CS208/204: Data Structures & Abstractions
Mahidol University International College

Assignment 6:
Written Answers

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June 28, 2023

Task 3: Quick Sort Recurrence

(ii)

$$g(n) = \frac{f(n)}{n+1}$$
$$f(n) = 2 + \frac{(n+1)\cdot f(n-1)}{n}$$

$$\begin{aligned} g(n) &= \frac{f(n)}{n+1} \\ &= \frac{2 + \frac{(n+1)\cdot f(n-1)}{n}}{n+1} \\ &= \frac{2}{n+1} + \frac{f(n-1)}{n} \\ &= \frac{2}{n+1} + g(n-1) \end{aligned}$$

$$g(n) = \frac{2}{n+1} + g(n-1)$$

(iii)

$$g(n) = \frac{2}{n+1} + g(n-1)$$

The recurrence expands to:

$$g(n) = \frac{2}{n+1} + \ \frac{2}{n} + \frac{2}{n-1} + \frac{2}{n-2} + \frac{2}{n-3} + \cdots + \frac{2}{2}$$

$$g(n) = 2(\frac{1}{n+1} + \ \underbrace{\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \frac{1}{n-3} + \cdots + \frac{1}{2}}_{h(n)-1})$$

We can substitute the series with the function $h(n) - 1$ harmonic numbers.

$$g(n) = 2(\frac{1}{n+1} + h(n) - 1)$$

This completes the form of g(n) as there is no longer a recurrence relation.

(iv)

Lets first write f(n) in terms of n:

$$f(n) = g(n) \cdot (n+1)$$

$$f(n) = 2(\frac{1}{n+1} + \underbrace{h(n)}_{ln(n)+1 \text{ from fact given.}} -1) \cdot (n+1)$$

$$f(n) = 2(\frac{1}{n+1} + (ln(n)+1) - 1) \cdot (n+1)$$

$$f(n) = 2(\frac{1}{n+1} + ln(n)) \cdot (n+1)$$

We **want to show** that $f(n)$ is $O(nln(n))$. This can be solved by the following limit:

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{f(n)}{nln(n)} \\
&= \lim_{n\to\infty} \frac{2(\frac{1}{n+1} + ln(n) \cdot (n+1))}{nln(n)} \\
&= \lim_{n\to\infty} \frac{2 + 2ln(n) \cdot (n+1)}{nln(n)} \\
&= \lim_{n\to\infty} \frac{2}{nln(n)} + \lim_{n\to\infty} \frac{2ln(n) \cdot (n+1)}{nln(n)} \\
&= \lim_{n\to\infty} \frac{2}{nln(n)} + \lim_{n\to\infty} \frac{2n+2}{n} \\
&= \lim_{n\to\infty} \frac{2}{nln(n)} + \lim_{n\to\infty} 2 + \frac{2}{n} \\
&= 2
\end{aligned}
$$

Notice when n approaches positive infinity the constant 2 is the only term remaining because other terms converge to 0. This shows that the function nln(n) times some constant can over-bound the function f(n) thus proving that f(n) is O(nln(n)).