + 1). \end{aligned}$$

(4.6) $T(n) = 2 \cdot T(n/2) + 1, T(1) = 1$

$$\begin{aligned} T(n) &= 2 \cdot T(n/2) + 1 \\ &= 2 \cdot T(n/4) + 2 \cdot 1 + 1 \\ &= 4 \cdot T(n/8) + 4 \cdot 1 + 2 \cdot 1 + 1 \\ &\vdots \\ &= n + \frac{n}{2} + \cdots + 2 + 1 = 2n - 1. \end{aligned}$$

(4.7) $T(n) = 2 \cdot T(n/2) + \lg n, T(1) = 1$

$$\begin{aligned} T(n) &= 2 \cdot T(n/2) + \lg n \\ &= 2 \cdot T(n/4) + 2 \lg \frac{n}{2} + \lg n \\ &= 4 \cdot T(n/8) + 4 \lg \frac{n}{4} + 2 \lg \frac{n}{2} + \lg n \\ &= \vdots \\ &= n + \frac{n}{2} \lg 2 + \cdots + 4 \lg \frac{n}{4} + 2 \lg \frac{n}{2} + \lg n \end{aligned}$$

Question 5: Asymptotics

(5.1) O, Ω, Θ

(5.2) O, o

(5.3) O, o

(5.4) ω, Ω

Explanation:

$$\lim_{n \to \infty} \frac{\sqrt{n}}{(\log n)^{999}} = \left(\lim_{n \to \infty} \frac{n^{1/(2 \cdot 999)}}{\log n} \right)^{999} = 0^{999} = 0$$

In general, $(\log n)^k = O(n^\epsilon)$ for any k and $\epsilon > 0$ [1]. Base of the logarithm also will not matter here.

[1] For proof see https://math.stackexchange.com/questions/1663818/does-the-logarithm-function-grow-slower-than-any-polynomial.

(5.5) ω, Ω

(5.6) ω, Ω

Explanation:

$$\lim_{n\to\infty} \frac{n^2}{a^n} = \lim_{n\to\infty} \frac{2n}{a^n \ln a} = \lim_{n\to\infty} \frac{2}{a^n (\lg a)^2} = 0$$

We had to use L'Hôpital's rule twice here.

Generally, all exponential functions grows faster than any polynomial functions, so any polynomial term, $n^x = O(a^n)$ for any real x. You to show this generally using a combination of induction and L'Hôpital's rule.

(5.7) O, Ω, Θ

(5.8) O, o

Question 6: More Recurrence

(6.1) $T_i = 5T_{i-1} - 6T_{i-2}; T_0 = 8, T_1 = 17$

Assume that T is in the form of x^i where x is some integer. Then

$$x^i = 5x^{i-1} - 6x^{i-2},$$

$$x^2 - 5x + 6 = 0$$

which means that $x = 2$ and $x = 3$. The actual answer should therefore be in the form of $T_i = a \cdot 2^i + b \cdot 3^i$ where a and b are constants subjected to the initial conditions.

By observation, we can *clearly* see that $a = 7$ and $b = 1$. This means that our final solution is

$$T_i = 7 \cdot 2^i + 3^i.$$

(6.2) $T_i = -T_{i-1} + 12T_{i-2}; T_0 = 0, T_1 = -7$

Assume that T is in the form of x^i where x is some integer. Then

$$x^i = -x^{i-1} + 12x^{i-2},$$

$$x^2 + x - 12 = 0$$

which means that $x = -4$ and $x = 3$. The actual answer should therefore be in the form of $T_i = a \cdot (-4)^i + b \cdot 3^i$ where a and b are constants subjected to the initial conditions.

By observation, we can again *clearly* see that $a = 1$ and $b = -1$. This means that our final solution is

$$T_i = (-4)^i - 3^i.$$

Question 7: Roulette Trick

(Part 1)

We prove that if $w+l=2$ then the outcome is the same regardless. We will do so by considering the case where we swap two of the outcome cards. Let's say we have a long outcome list of cards $\ldots, X, Y, \ldots$. We want to swap the cards X and Y in our outcome lists. There are two possibilities.

- X and Y are of the same colours. In this case the outcome should still be the same regardless since the only thing that matters in the game is the sequence of colour that appears, and this would be unchanged if we swap two cards of the same colour.
- X and Y are of different colours. This divides into two subcases. For the purpose of the proof, let's say the money right before the swap position is given by m and the next betting amount is k (which both should be the same since before the swap point, card sequence appears in the same order).

  If X is red and Y is black, then right after the red card, the new balance is $m+k$ and the preceding bet is wk. The next card is black, so the new balance is $m+k-wk$ and the next bet is wlk.

  Instead, if X is black and Y is red, then right after the black card, the new balance is $m-k$ and the preceding bet is lk. The next card is black, so the new balance is $m-k+lk$ and the next bet is wlk.

  We can see that the next bet right after the swap will be the same in either cases. We can also see that

  $$m+k-wk = m+k-(2-l)k = m-k+lk$$

  which shows that the balance after the round will also be the same.

We have shown that the outcome is invariant after one swap.

We can also see that two permutations of outcomes of the cards will only be different through a sequence of swaps. We can say this for all permutations of outcomes, meaning that all permutations of cards will give the same outcomes.

(Part 2 and 3)

Left as exercise for reader. Breifly, for Part 2, since permutation doesn't matter, you can pretend that all the first r cards are red, then the next b cards being black. This should make the summation easier (since you can use geometric sum formula).

Question 8: Coupon Bond

If the coupon bond is too expensive, then you will make less profit buying it. If the amount of money you get from buying it is small, then it's not worth buying the coupon bond, because you are better off just storing the money in the bank and gaining the extra money through interest rates instead.

This means the money you gain from the bond should be more than (or equal to) the money you would have gained from simply putting the money in the bank, since if it makes any less then it's not worth buying the bond.

For the bond of N years costing F, the money you get after N years is $F + NP$. If you deposit money of F to the bank, in N years, the bank will give you $F(1 + r)^N$ in N years. We know this as we find that after each passing year, the money increases by $1 + r$ times, which is like multiplying the amount of money by $1 + r$ for each passing year.

This means that if F is the cost for the bond, then

$$F + NP \geq F(1 + r)^N$$

and to find the actual amount, you solve for F (left as an exercise for the reader).